Today

- General solution for complex eigenvalues case.
- Shapes of solutions for complex eigenvalues case.

Calculating eigenvalues - trace/det shortcut

- For the general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- find the characteristic equation and solve it to find the eigenvalues.

(A) $\lambda^2 + (ad - bc)\lambda + a + d = 0$

(B) $\lambda^2 + (b + c)\lambda + ac - bd = 0$

(C) $\lambda^2 - (a + d)\lambda + ad - bc = 0$

(D) $\lambda^2 + (a - d)\lambda + ad + bc = 0$

(E) I don’t know how to find eigenvalues.

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$$\lambda^2 - \mathrm{tr(A)}\lambda + \det(A) = 0$$

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Complex eigenvalues - example

- Find the general solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x}$.
  - The eigenvalues are

(A) $\lambda = 1 \pm 2i$

(B) $\lambda = -1, 3$

(C) $\lambda = 2 \pm 4i$

(D) $\lambda = -2, 6$

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  - The eigenvectors are . . .

$$A - \lambda_1 I = \begin{pmatrix} 1-(1+2i) & 1 \\ -4 & 1-(1+2i) \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \times \tfrac{1}{2}i$$

$$\sim \begin{pmatrix} -2i & 1 \\ -2i & 1 \end{pmatrix}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

Complex eigenvalues - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

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Complex eigenvalues - general case

- Find e-values, $\lambda = \alpha \pm \beta i$, and e-vectors, $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

- Using method on previous slide, you should get:

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$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(\beta t) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin(\beta t) \right) \right.$$

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Complex eigenvalues - example

- Suppose you find eigenvalue $\lambda = 2\pi i$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

  Which of the following is a solution to the original equation?

(A) $\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$

(B) $\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t)$

(C) $\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t)$

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- Sum and difference trick lets us take the Real and Imaginary parts as two indep. solutions

Complex eigenvalues - example

- But what about $\lambda_2 = -2\pi i$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$?

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\begin{aligned}
\overline{\mathbf{x}}(\mathbf{t}) &= e^{-2\pi i t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= (\cos(-2\pi t) + i\sin(-2\pi t)) \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
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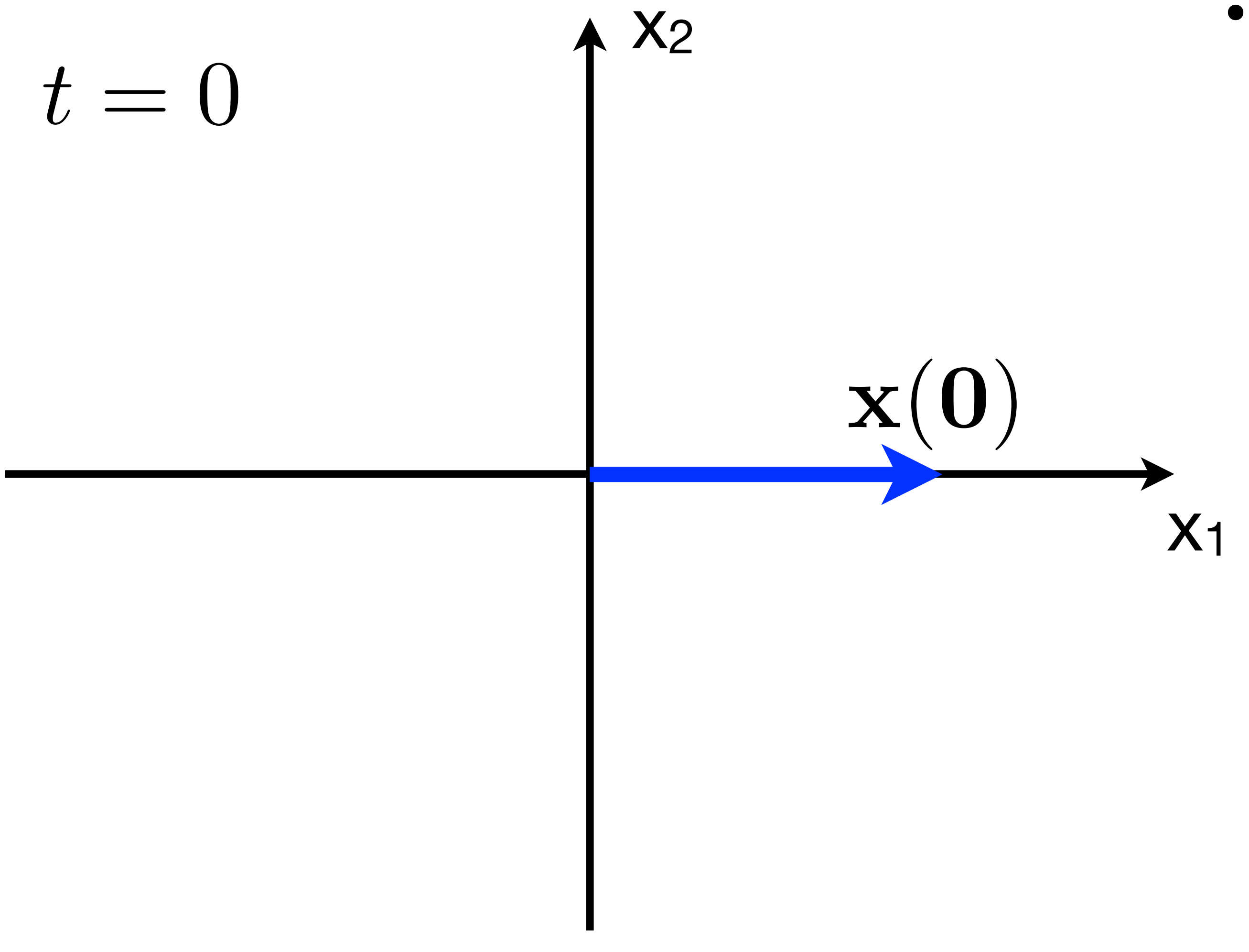
Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



- What happens as t increases?

Complex eigenvalues - example

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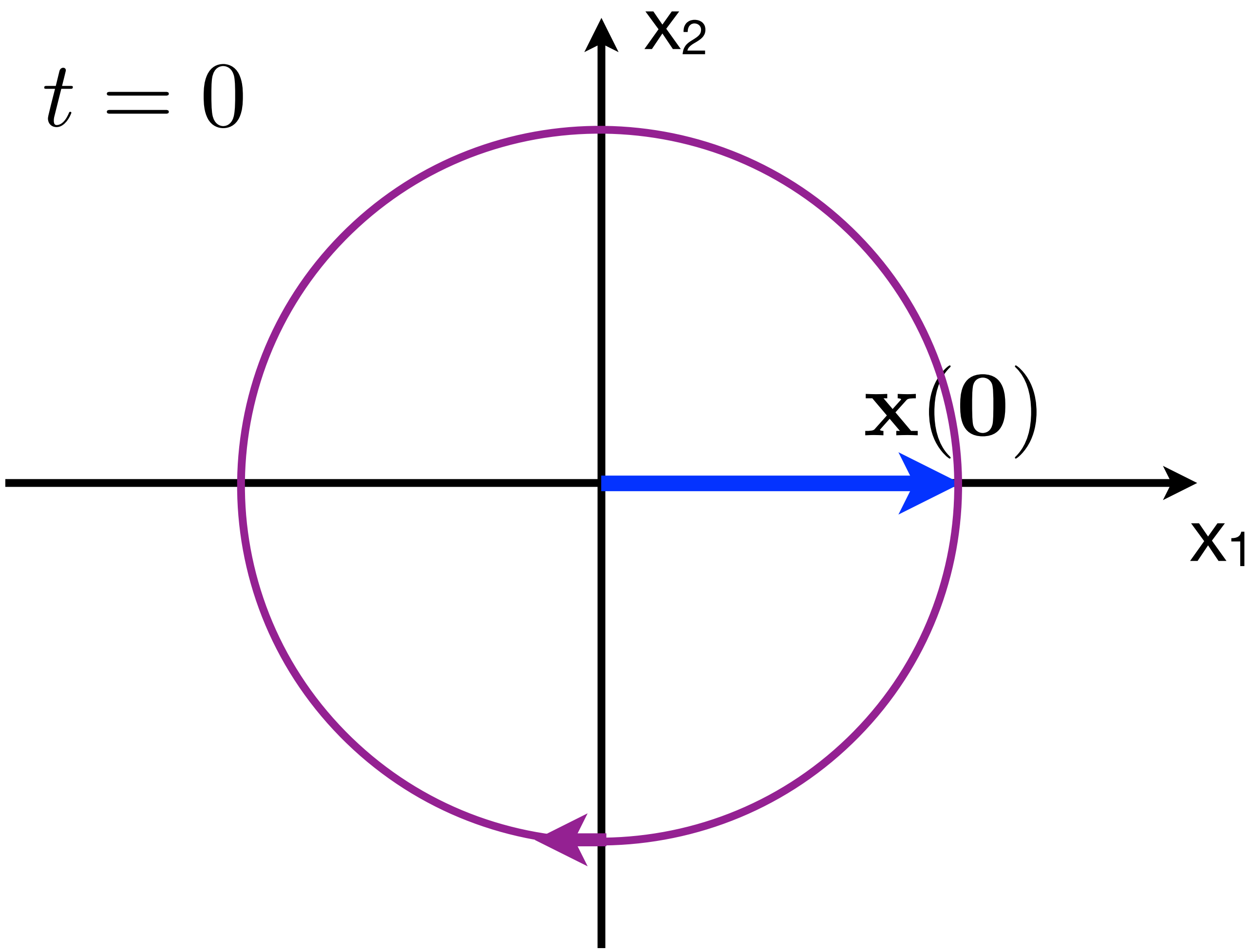

- What happens as t increases?

  (A) The vector rotates clockwise.

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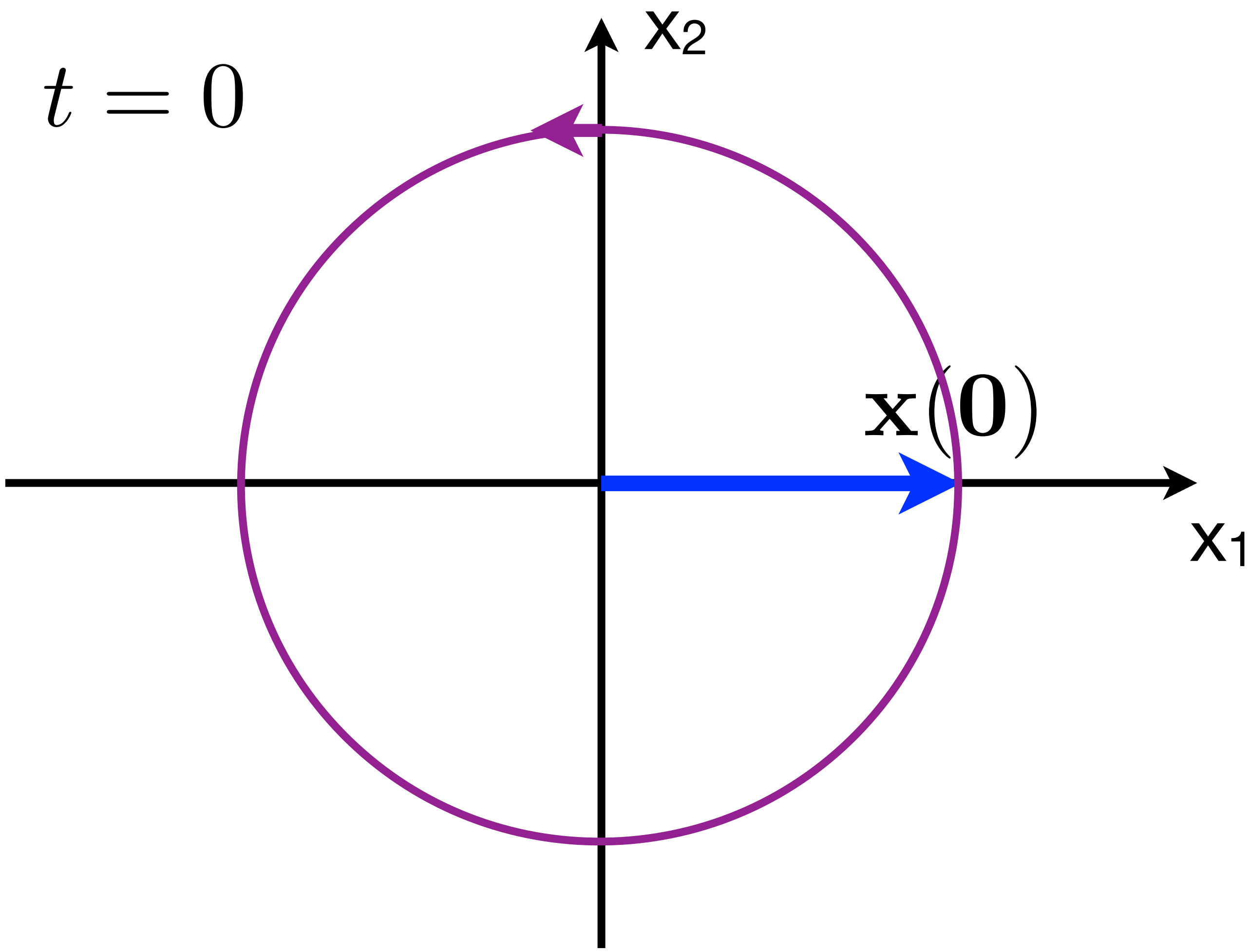

