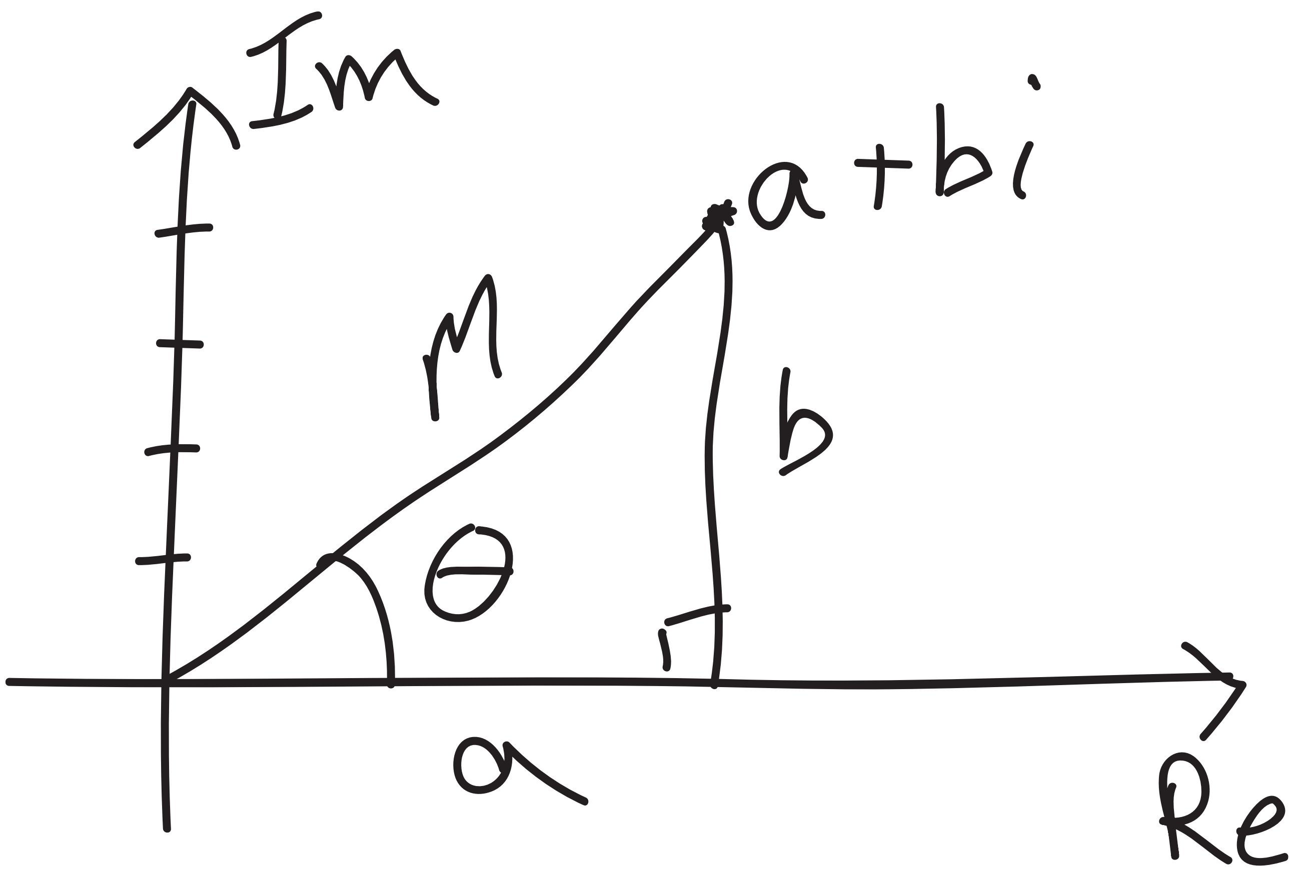
Today

- Euler's formula
- Complex case
- Repeated roots
- The geometry of homogeneous and nonhomogeneous matrix equations
- Solving nonhomogeneous equations
    - Method of undetermined coefficients

