Find the solution to the equation

$$u_t = 4u_{xx}$$

with IC $u(x,0) = 0$

and BCs $u(0,t) = 2$

$u(3,t) = 5$

$u_{ss}(x) = ax + b$

$u_{ss}(0) = b = 2$

$u_{ss}(3) = 3a + 2 = 5 \Rightarrow a = 1$

$u_{ss}(x) = x + 2$

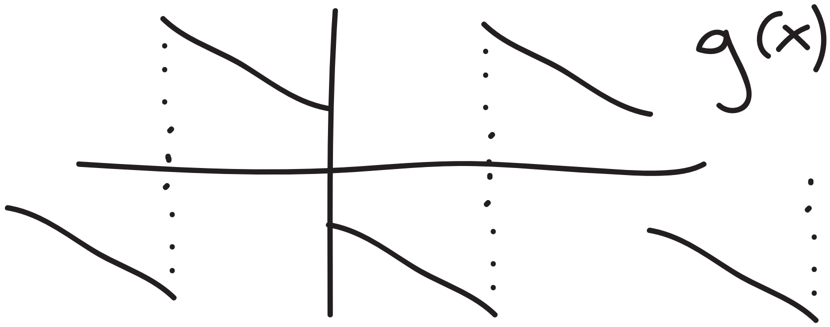
$$u(x,t) = x + 2 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t} \sin \frac{n\pi x}{3}$$

$$u(x,0) = x + 2 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} = 0$$

Choose b_n so that

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} = -x - 2 \quad \text{on } (0,3).$$

Extend $-x-2$ as odd about $x=0$ and then periodic with period 6.



$$b_n = \frac{1}{3}\int_{-3}^{3} g(x)\sin\frac{n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 (-x-2)\sin\frac{n\pi x}{3}\,dx$$

$$= \cdots = \begin{cases} 2\left(\dfrac{-2+5(-1)^n}{n\pi}\right) & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

$$u(x,t) = x+2 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t} \sin\frac{n\pi x}{3}$$

$$= x+2 + \sum_{k=1}^{\infty} b_{2k-1} e^{-\frac{4}{9}(2k-1)^2\pi^2 t} \sin\frac{(2k-1)\pi x}{3}$$

Check: $$u_t = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{9}\cdot 4\right) b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t} \sin\frac{n\pi x}{3}$$

$$4u_{xx} = 4\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{9}\right) b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t} \sin\frac{n\pi x}{3}$$

So $u_t = 4u_{xx}$

$$u(x,0) = x+2 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3} = x+2+(-x-2)$$
$$= 0$$

$u(0,t) = 2$, $u(3,t) = 5$.

Find the solution to the equation

$$u_t = 4u_{xx}$$

with IC $u(x,0) = 6x$

and BCs $u_x(0,t) = 4$

$u_x(1,t) = 4$

$u_{ss}(x) = ax + b$

$\frac{\partial u_{ss}}{\partial x}(0) = a = 4 = \frac{\partial u_{ss}}{\partial x}(5)$

To determine b, note that

$J_0 = -Du_x(0,t) = -16 = -Du_x(1,t) = J_1$

so that the total amount of mass (assuming $u(x,t)$ is mass/unit length) inside the domain $[0,1]$ is constant in time. That is,

$$\int_0^1 u(x,t)\,dx = \text{Total mass} = \text{constant.}$$

So initial mass = mass at steady state

$$\underbrace{\int_0^1 u(x,0)\,dx = \int_0^1 6x\,dx}_{} = \underbrace{\int_0^1 (4x+b)\,dx}_{}$$

$$3x^2\Big|_0^1 = 2x^2 + bx\Big|_0^1$$

$$3 = 2 + b \longrightarrow b = 1$$

$$u_{ss}(x) = 4x+1$$

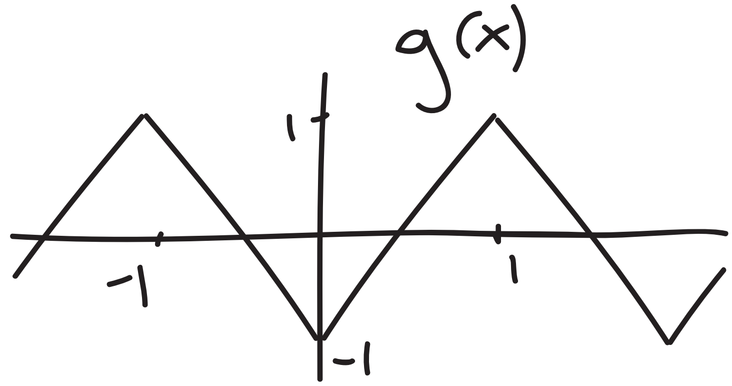
$$u(x,t) = 4x+1+\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n e^{-4n^2\pi^2 t}\cos n\pi x$$

$$u(x,0) = 4x+1+\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x = 6x$$

Choose a_n so that

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x = 2x-1 \text{ on } (0,1).$$

Extend $2x-1$ as an even function about $x=0$. Then extend that function as a periodic function with period 2.



$$a_0 = \int_{-1}^{1} g(x)\,dx = 2\int_0^1 (2x-1)\,dx = 0$$

Note: This is zero precisely because we subtracted the s.s.

$$a_n = \int_{-1}^{1} g(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 (2x-1)\cos n\pi x\,dx$$

$$= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

This is zero for n even because $2x-1$ happens to be odd about $x=\frac{1}{2}$.

$$u(x,t) = 4x+1 + \sum_{n=1}^{\infty} a_n e^{-4n^2\pi^2 t} \cos n\pi x$$

where $a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right).$