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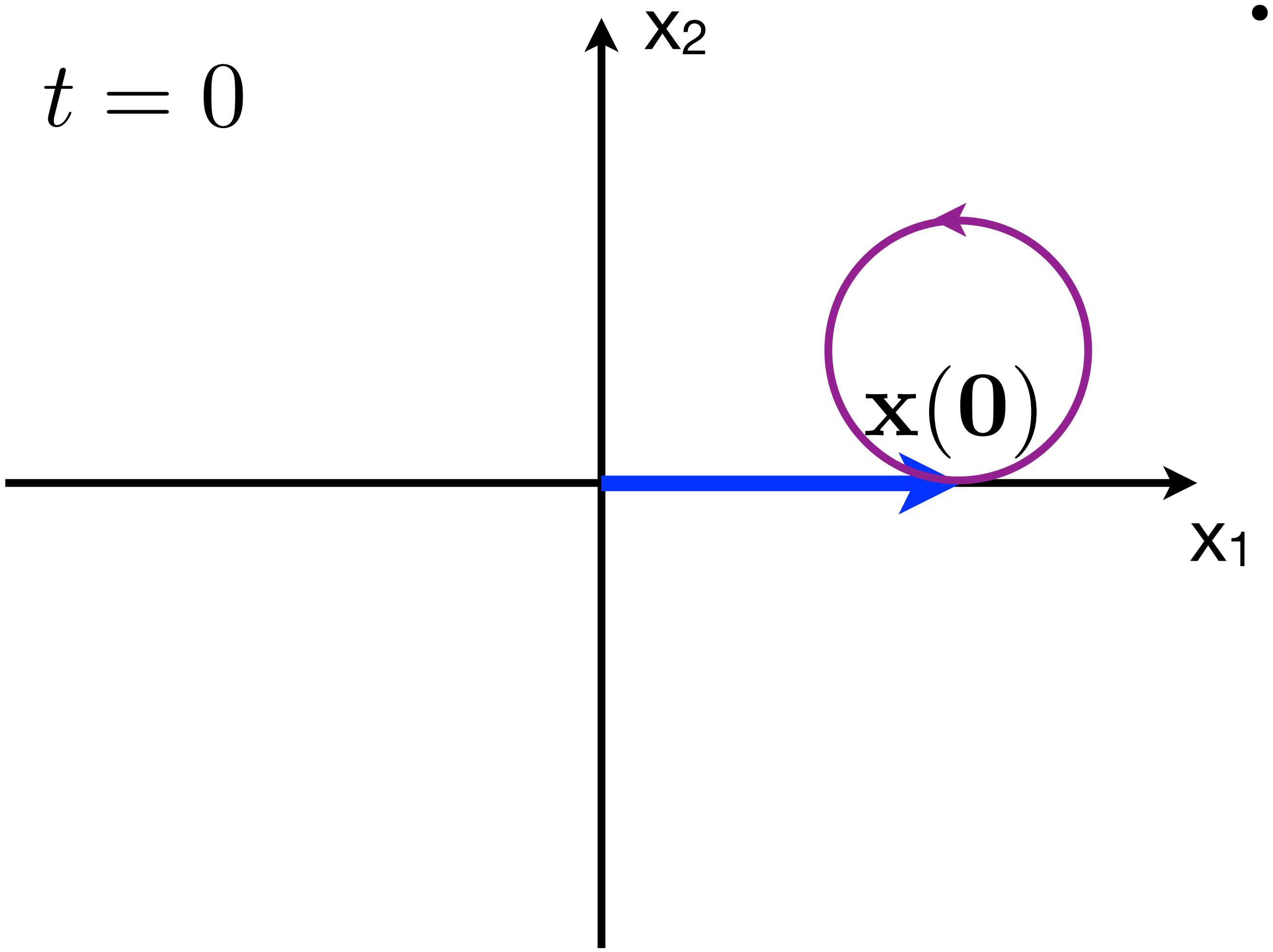

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  - (A) The vector rotates clockwise.
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  - (C) The tip of the vector maps out a circle in the first quadrant.

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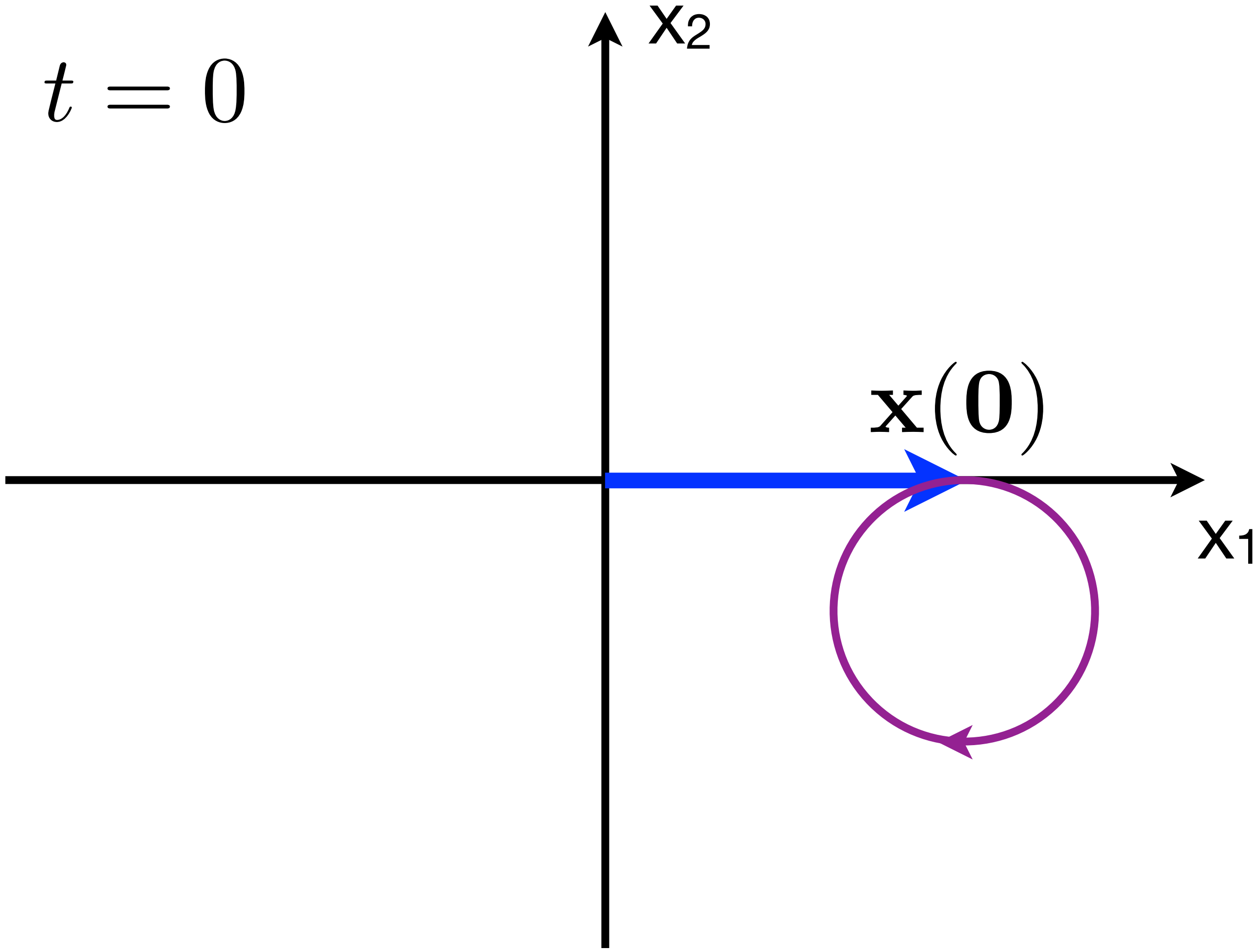
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$t = 0$



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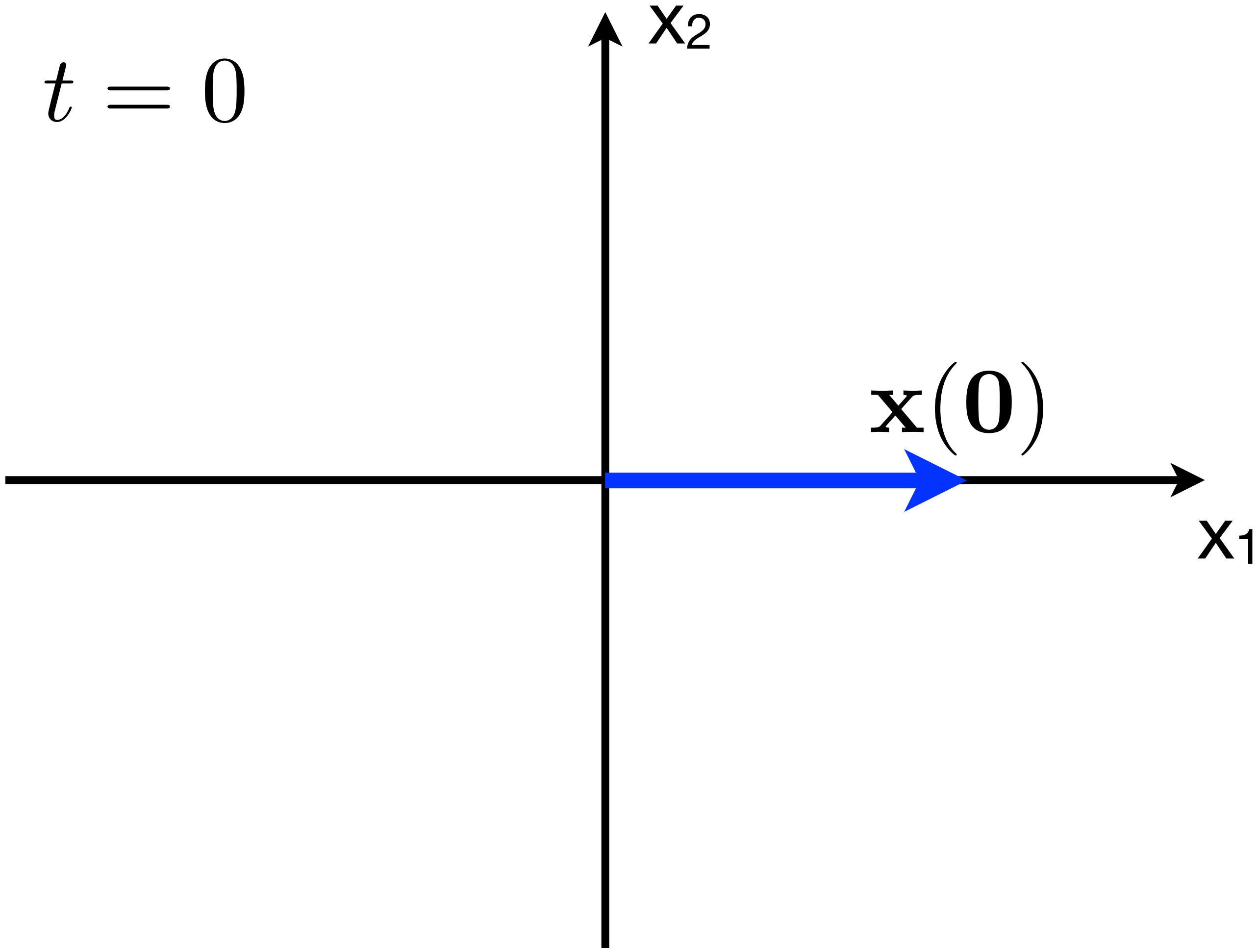
  (B) The vector rotates counter-clockwise.

  (C) The tip of the vector maps out a circle in the first quadrant.

  (D) The tip of the vector maps out a circle in the fourth quadrant.

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



- What happens as t increases?

  (A) The vector rotates clockwise.

  (B) The vector rotates counter-clockwise.

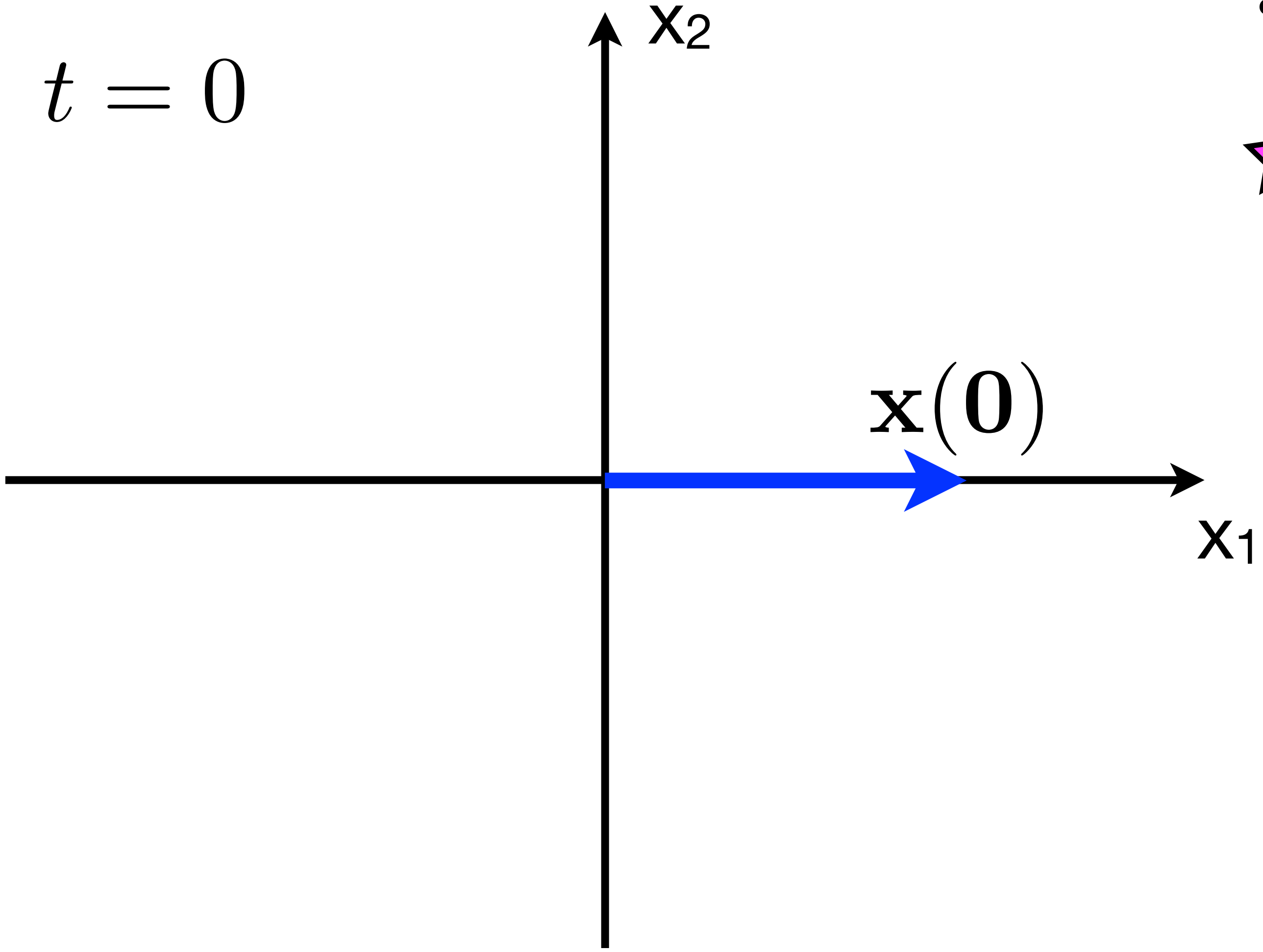
  (C) The tip of the vector maps out a circle in the first quadrant.