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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$$a = M\cos\theta$$
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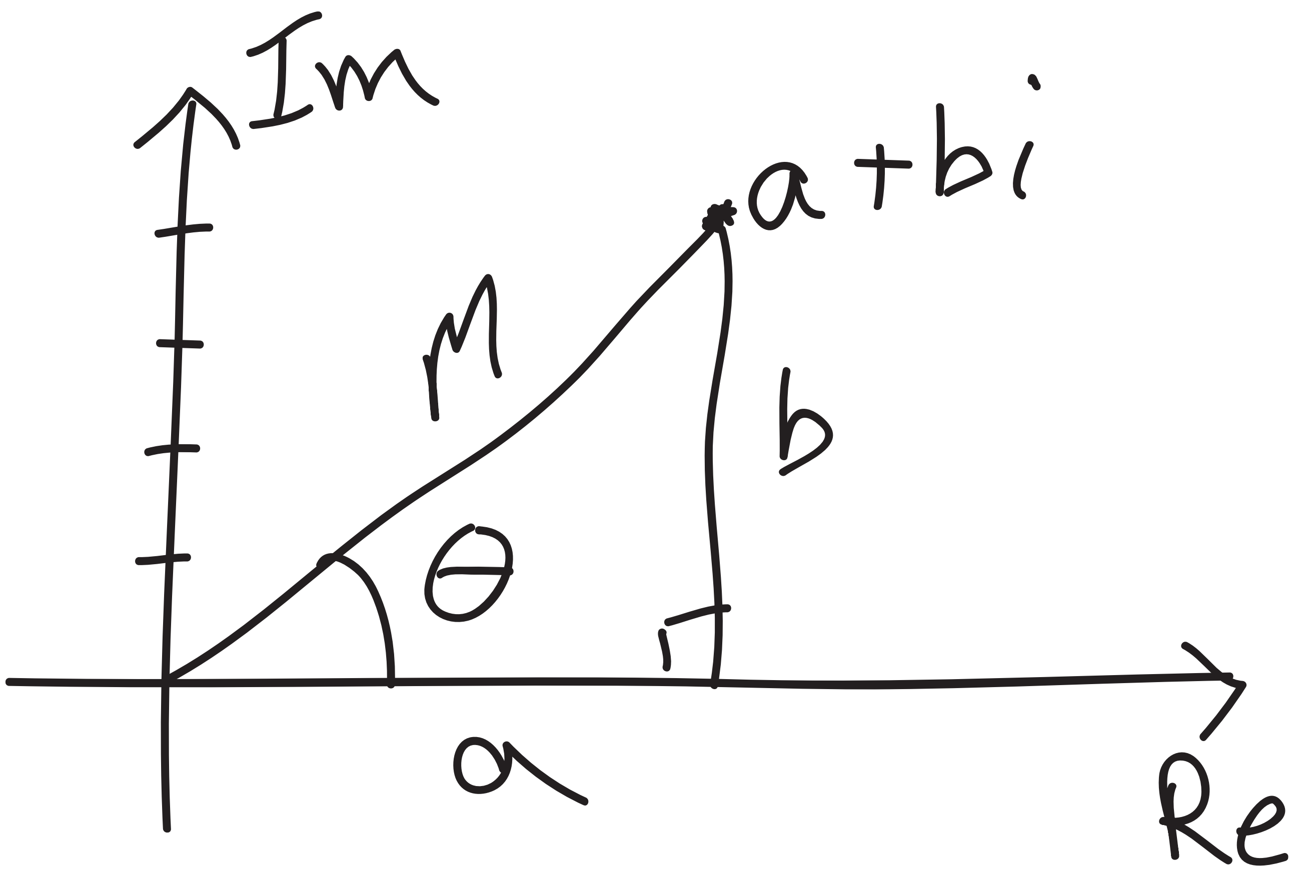

