Today

- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

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- $m=2000$. Called steady state - a constant solution.
- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

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- A couple questions/examples to explore on your own:
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    - Example: $$\frac{dy}{dt} = y^2, \quad y(0) = 1$$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

Second order linear equations

- The general form for a second order linear equation:

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- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

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  (B) $y(t) = y_1(t)+y_2(t)$

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- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

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- One case where the arbitrary constants DO appear as we calculate:

$$y'' + y' = 0$$

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$$(e^{rt})'' + (e^{rt})' = 0$$
$$r^2 e^{rt} + r e^{rt} = 0$$
$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$r = 0, \quad r = -1$$

$$y = C_1 e^0 + C_2 e^{-t}$$
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(C) $y(t) = \frac{7}{4} e^{4t} + \frac{5}{4} e^{-4t}$

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- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0,\ y'(0) = v_0$ with these two solutions?

- Solve this system for C_1, C_2...

- Can’t do it. Why?

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$$\det \begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} = 0$$

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$$\det\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

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- For ICs other than t_0=0, we require that

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Independence and the Wronskian (Section 3.2)

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$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

- For ICs other than t_0=0, we require that

$$y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

- This quantity is called the Wronskian.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

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  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

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(A) $C_1 = e^{-2t-3}, \ C_2 = -e^{-2t+3}$

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- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

  is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

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- So for case i (distinct roots), can we form a general solution from

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$$= (r_1 - r_2) e^{r_1 t} e^{r_2 t} \neq 0$$

So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.

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(A) ...is unbounded for all ICs.

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Challenge: come up with an initial condition for (iii) that has a bounded solution.

Complex roots (Section 3.3)

- Complex number review (Euler's formula)
- Complex roots of the characteristic equation
- From complex solutions to real solutions

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- For any equation, $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$, the solutions have the form $x = \alpha \pm \beta i$ where α and β are both real numbers.
- For α+βi, we call α the real part and β the imaginary part.

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$$(a + bi) + (c + di) = a + c + (b + d)i$$

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- What is the inverse of c+di written in the usual complex form p+qi?

(A) $c - di$

(B) $\frac{c + di}{c^2 + d^2}$

(C) $\frac{c - di}{c^2 + d^2}$

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Complex number review

- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$