  (D) The tip of the vector maps out a circle in the fourth quadrant.

  (E) Explain please.

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



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(A) The vector rotates clockwise.

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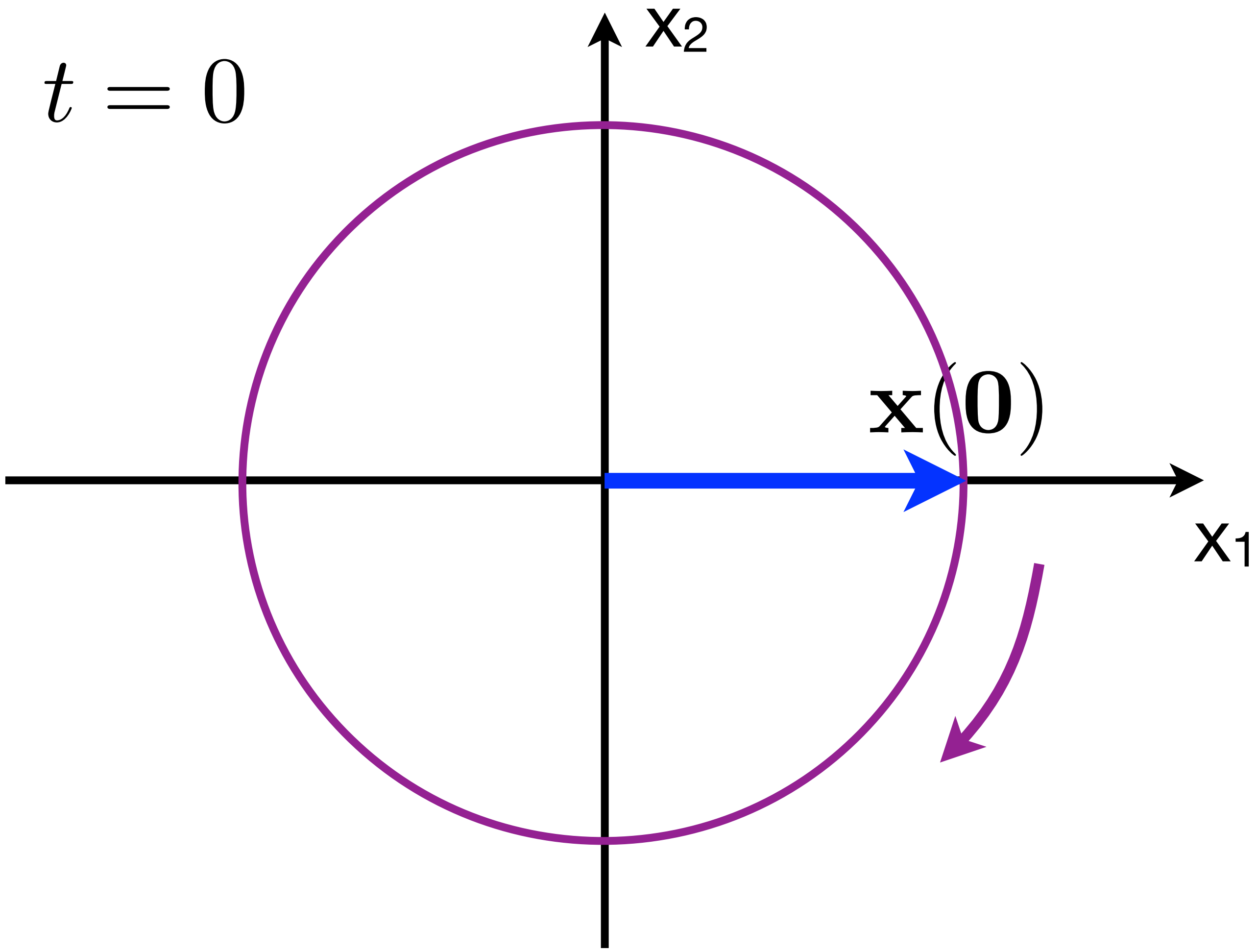
(E) Explain please.

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

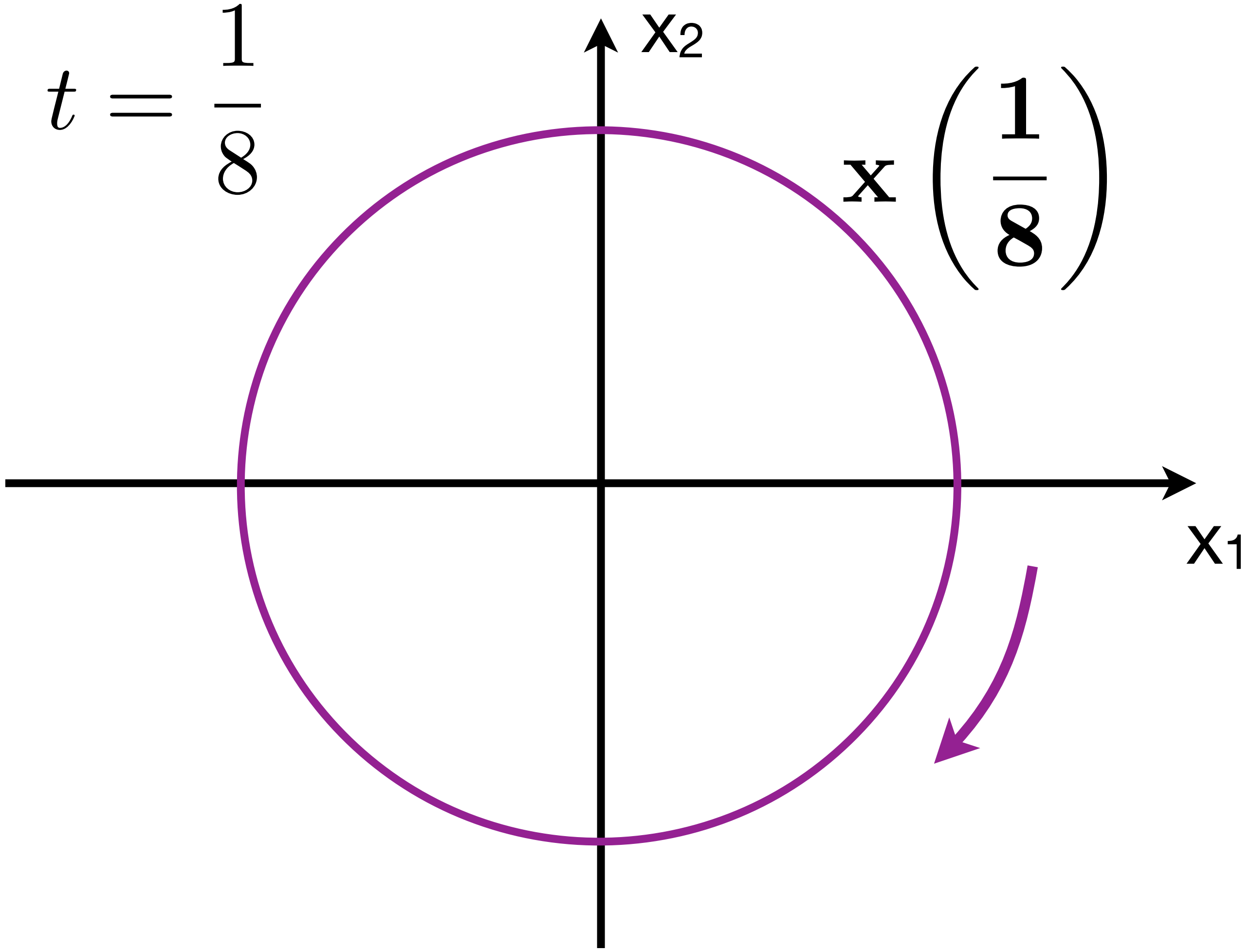
$t = 0$



Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$t = \frac{1}{8}$

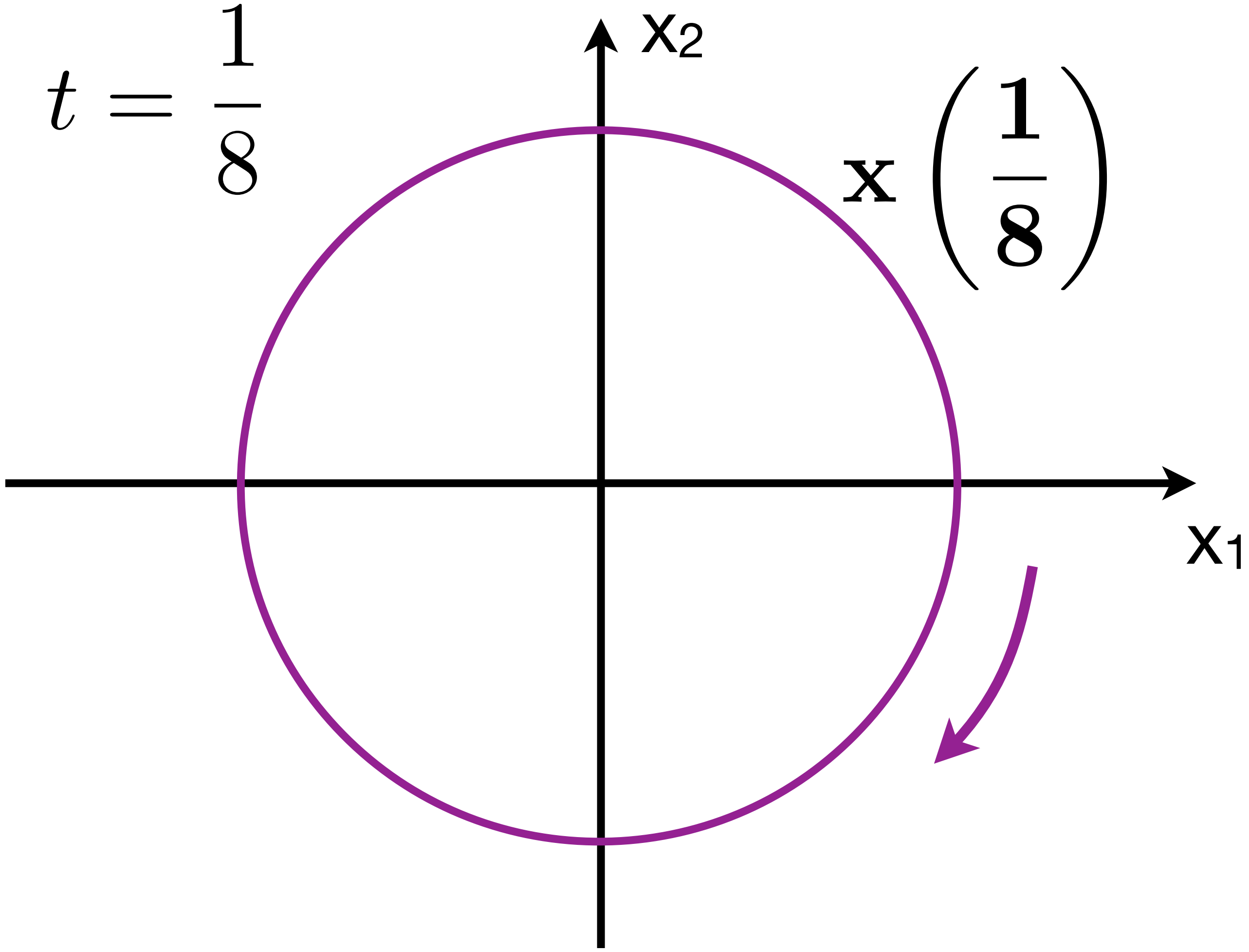


$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right)$

Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$$t = \frac{1}{8}$$



$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right)$$

Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

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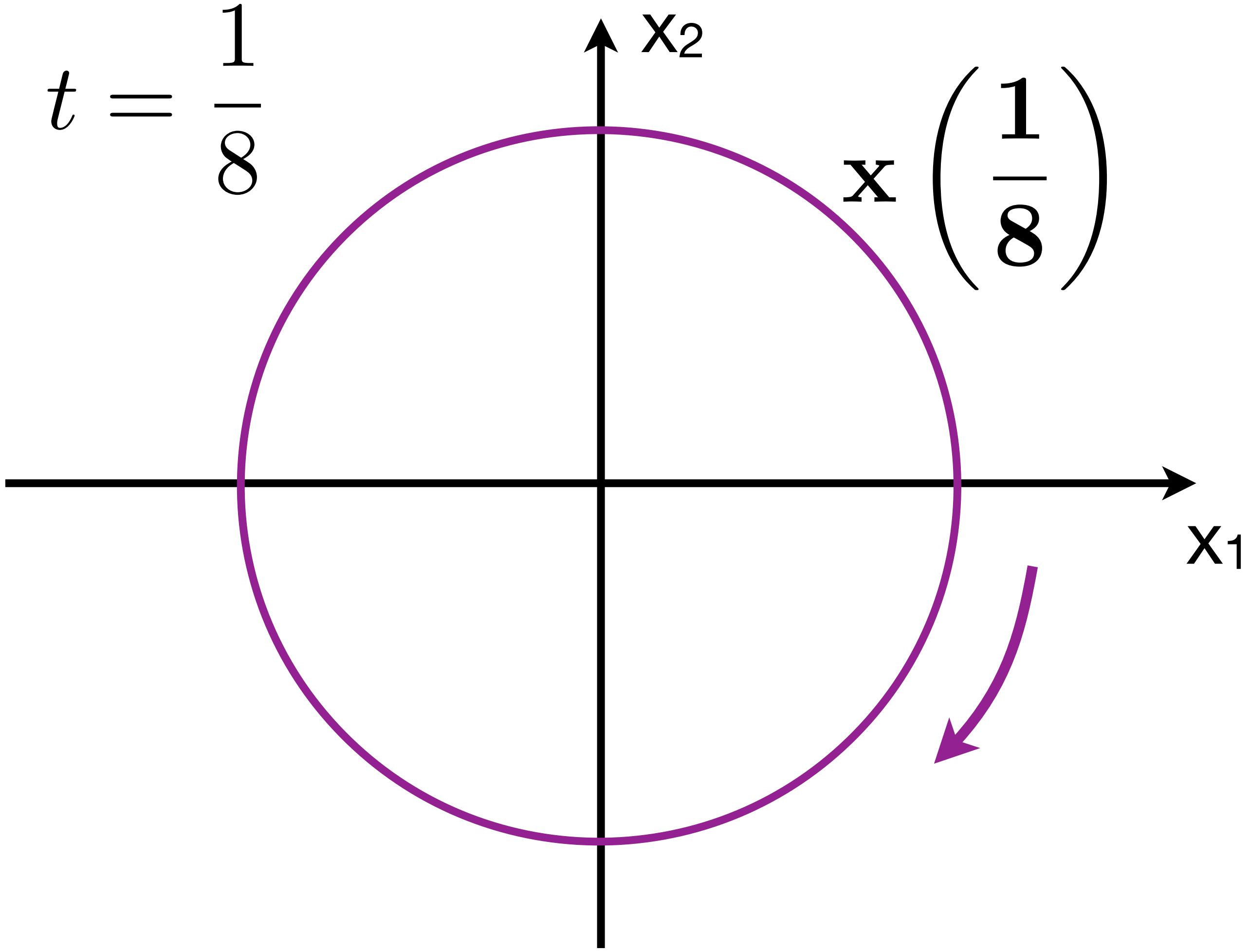