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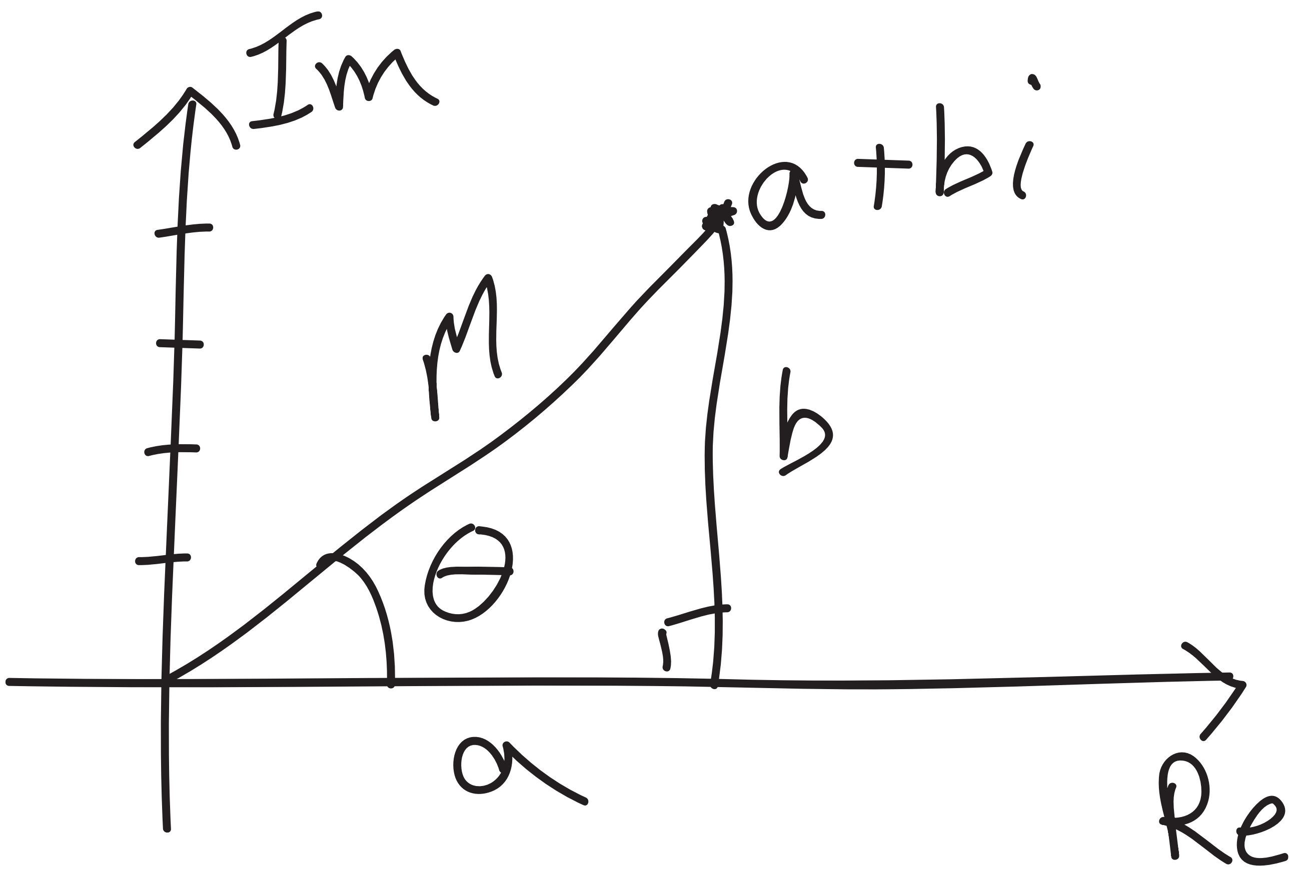

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$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

