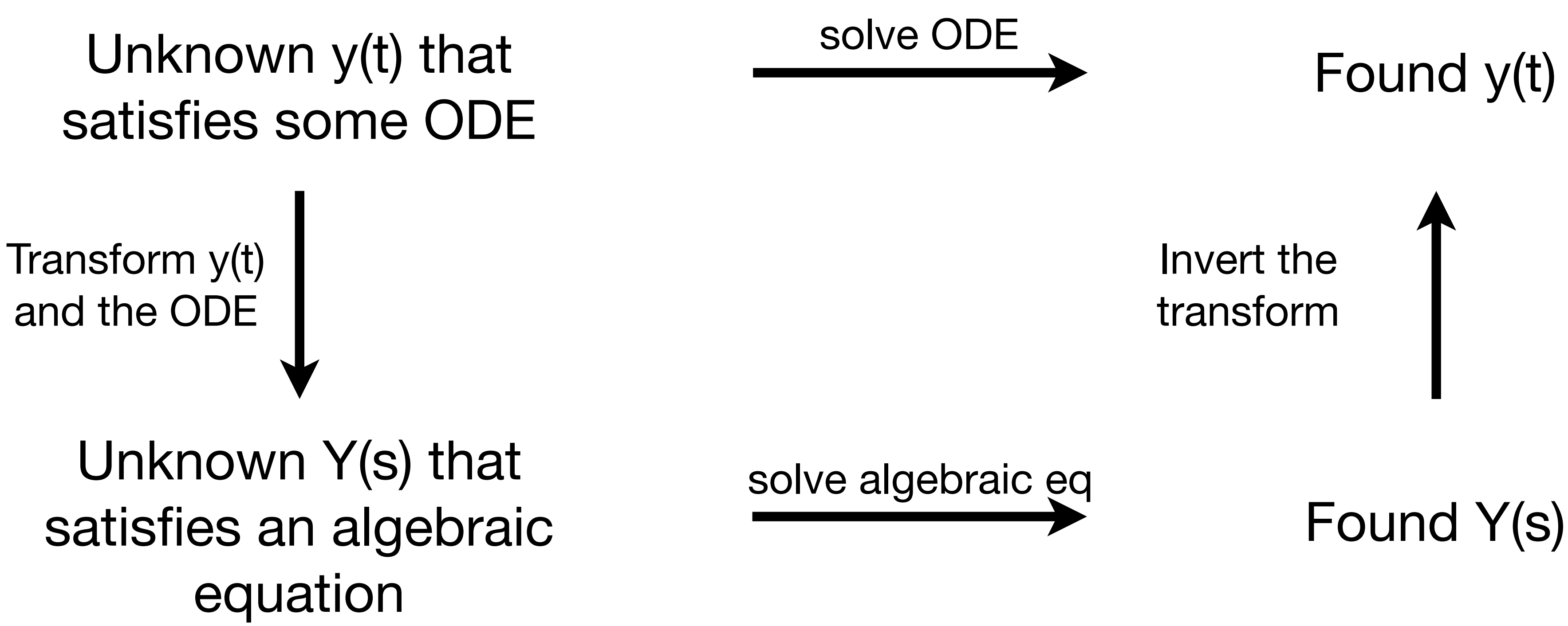
Today

- Intro to the Laplace Transform
- Solving ODEs with forcing terms using Laplace transforms - examples

Laplace transforms - intro

- Using the Laplace Transform to solve (linear) ODEs.

- Idea:



- Laplace transform of y(t): $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\ dt$

Laplace transforms - examples

- What is the Laplace transform of $y(t) = 3$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} 3 \ dt$$