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos\left(\frac{\pi}{4}\right) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{4}\right)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix}1\\0\end{pmatrix}\cos(2\pi t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(2\pi t)$$

$$\mathbf{x}\left(\mathbf{\frac{1}{8}}\right) = \begin{pmatrix}1\\0\end{pmatrix}\cos\left(\frac{\pi}{4}\right) - \begin{pmatrix}0\\1\end{pmatrix}\sin\left(\frac{\pi}{4}\right)$$

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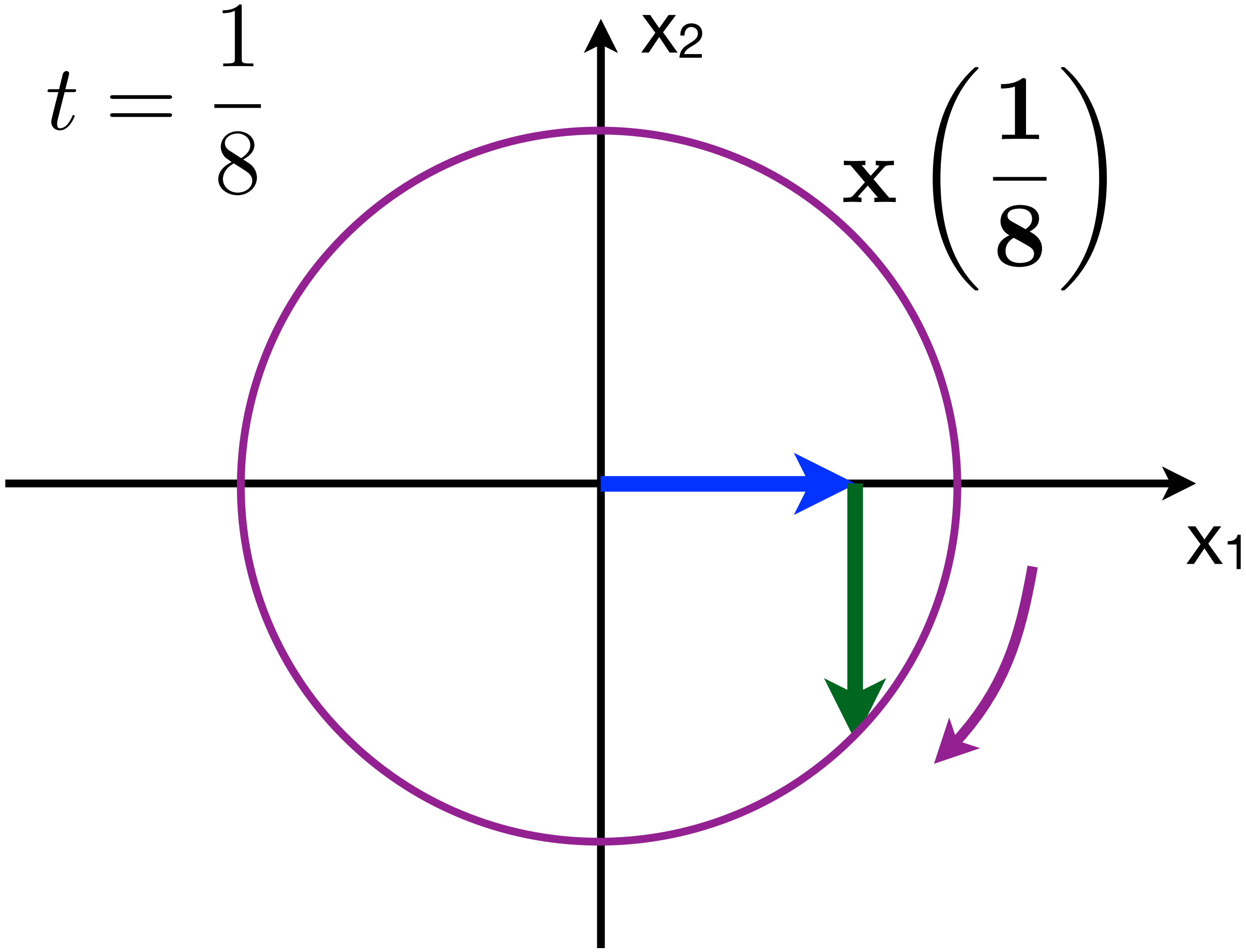
$$t = \frac{1}{8}$$

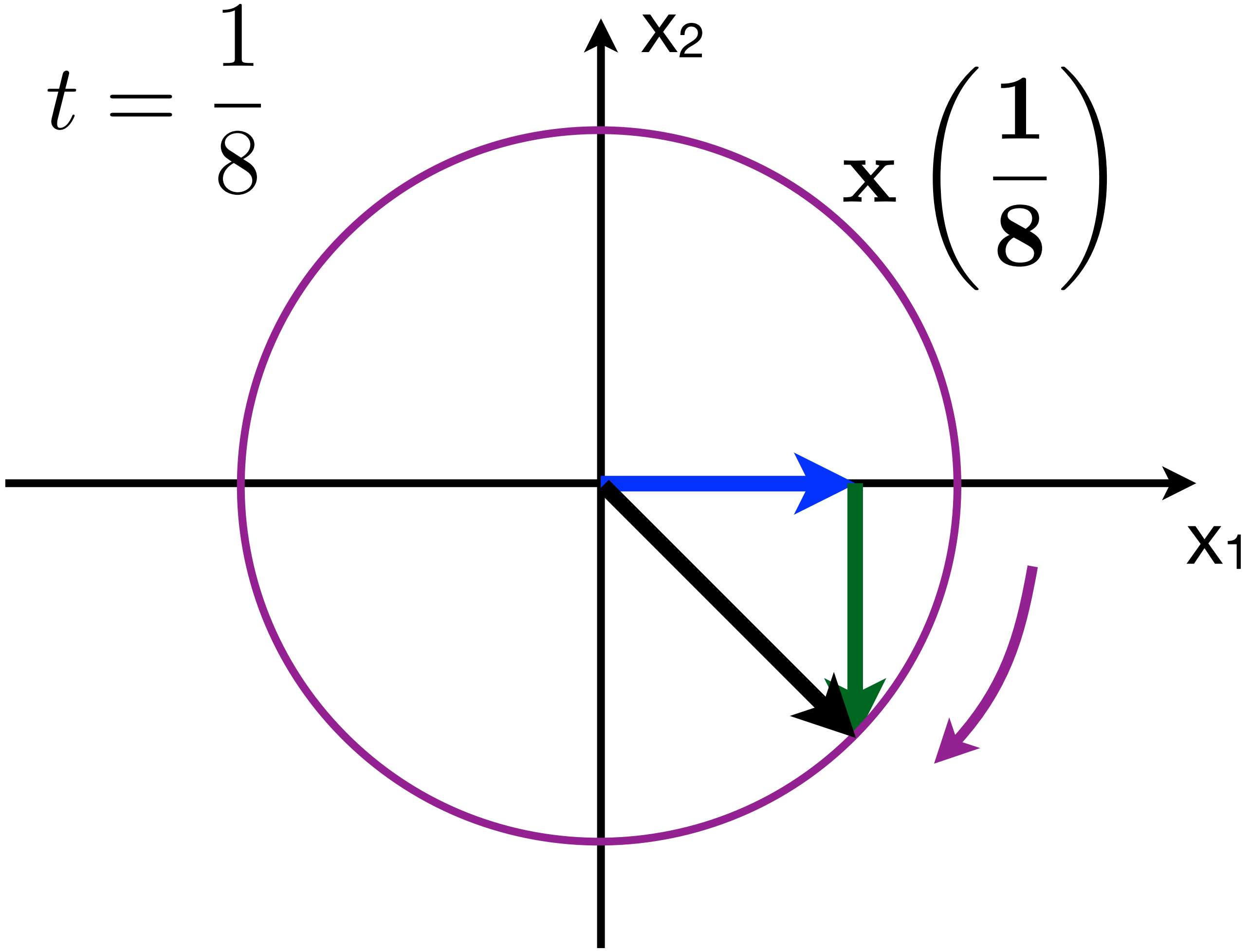
$$\mathbf{x}\left(\mathbf{\frac{1}{8}}\right)$$



Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix}1\\0\end{pmatrix}\cos(2\pi t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(2\pi t)$$

$$t = \frac{1}{8}$$



$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) = \begin{pmatrix}1\\0\end{pmatrix}\cos\left(\frac{\pi}{4}\right) - \begin{pmatrix}0\\1\end{pmatrix}\sin\left(\frac{\pi}{4}\right)$$

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Complex eigenvalues - example

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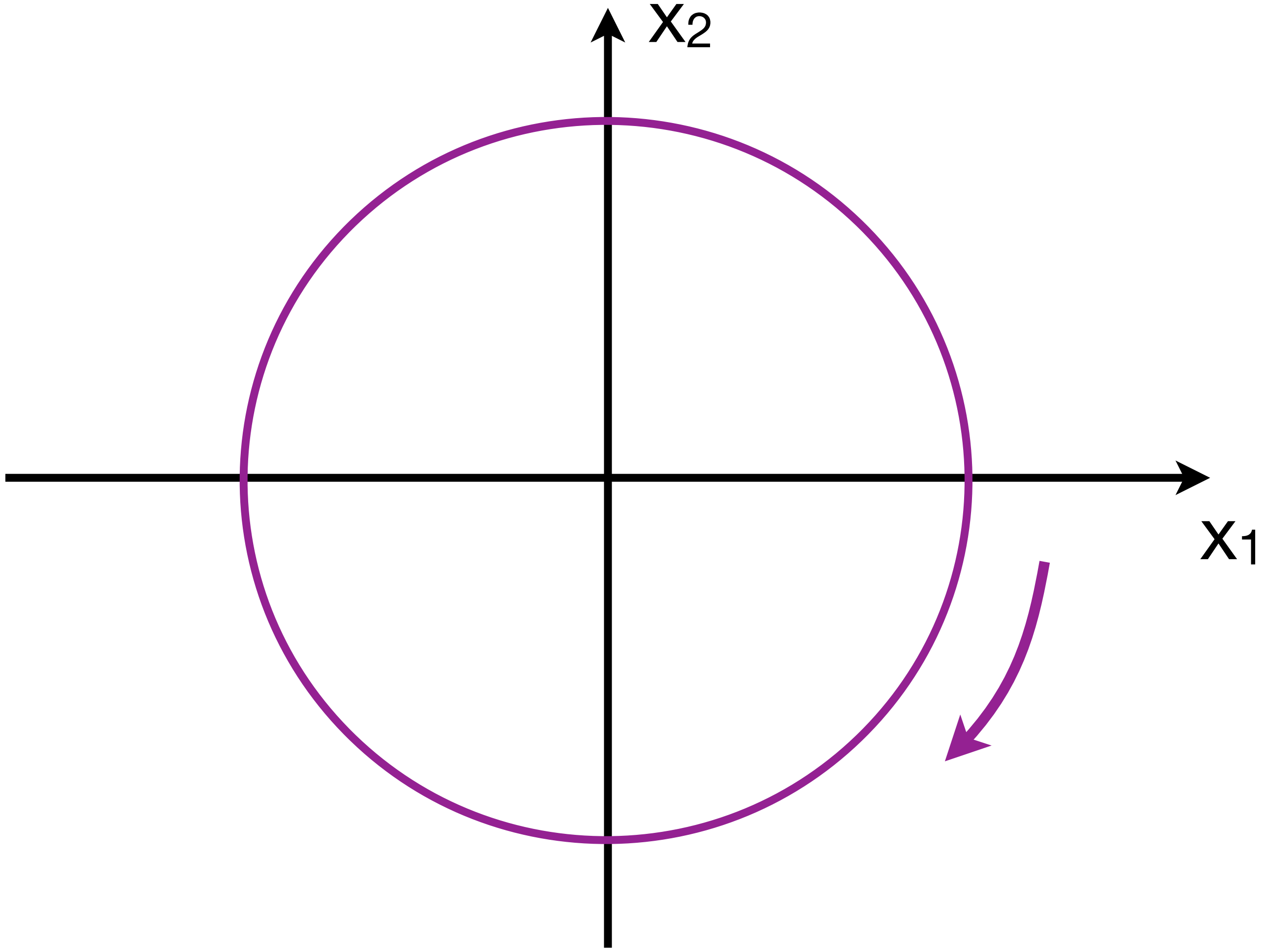
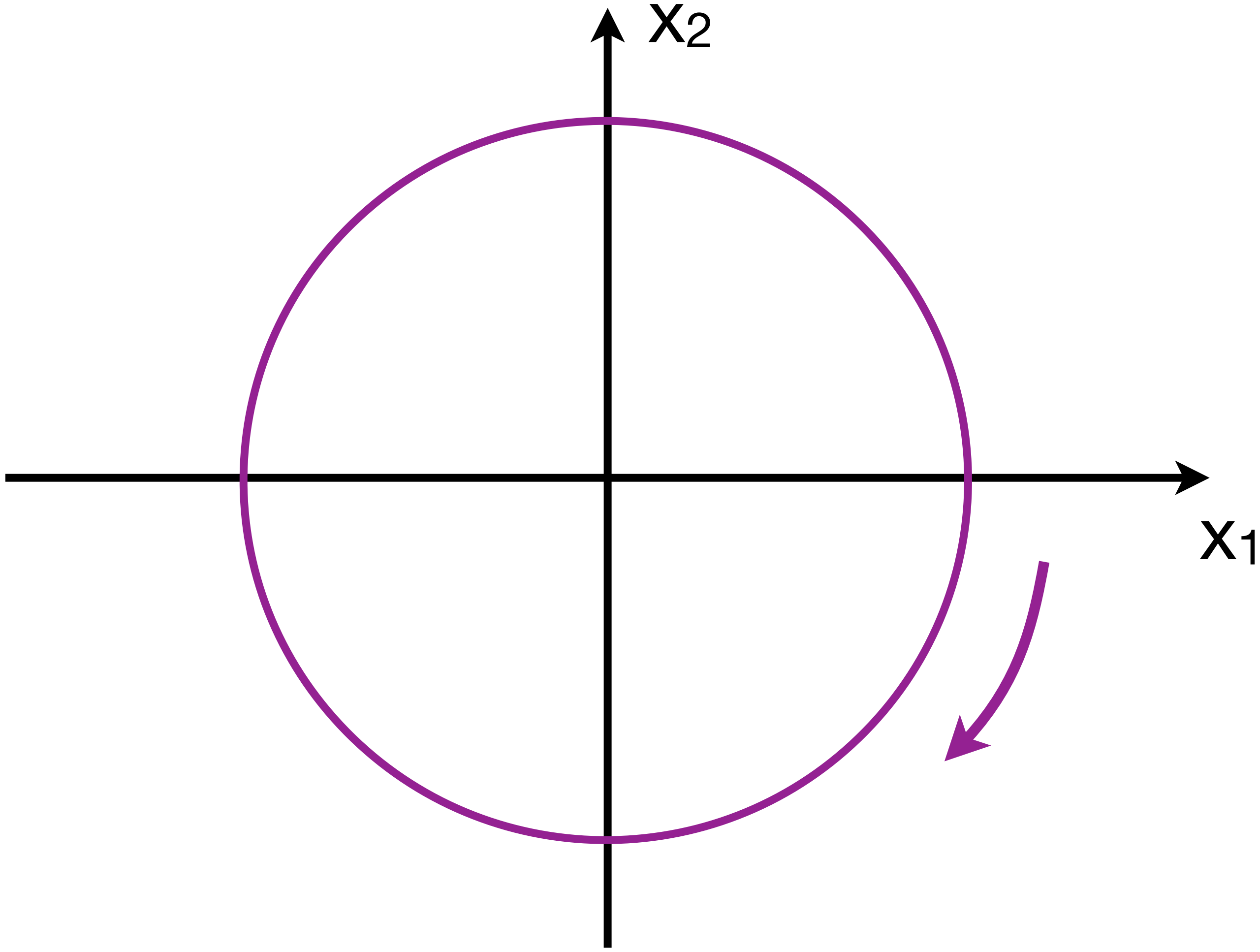


Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



Complex eigenvalues - example

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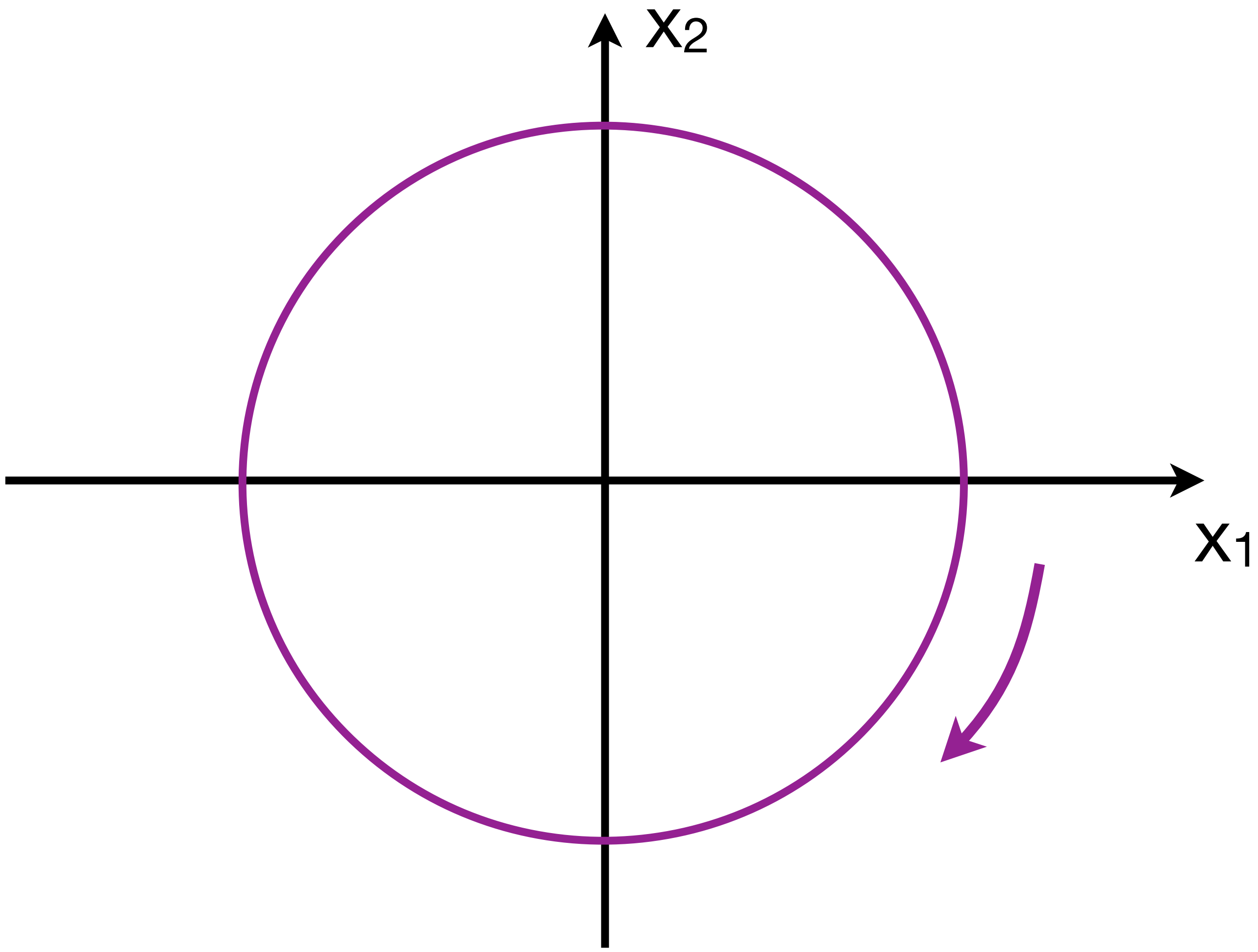
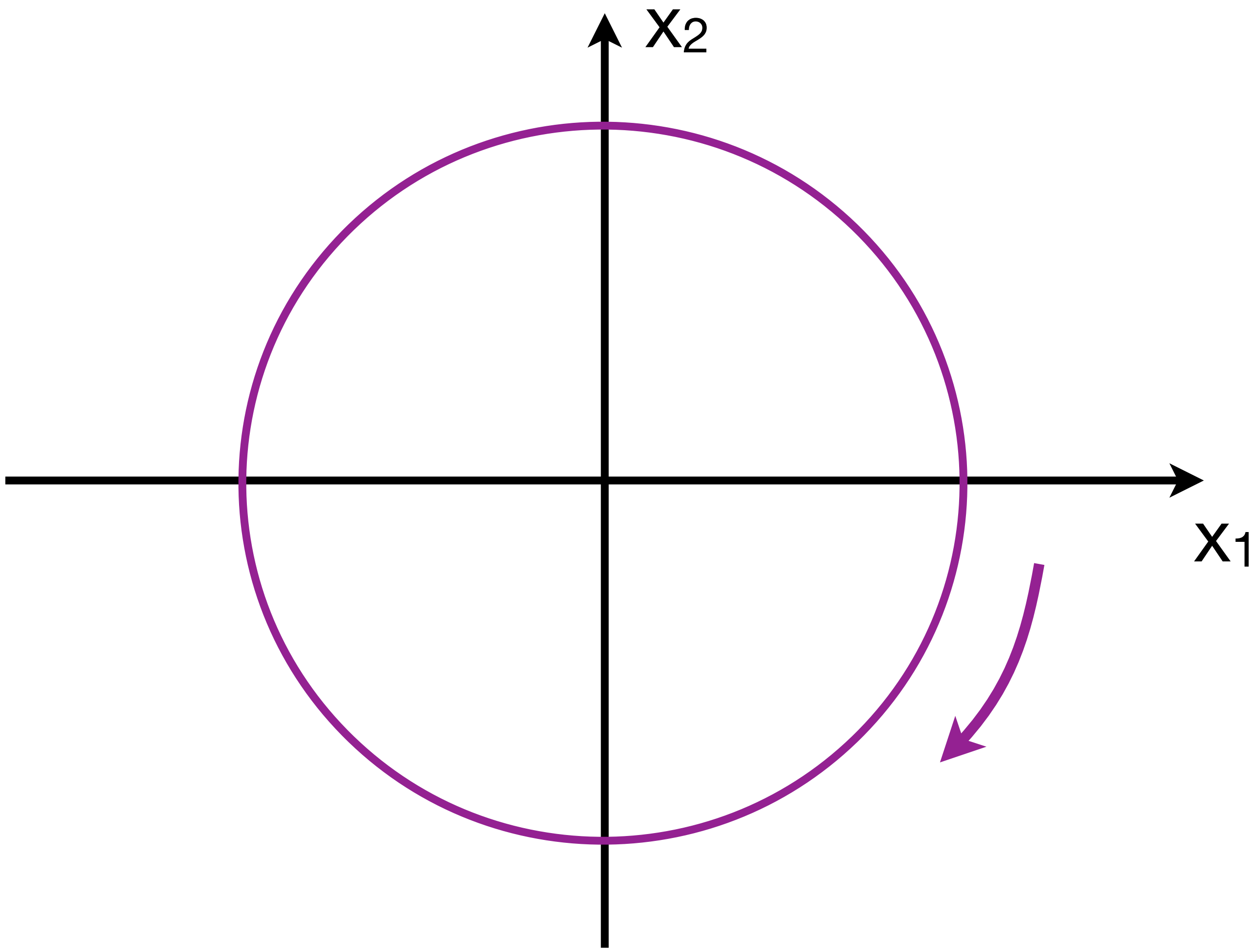
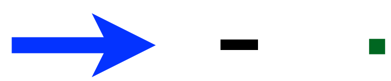


$$\mathbf{x}(\mathbf{0}) =$$

Complex eigenvalues - example

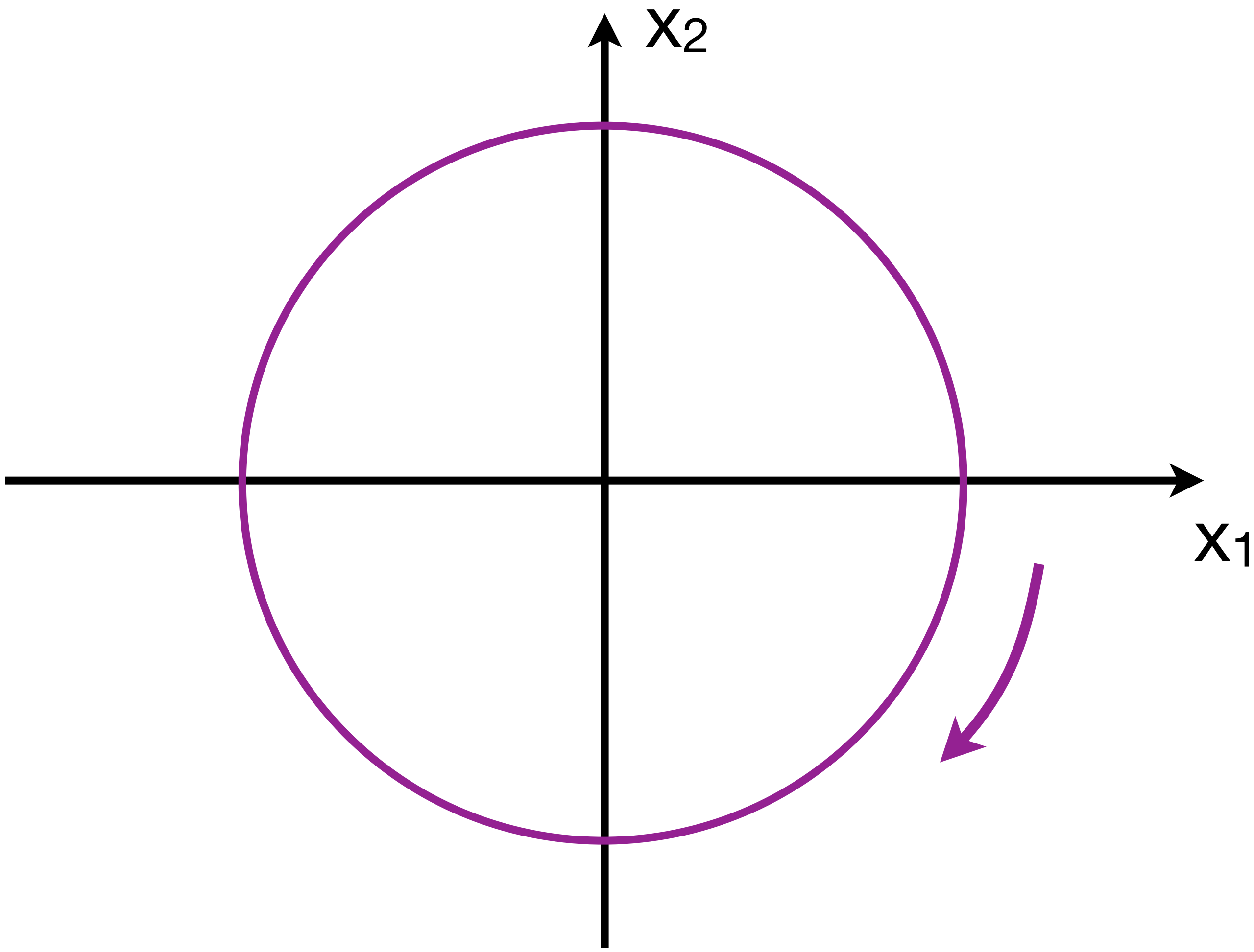
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

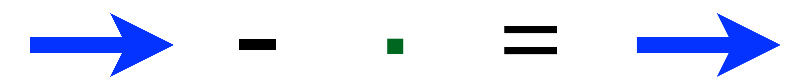


$\mathbf{x}(\mathbf{0}) =$ 

Complex eigenvalues - example

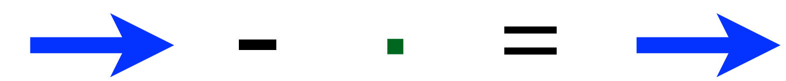
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