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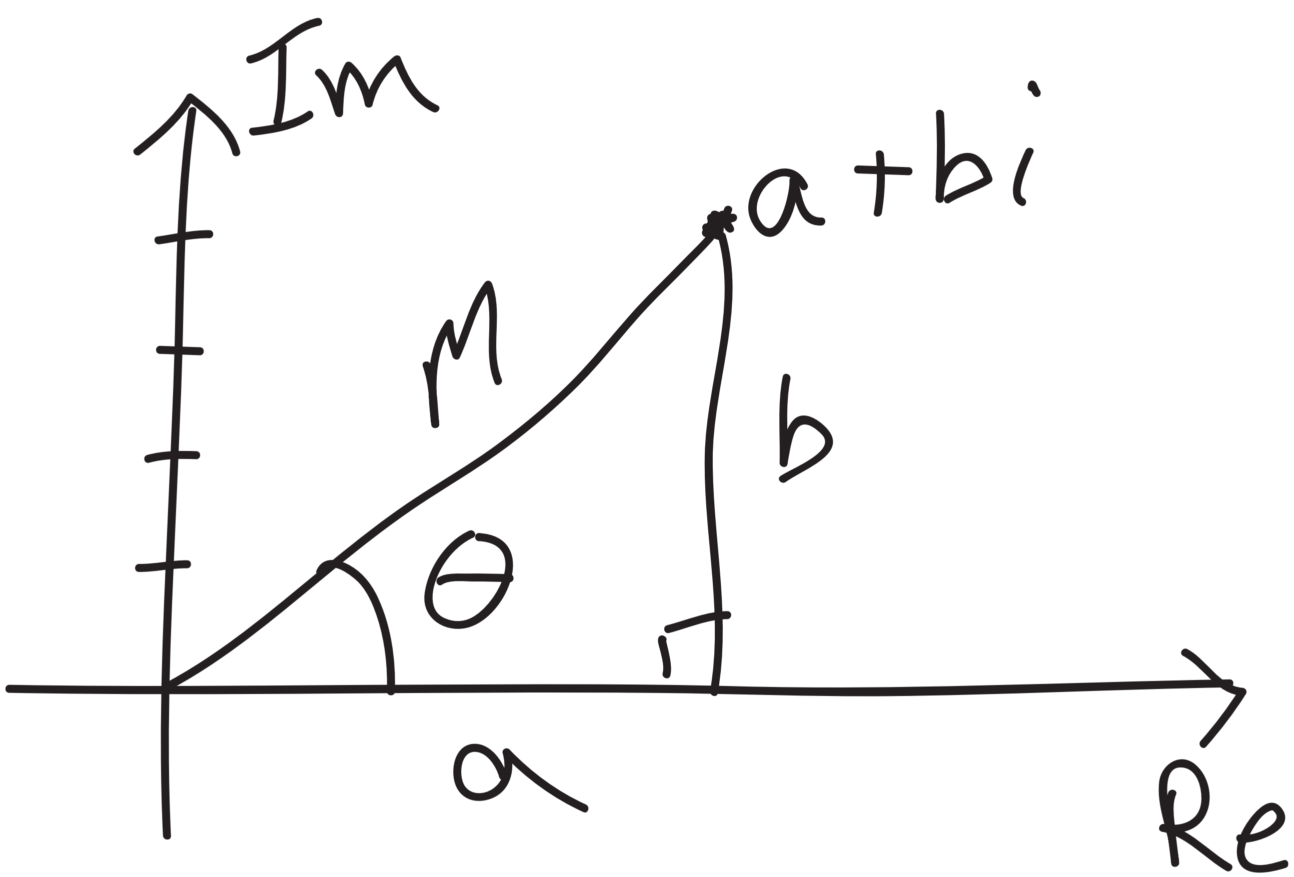