$$= -\frac{3}{s} e^{-st} \Big|_0^\infty$$

$$= \lim_{A \to \infty} -\frac{3}{s} e^{-st} \Big|_0^A$$

$$= -\frac{3}{s} \left(\lim_{A \to \infty} e^{-sA} - 1 \right)$$

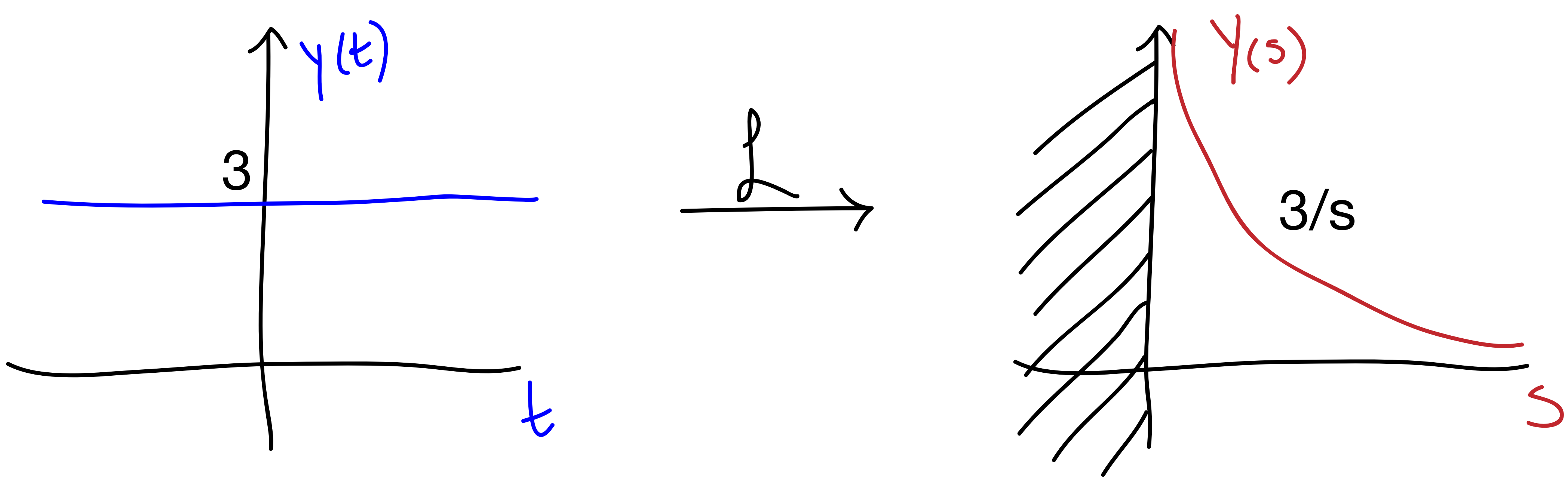
$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

Laplace transforms - examples

- What is the Laplace transform of $y(t) = 3$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} 3 \; dt$$