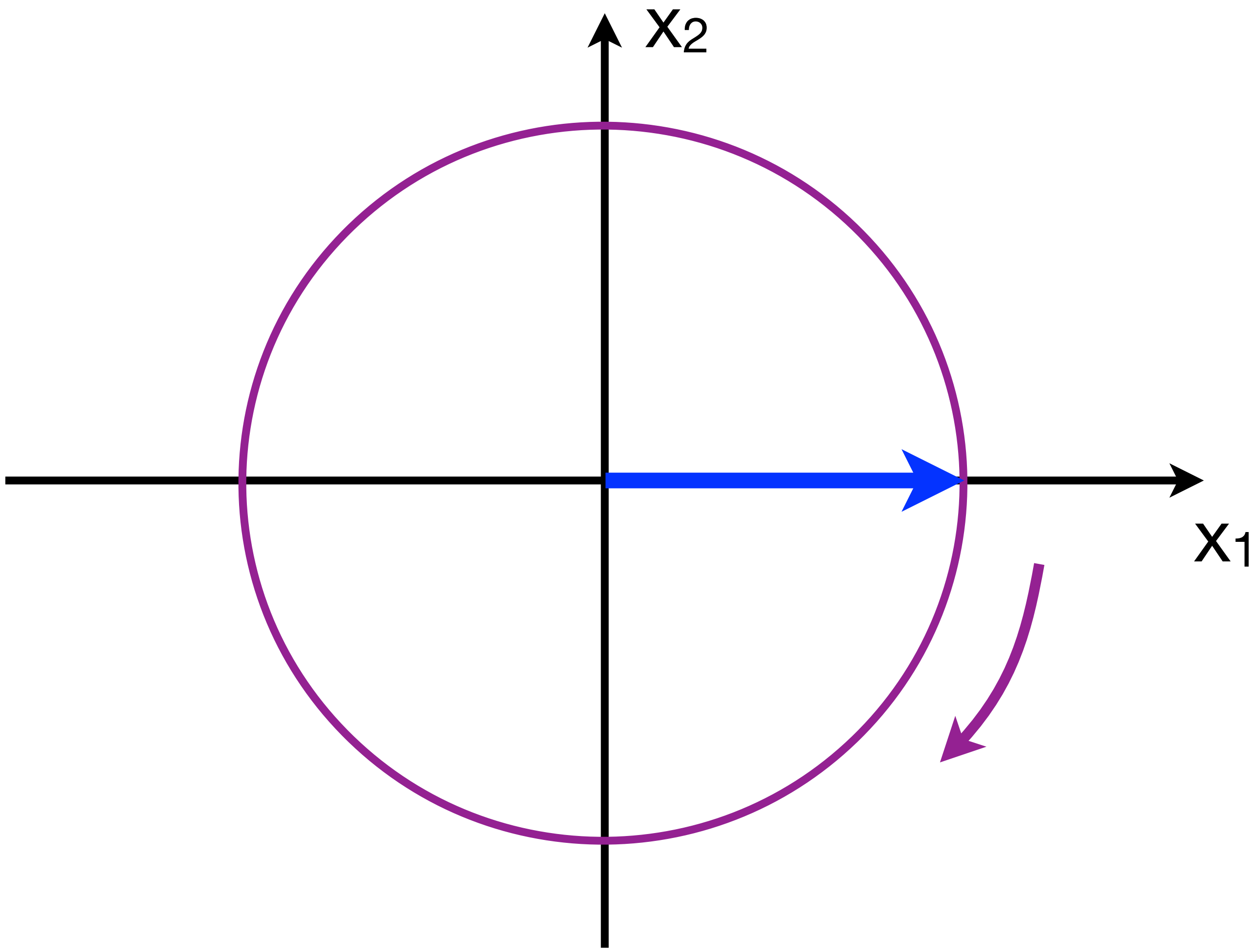


$\mathbf{x}(\mathbf{0}) =$ 

Complex eigenvalues - example

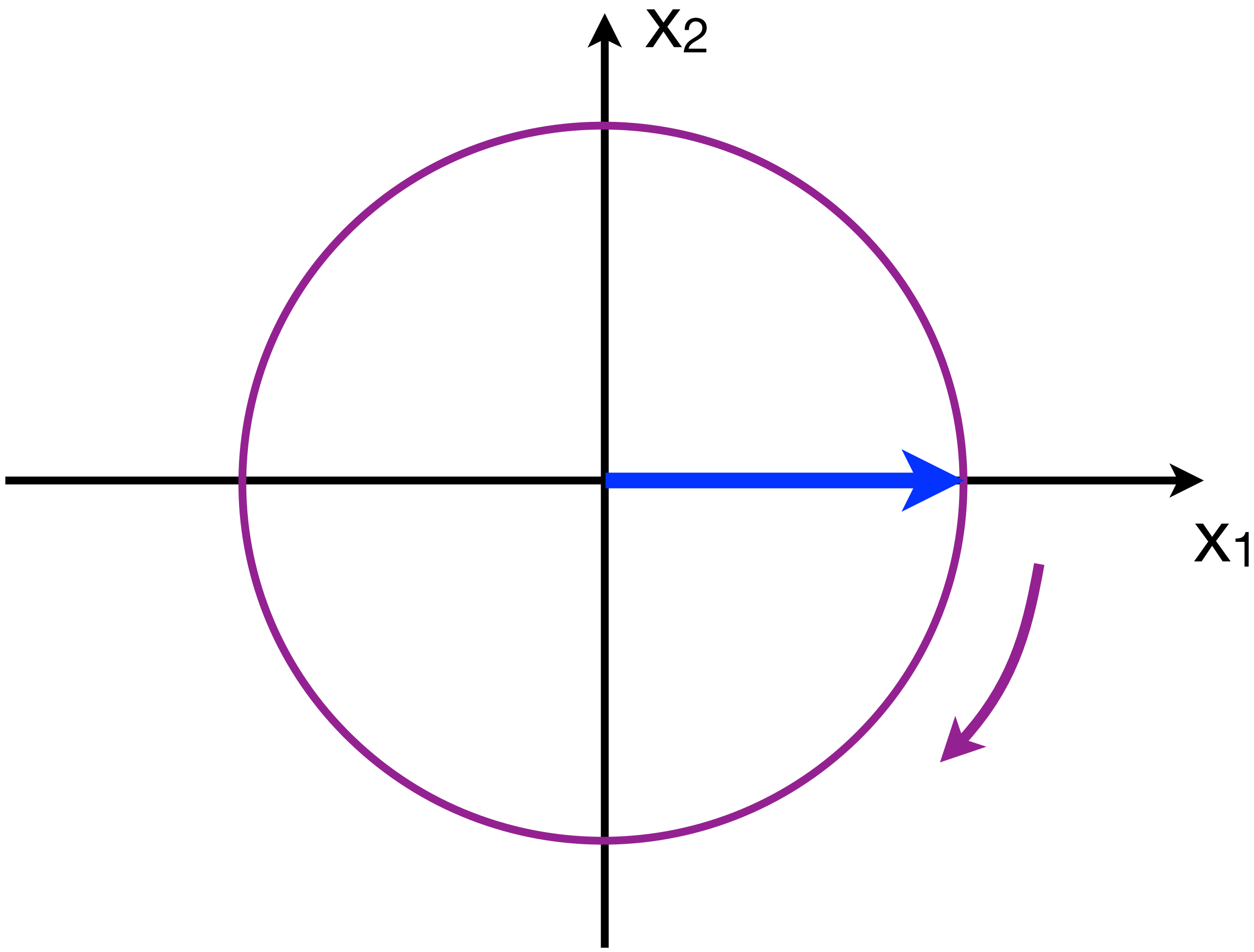
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$\mathbf{x}(\mathbf{0}) =$ 



Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix}1\\0\end{pmatrix}\cos(2\pi t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(2\pi t)$$

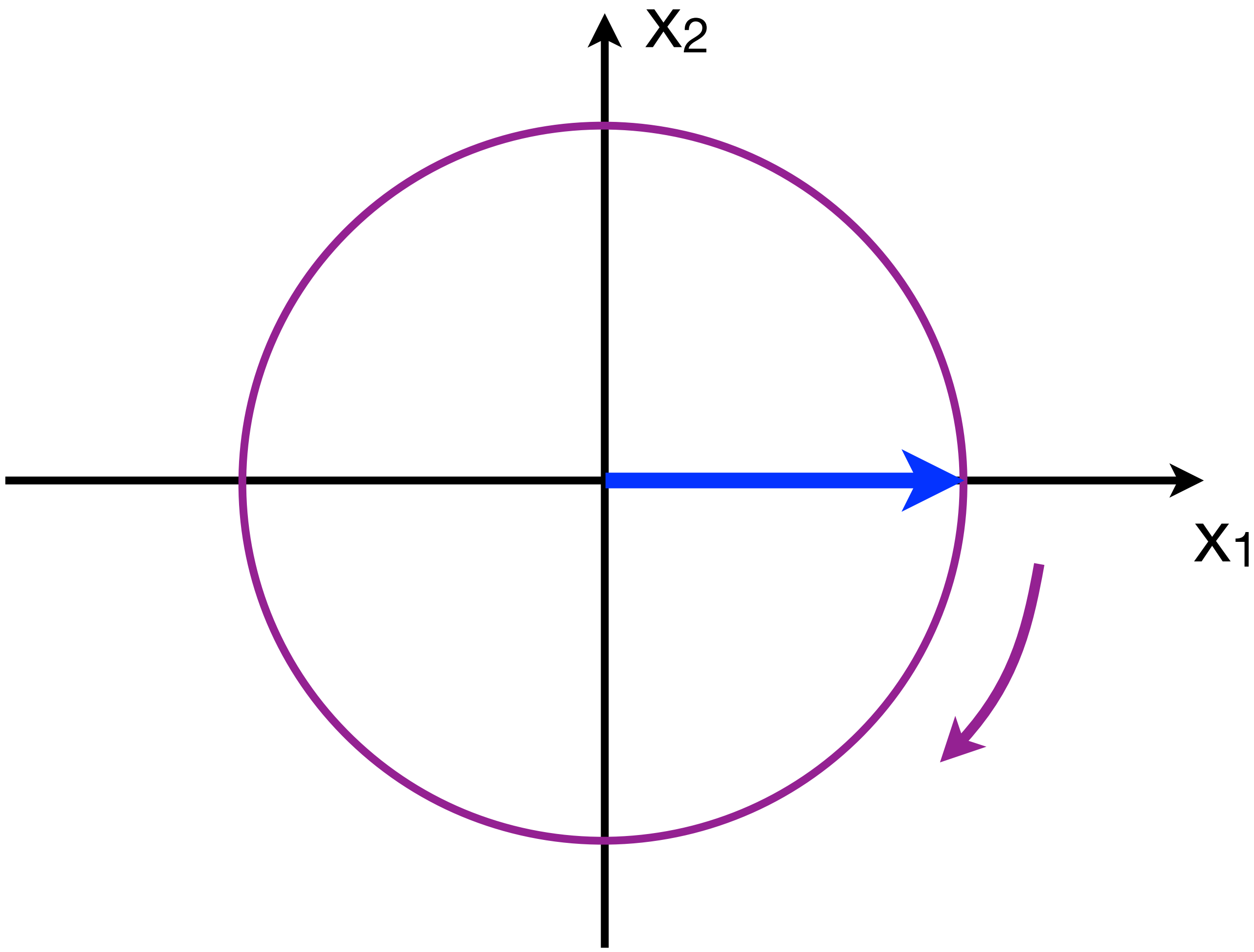


$$\mathbf{x}(\mathbf{0}) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

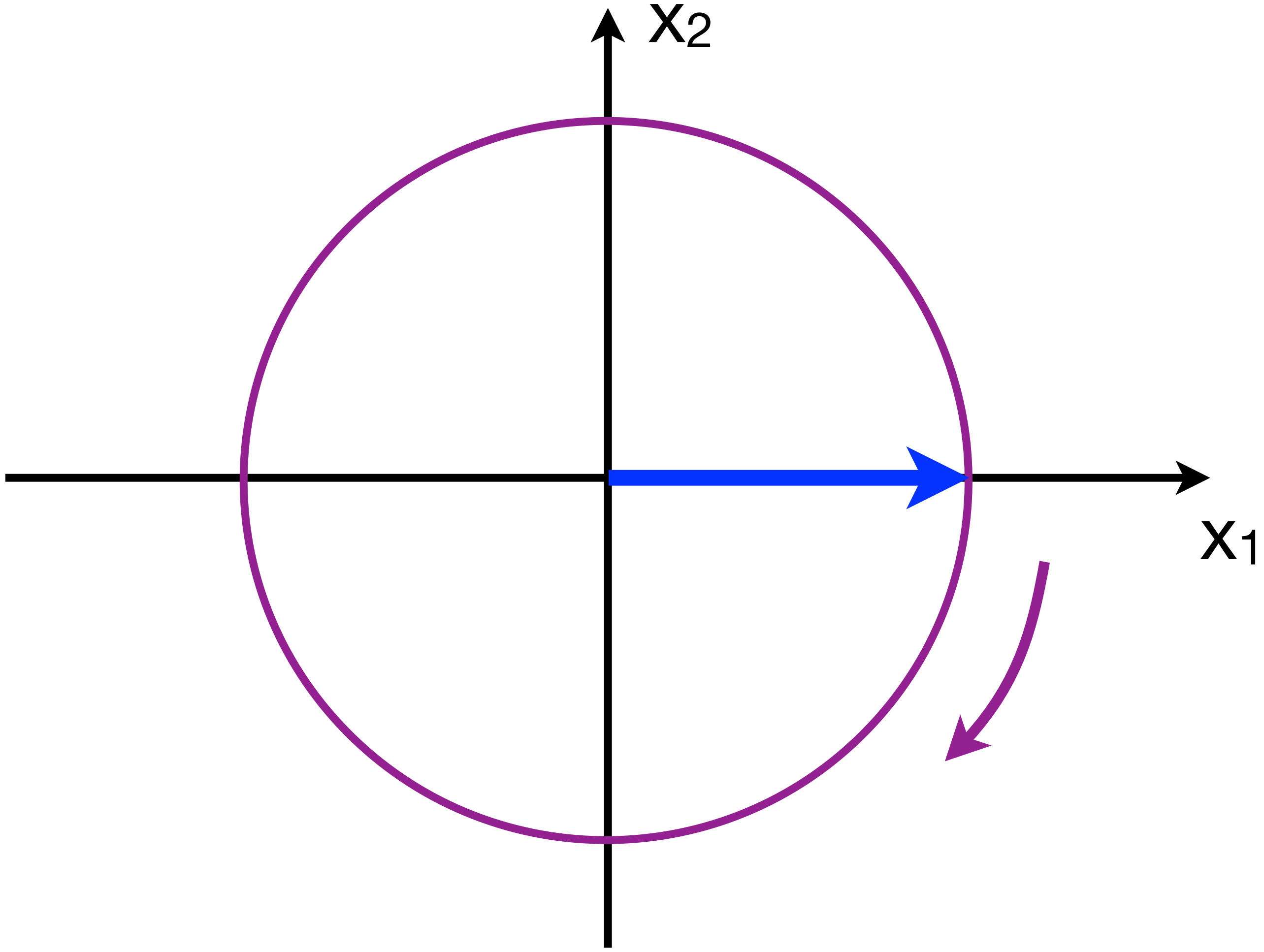


$$\mathbf{x}(\mathbf{0}) = \quad - \quad =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) = \quad -$$