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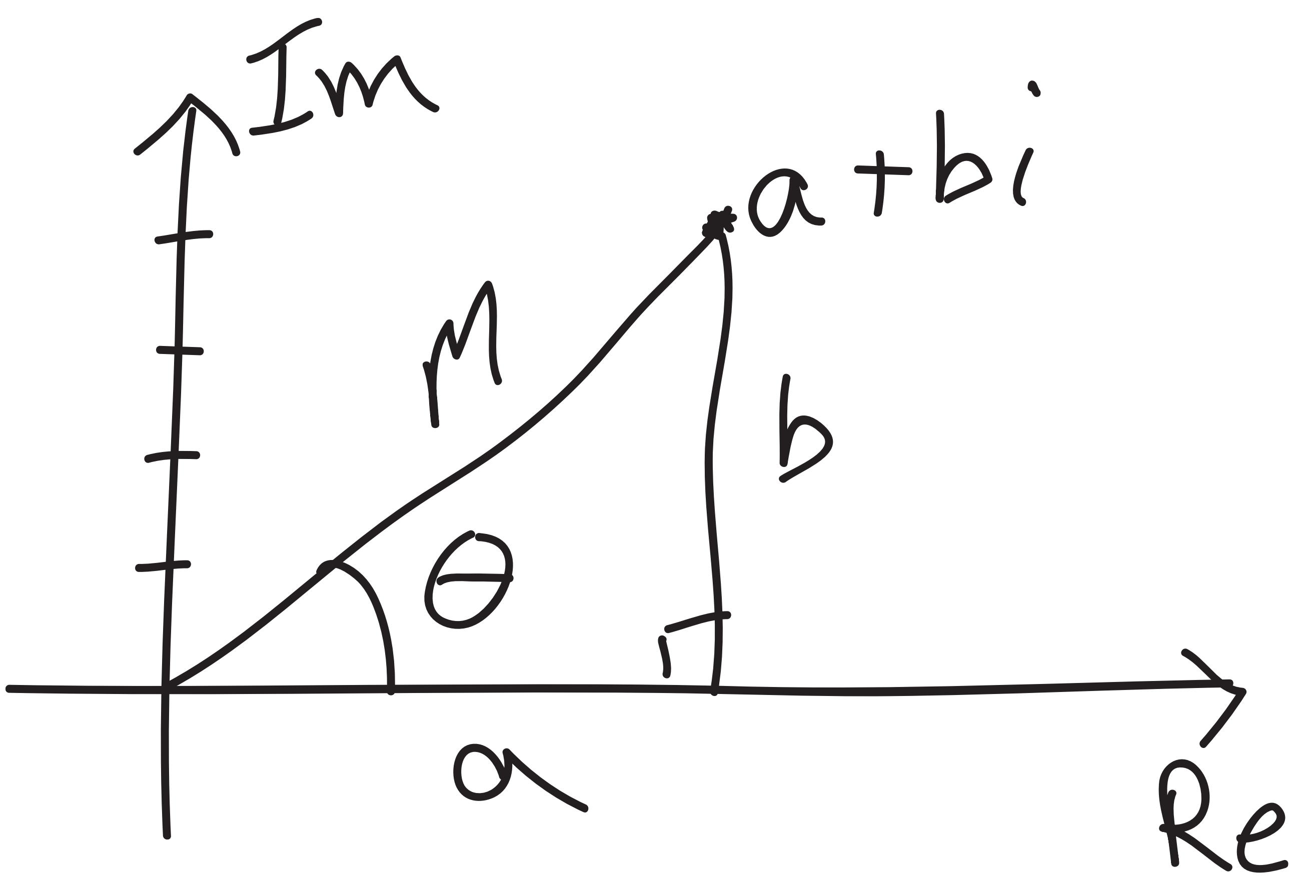

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θ is sometimes called the argument or phase of $a + bi$.