$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

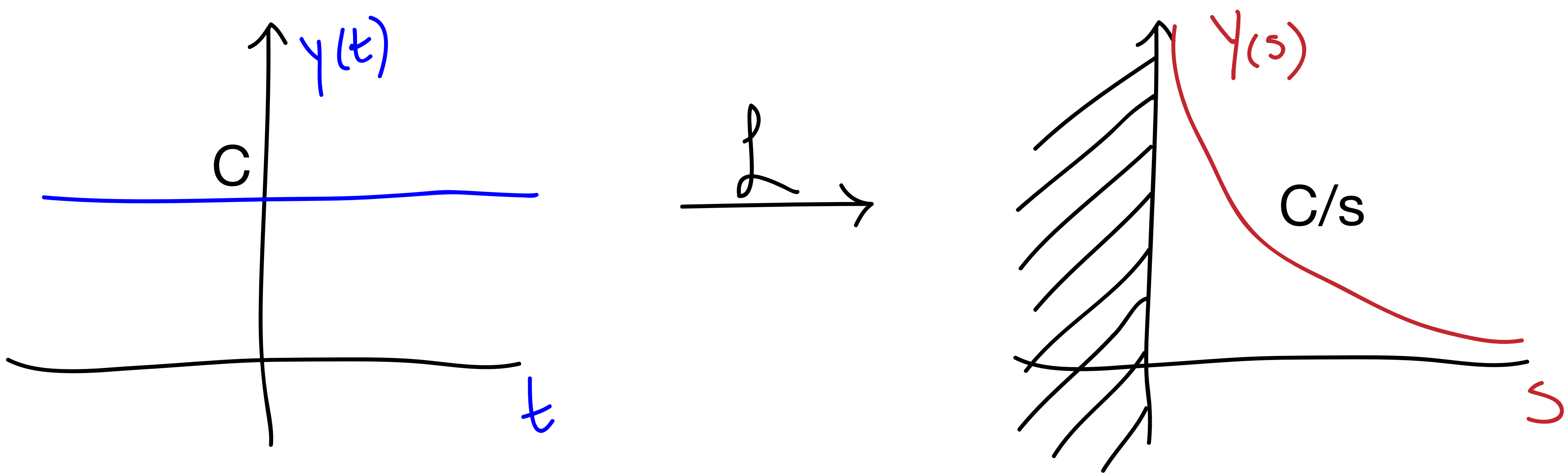


Laplace transforms - examples

- What is the Laplace transform of $y(t) = C$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} C \, dt$$

$$= \frac{C}{s}$$ provided $s > 0$ and does not exist otherwise.



Laplace transforms - examples

- What is the Laplace transform of $y(t) = e^{6t}$?

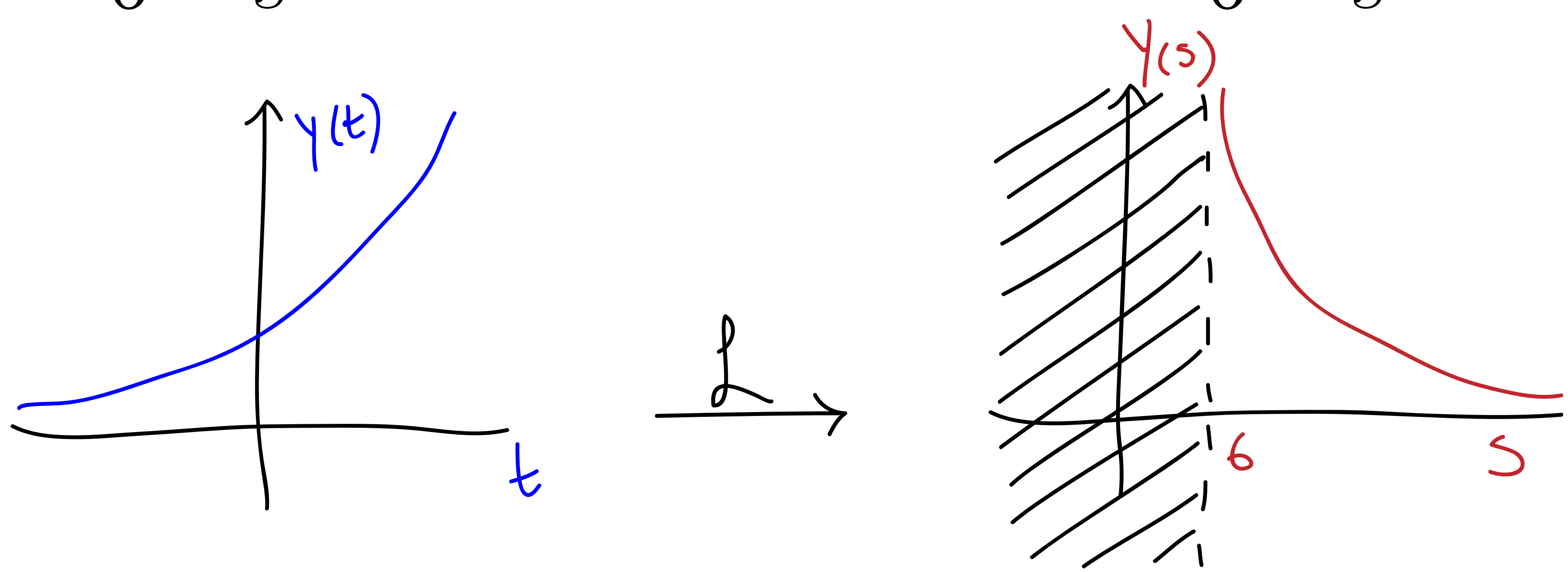
$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} e^{6t} \, dt$$