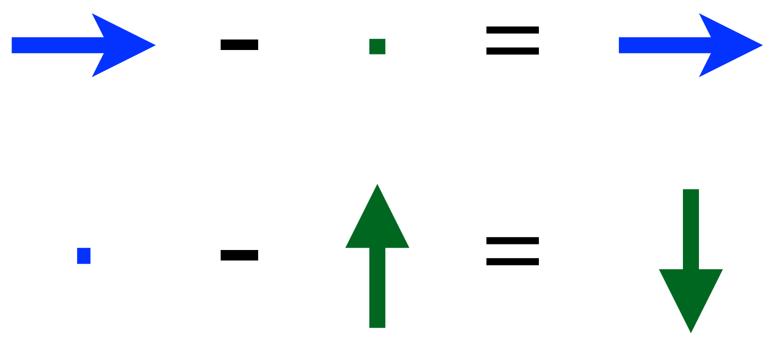
Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



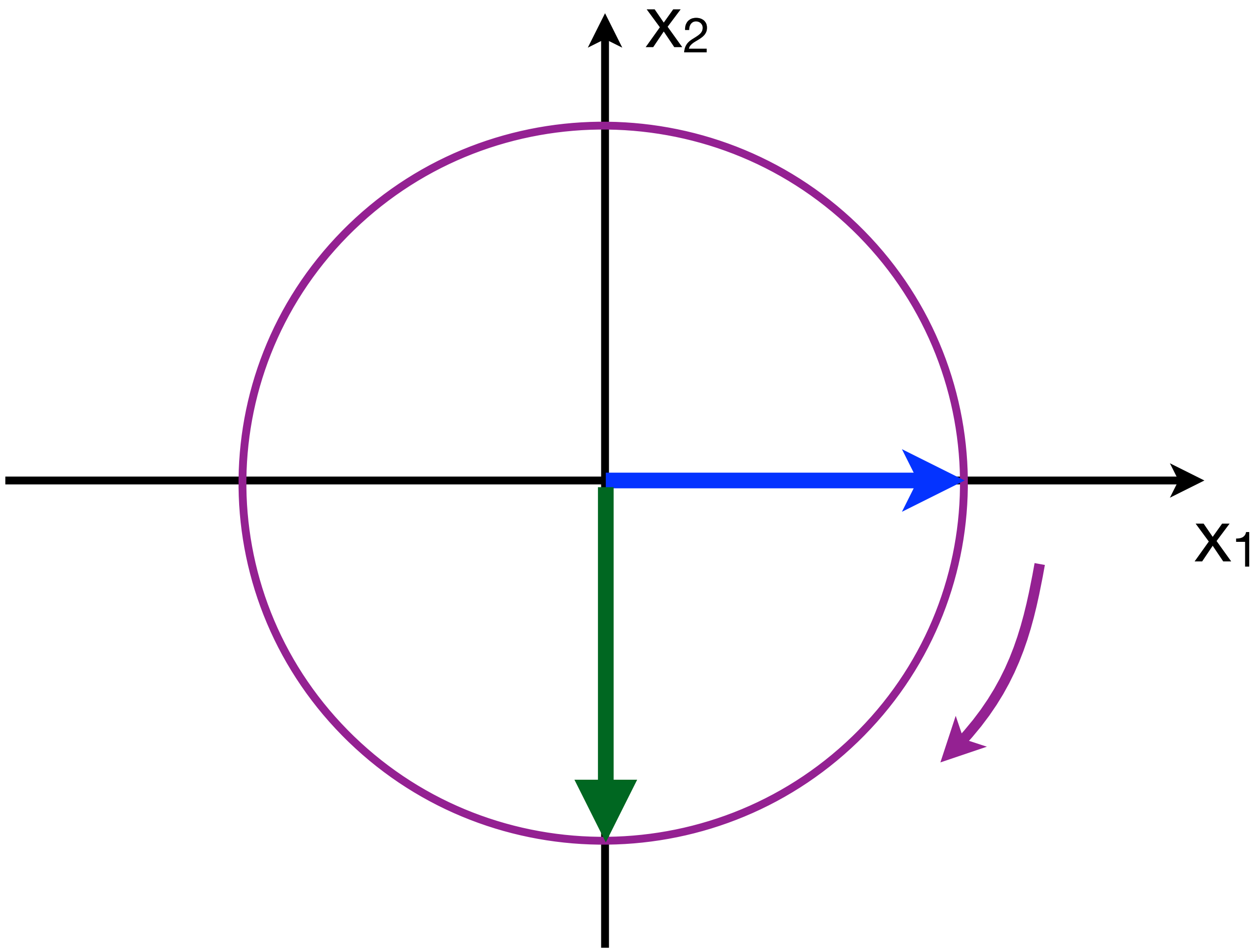
$$\mathbf{x(0)} =$$

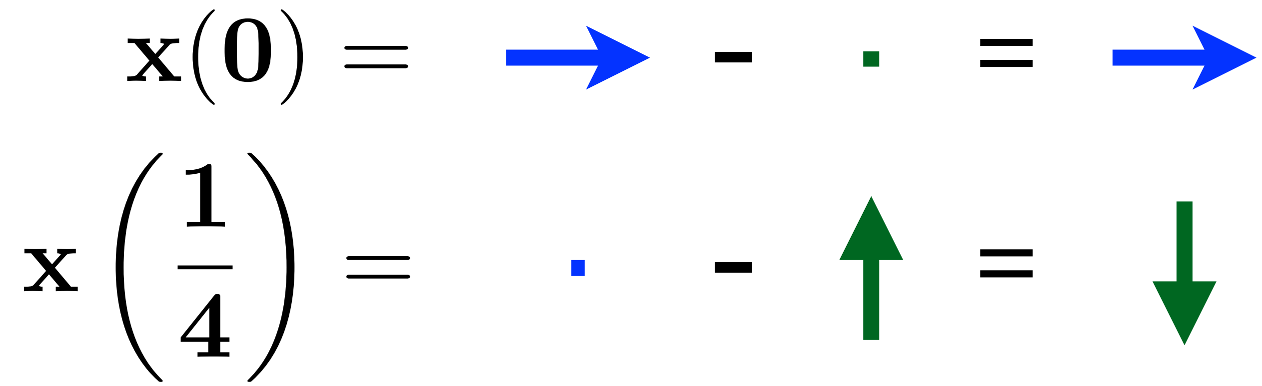
$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$



Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$





Complex eigenvalues - example

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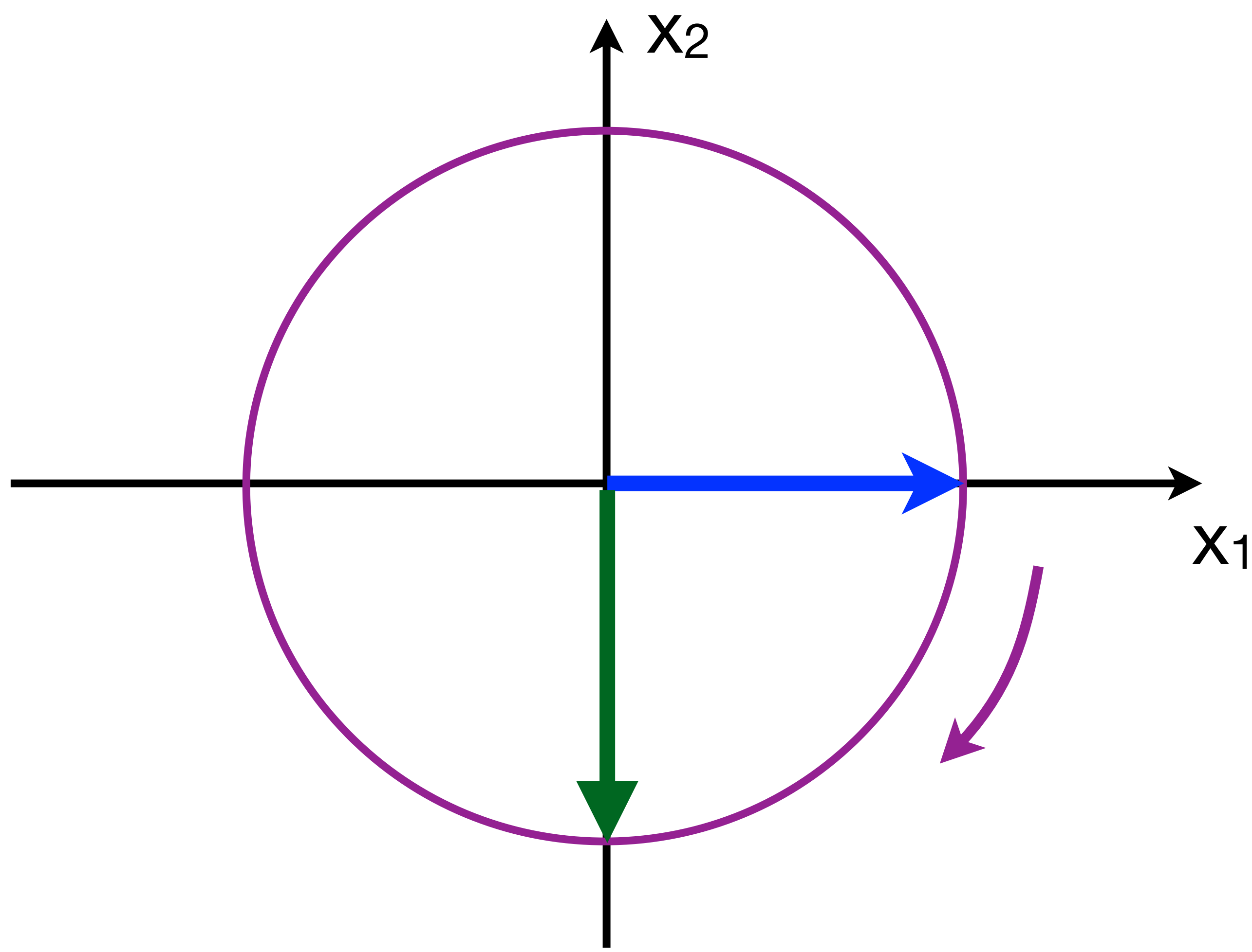
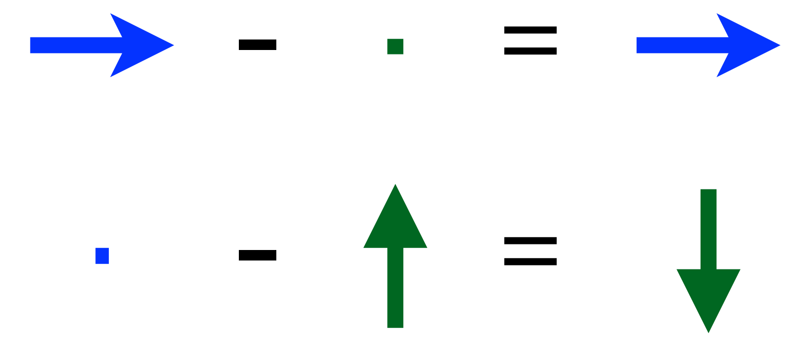


$$\mathbf{x}(\mathbf{0}) =$$

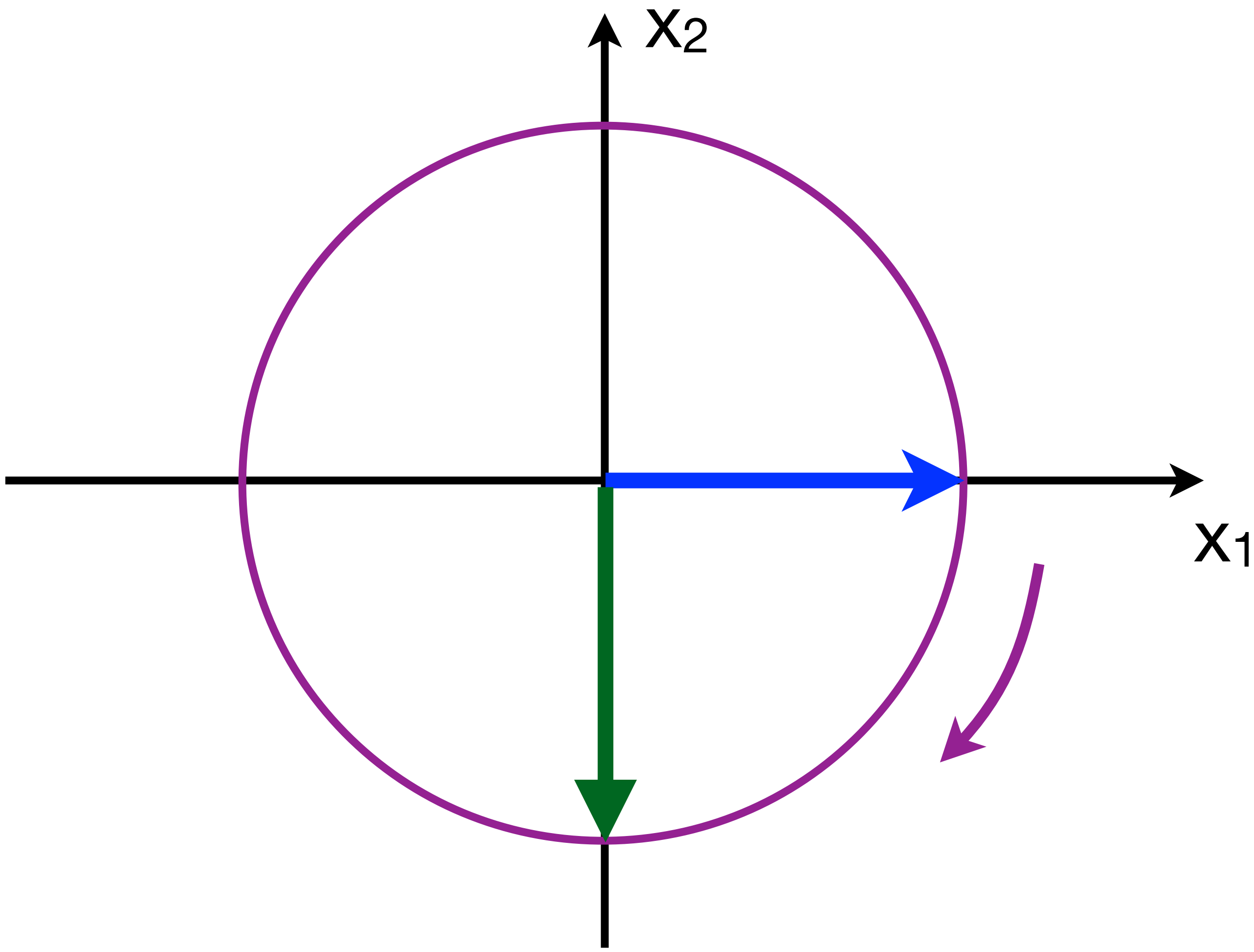
$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