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(A) cos x

(C) e^x

(B) sin x

(D) ln x

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

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$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

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$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta$$

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Euler's formula:

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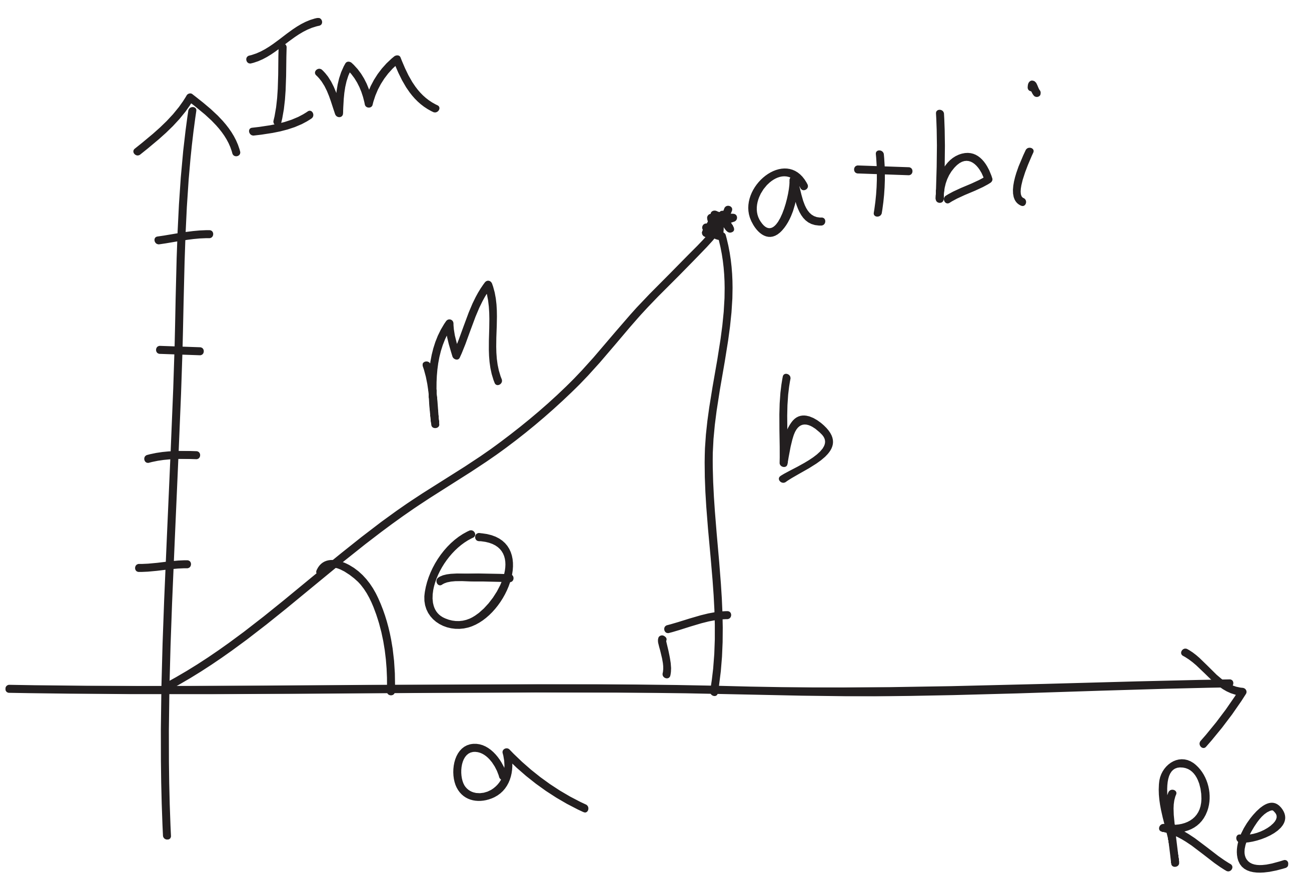