(A) $Y(s) = \dfrac{1}{s-6} \qquad s > 0$

★(C) $Y(s) = \dfrac{1}{s-6} \qquad s > 6$

(B) $Y(s) = \dfrac{1}{6-s} \qquad s > 6$

(D) $Y(s) = \dfrac{1}{6-s} \qquad s > 0$



Laplace transforms - examples

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underset{u}{e^{-st}} \underset{dv}{\sin t \, dt}$$

$$= e^{-st}(-\cos t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}(-\cos t)\, dt$$

$$= \lim_{A\to\infty} e^{-sA}(-\cos A) - (-1) - \int_0^\infty (-s)e^{-st}(-\cos t)\, dt$$

$$= 1 - s\int_0^\infty e^{-st}\cos t \, dt \qquad s > 0$$

$$= 1 - s\left(\underbrace{e^{-st}\sin t\Big|_0^\infty}_{0} + s\underbrace{\int_0^\infty e^{-st}\sin t\, dt}_{F(s)} \right)$$

$$= 1 - s^2 F(s) \qquad s > 0$$

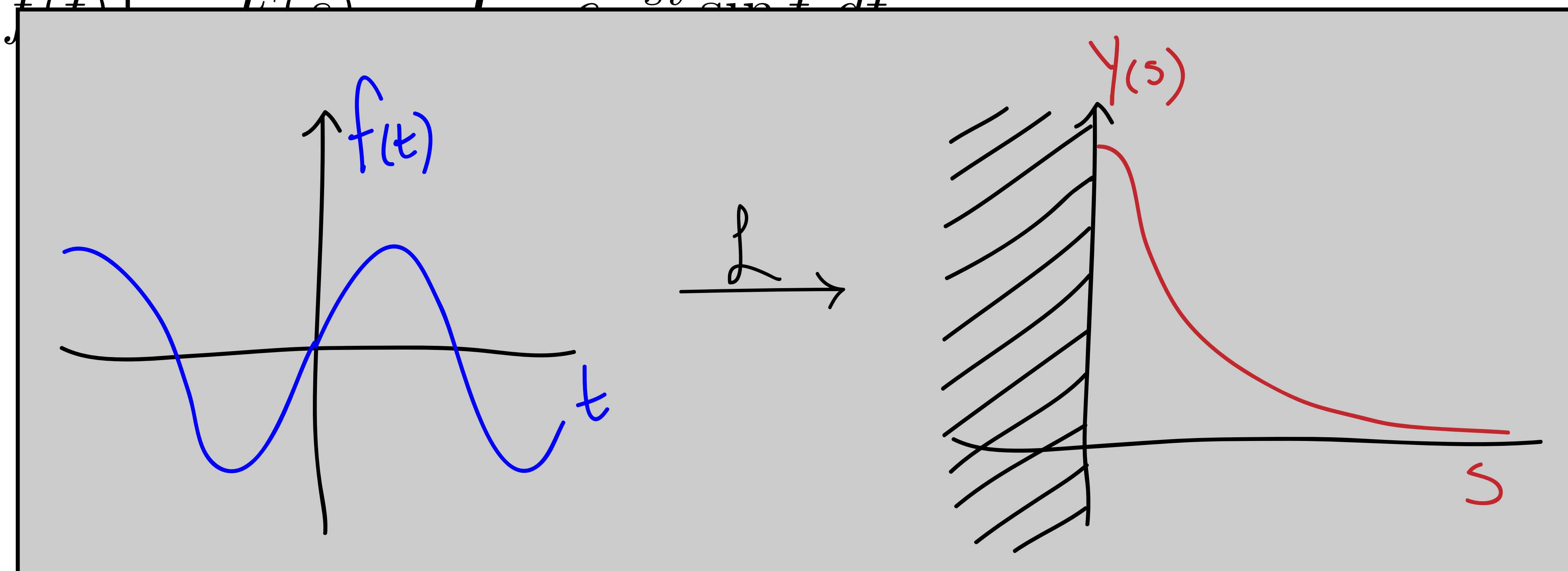
$$(1+s^2)F(s) = 1$$

$$F(s) = \frac{1}{1+s^2} \qquad s > 0$$

Laplace transforms - examples

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int^{\infty} e^{-st} \sin t \, dt$$



$$dt$$

$$= 1 - s\left(e^{-st} \sin t \Big|_0^\infty + s \int_0^\infty e^{-st} \sin t \, dt \right)$$

$$= 1 - s^2 F(s) \qquad s > 0$$

$$(1 + s^2) F(s) = 1$$

$$F(s) = \frac{1}{1 + s^2} \qquad s > 0$$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = \sin(\omega t)$? $\quad (\omega > 0)$