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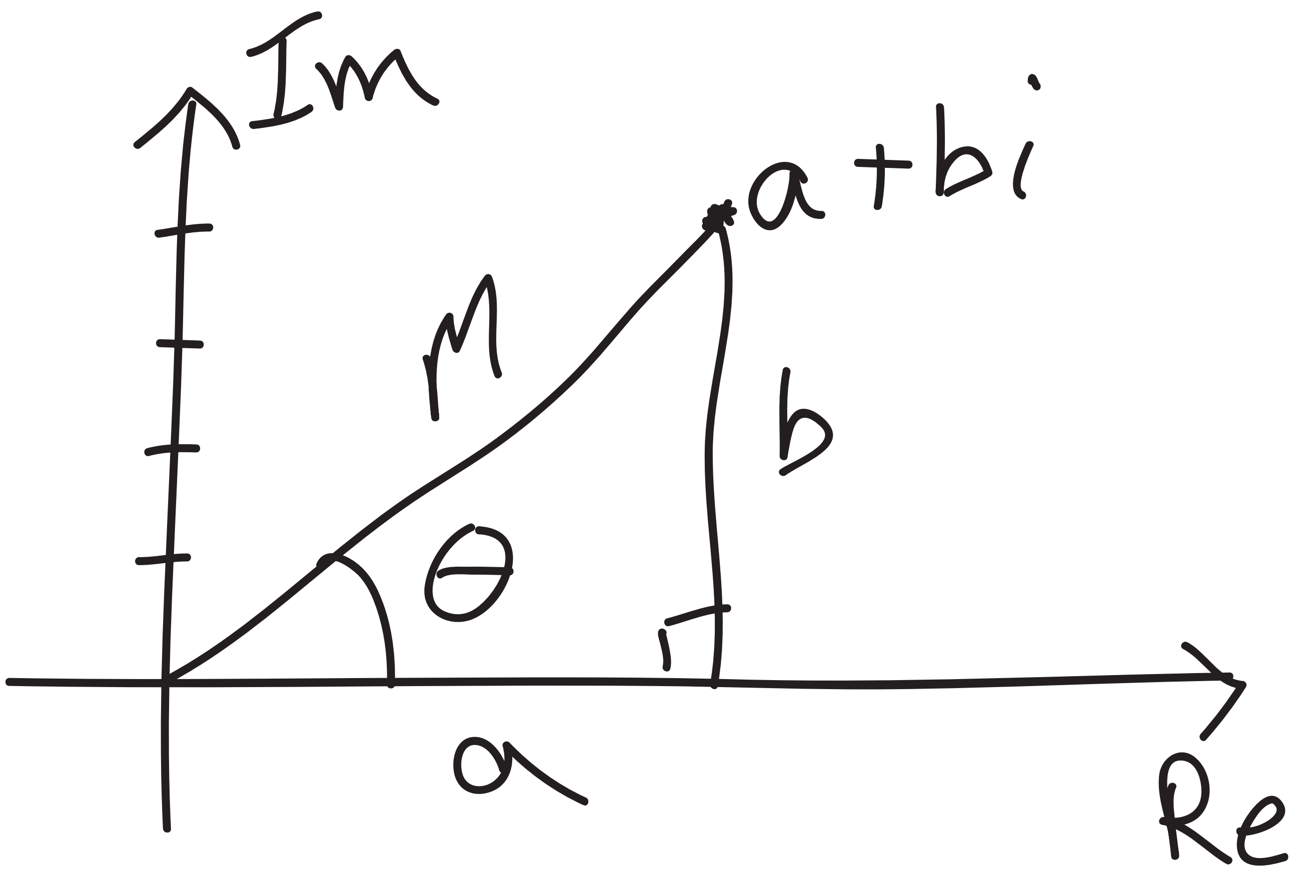

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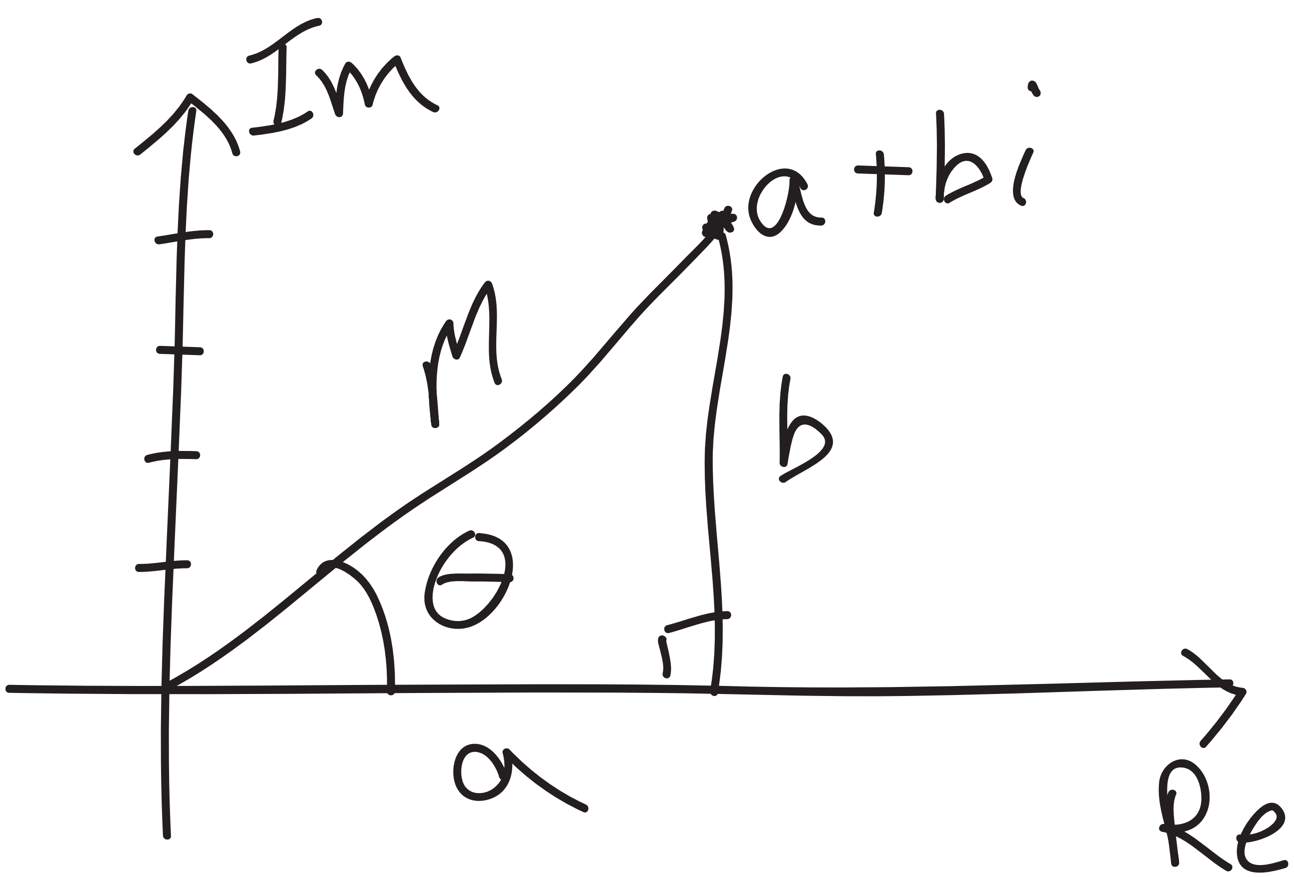

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(Polar form makes multiplication much cleaner)

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- General solution:

$$y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$

Complex roots

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i$, $r_2 = 1 - 2i$

(B) $r_1 = -1 + 2i$, $r_2 = -1 - 2i$

(C) $r_1 = 1 - 2i$, $r_2 = -1 + 2i$

(D) $r_1 = 2 + 4i$, $r_2 = 2 - 4i$

(E) $r_1 = -2 + 4i$, $r_2 = -2 - 4i$

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- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

(A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

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- General solution: $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

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(C) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

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