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st} \sin(\omega t)\ dt$$

- Hint: $u = \omega t$
  $du = \omega dt$

(A) $H(s) = \dfrac{\omega}{\omega^2 + s^2}$

(B) $H(s) = \dfrac{1}{1 + \left(\frac{s}{w}\right)^2}$

(C) $H(s) = \dfrac{1}{\omega} \dfrac{1}{1 + s^2}$

(D) $H(s) = \dfrac{1}{1 + s^2}$

(E) Huh?

$$H(s) = \int_0^\infty e^{-s\frac{u}{\omega}} \sin u \ \frac{du}{\omega}$$

$$= \frac{1}{\omega} \int_0^\infty e^{-\frac{s}{\omega} u} \sin u\ du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

$$= \frac{1}{\omega}\ \frac{1}{1 + \left(\frac{s}{\omega}\right)^2} \qquad s > 0$$

Laplace transforms - examples

- What is the Laplace transform of $g(t) = \cos t$?

- Could calculate directly but note that g(t) = f'(t) where f(t)=sin t.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt$$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\ dt}_{dv}$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\ dt$$

$$= -f(0) + sF(s) \qquad s > 0$$

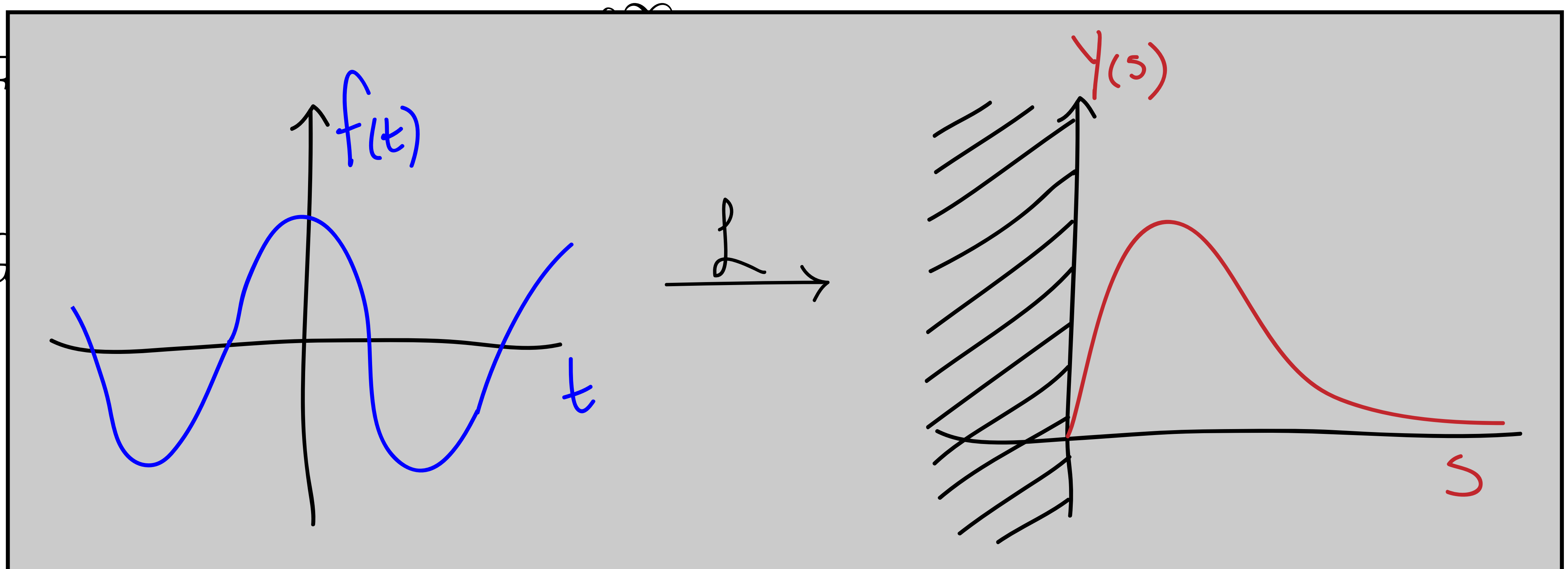
$$= -0 + s\frac{1}{1+s^2} \ = \frac{s}{1+s^2}$$

Laplace transforms - examples

- What is the Lapla

$$G(s) = \mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

- Could calculate directly but note that g(t) = f'(t) where f(t)=sin t.



$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

(A) $H(s) = \omega \; F(s)$

☆ (B) $H(s) = \dfrac{1}{\omega} \; F\left(\dfrac{s}{\omega}\right)$

(C) $H(s) = \omega \; F\left(\dfrac{s}{\omega}\right)$

(D) $H(s) = \dfrac{1}{\omega} \; F(s)$

(E) Don’t know.

$$\mathcal{L}\{f(\omega t)\} = \int_0^\infty e^{-st} f(\omega t) \; dt$$

- Hint: $u = \omega t$
  $du = \omega dt$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

- Recall two examples back:

$$u = \omega t$$
$$du = \omega dt$$

$$\begin{aligned}
\mathcal{L}\{h(t)\} = H(s) &= \int_0^\infty e^{-st} \; f(\omega t) \; dt \\
&= \int_0^\infty e^{-s\frac{u}{\omega}} \; f(u) \; \frac{du}{\omega} \\
&= \frac{1}{\omega} \int_0^\infty e^{-\frac{s}{\omega}u} \; f(u) \, du \\
&= \frac{1}{\omega} \; F\left(\frac{s}{\omega}\right)
\end{aligned}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\begin{aligned}
&\mathcal{L}\{\cos(\omega t)\} \\
&\qquad = \frac{1}{\omega} \; \frac{\frac{s}{\omega}}{1 + \left(\frac{s}{\omega}\right)^2} \\
&\qquad = \frac{s}{\omega^2 + s^2}
\end{aligned}$$

Laplace transforms - examples

- What is the Laplace transform of $k(t) = e^{at} f(t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

$$\mathcal{L}\{k(t)\} = K(s) = \int_0^\infty e^{-st} e^{at} f(t)\ dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\ dt$$

$$= F(s-a)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{e^{-3t} \cos t\} =$$

(A) $\dfrac{s}{1+(s+3)^2}$

(B) $\dfrac{1}{1+(s+3)^2}$

★(C) $\dfrac{s+3}{s^2+6s+10}$

(D) $\dfrac{1}{s^2+6s+10}$

Solving IVPs using Laplace transforms

- Solve the equation $ay'' + by' + cy = 0$ using Laplace transforms.
- Recall that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.
- Applying this to f”, we find that

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(sF(s) - f(0)) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

- Transforming both sides of the equation,

$$\mathcal{L}\{ay'' + by' + cy\} = 0$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = 0$$

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0$$

$$(as^2 + bs + c)Y(s) = asy(0) + ay'(0) + by(0)$$

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 4y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0$$
$$s^2Y(s) - s - 0 + 4Y(s) = 0$$
$$s^2Y(s) + 4Y(s) = s$$
$$Y(s) = \frac{s}{s^2+4}$$

- To find y(t), we have to invert the transform. What y(t) would have Y(s) as its transform?

- Recall that $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{\omega^2 + s^2}$. So $y(t) = \cos(2t)$.

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{\ldots}$$

$$\lambda = \frac{-6 \pm i\sqrt{52-36}}{2} = -3 \pm 2i$$

- To find y(t), we hav … would have Y(s) as its transform?

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{e^{-3t}\cos t\} = \frac{s+3}{1+(s+3)^2}$$

$$Y(s) = \frac{s+3+3}{s^2+6s+9+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{2}\frac{2}{(s+3)^2+4}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13} = \frac{s+6}{s^2+6s+9+4} = \frac{s+6}{(s+3)^2+4} = \frac{s+3+3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+2^2} + \frac{3}{2}\frac{2}{(s+3)^2+2^2}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

1. Does the denominator have real or complex roots? Complex.
2. Complete the square.
3. Put numerator in form (s+α)+β where (s+α) is the completed square.
4. Fix up coefficient of the term with no s in the numerator.
5. Invert.