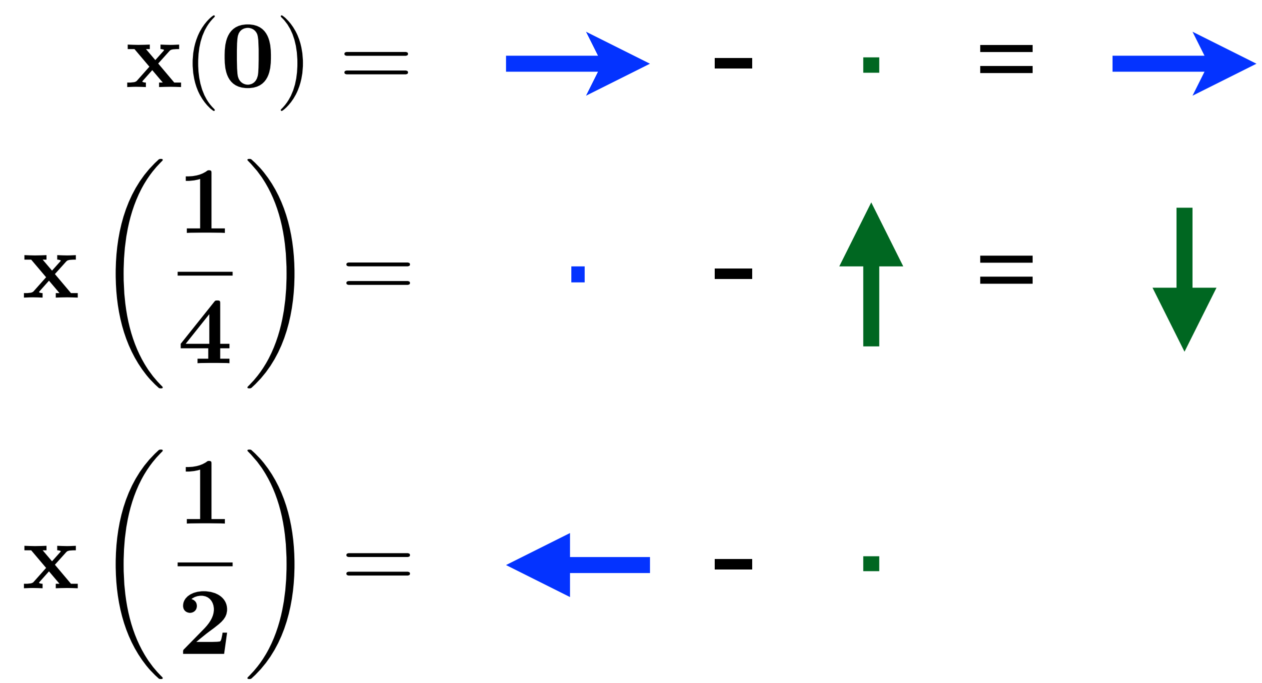


$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) =$$

Complex eigenvalues - example

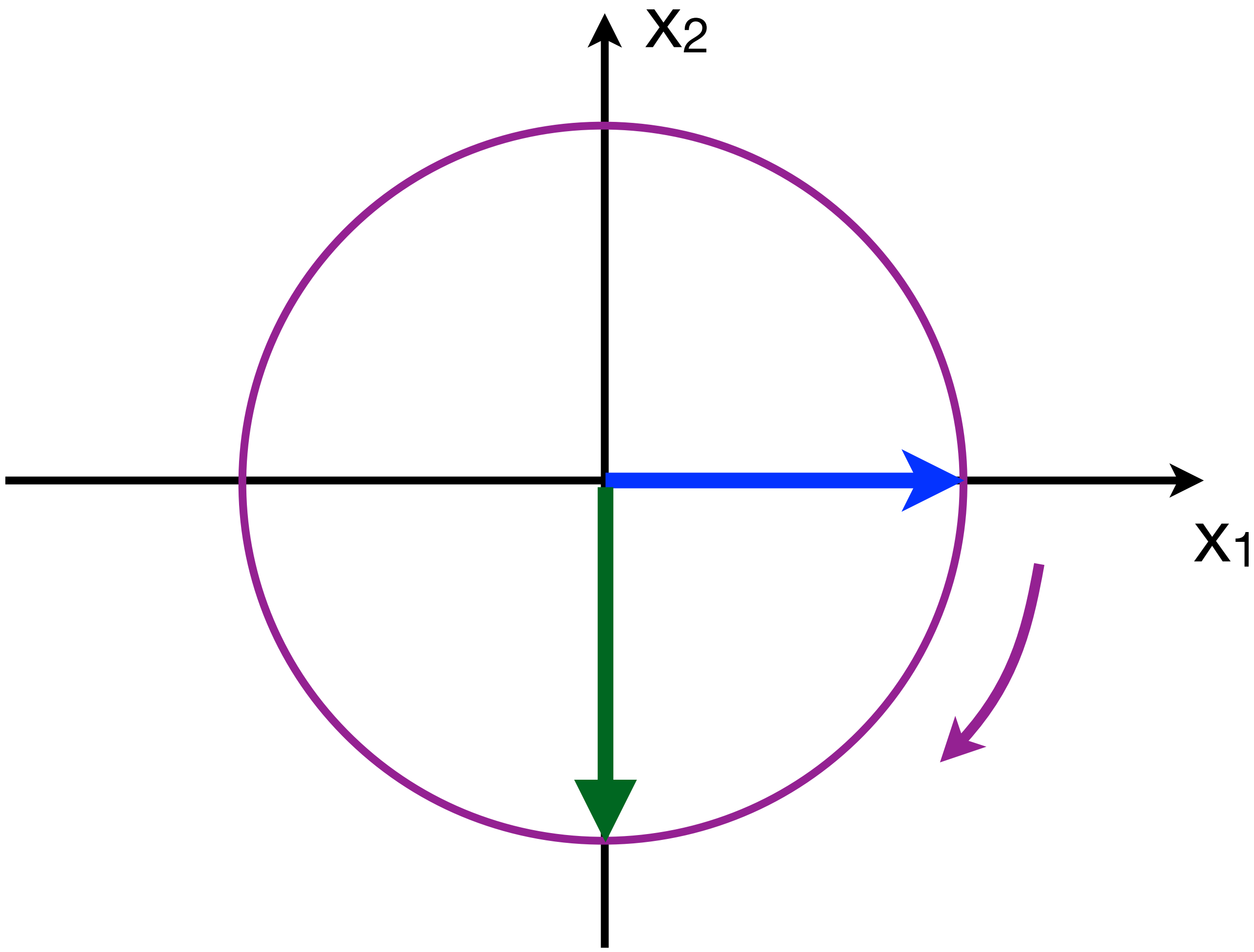
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$





Complex eigenvalues - example

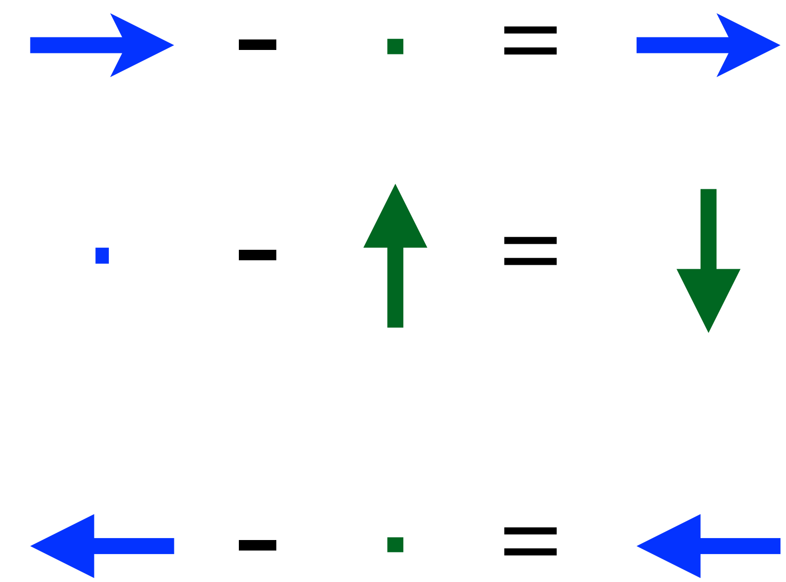
$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



$$\mathbf{x(0)} =$$

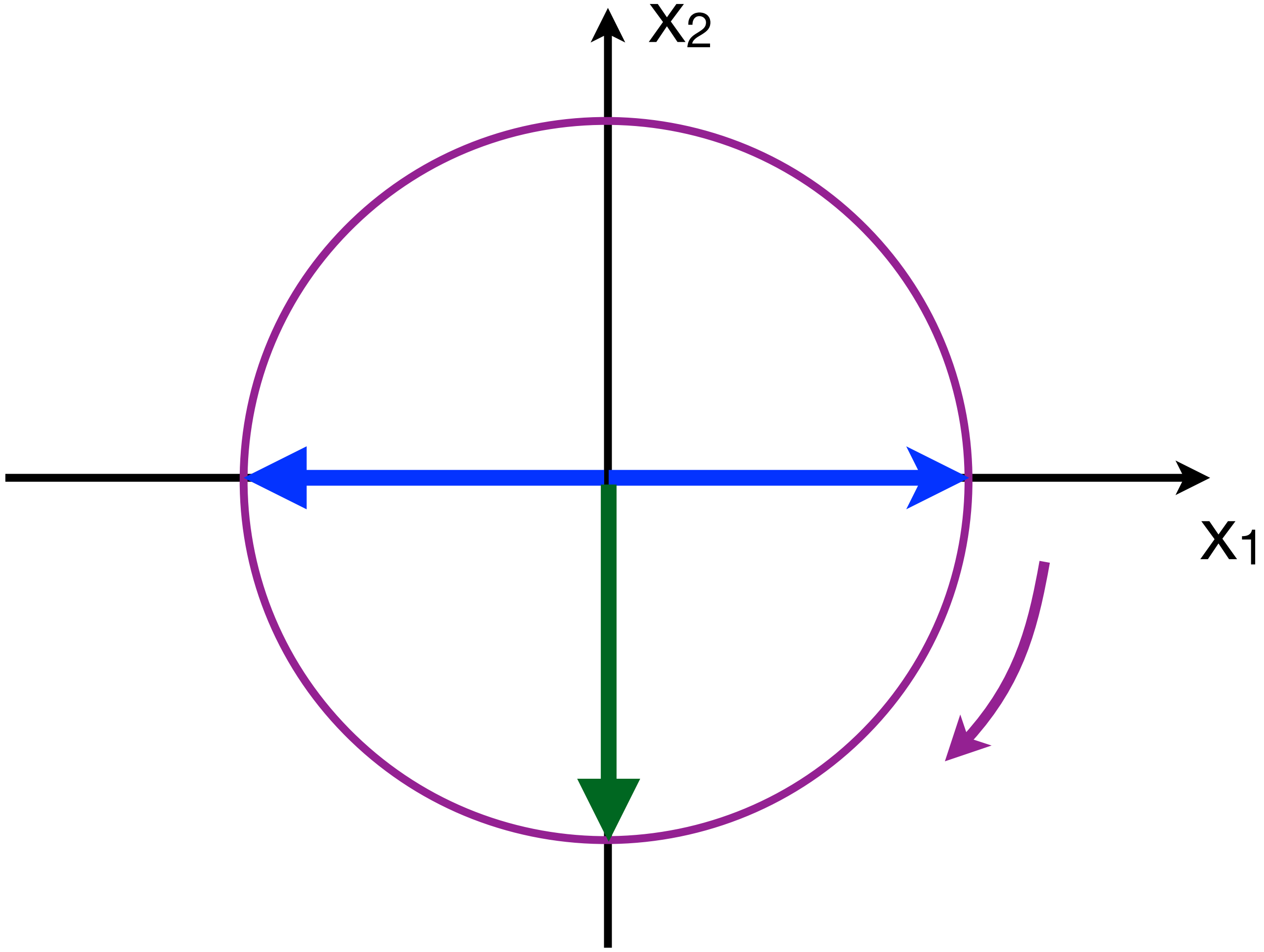
$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) =$$



Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



$$\mathbf{x}(\mathbf{0}) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

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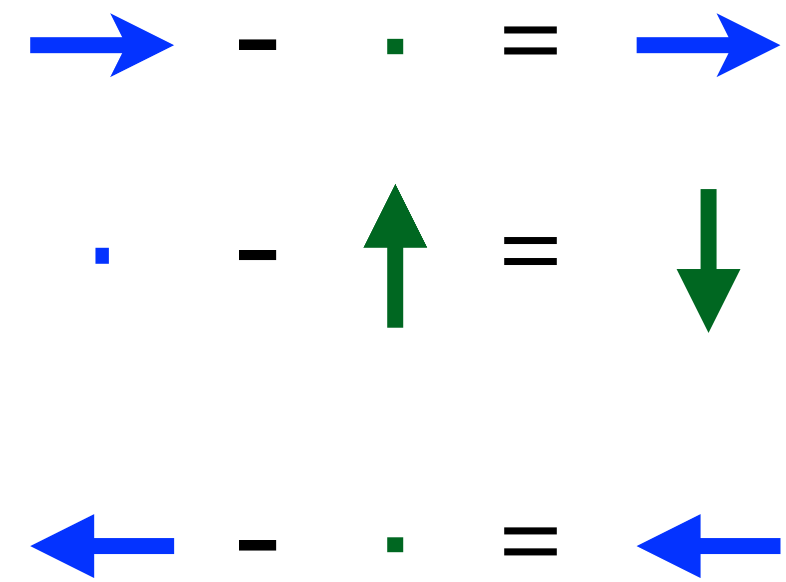
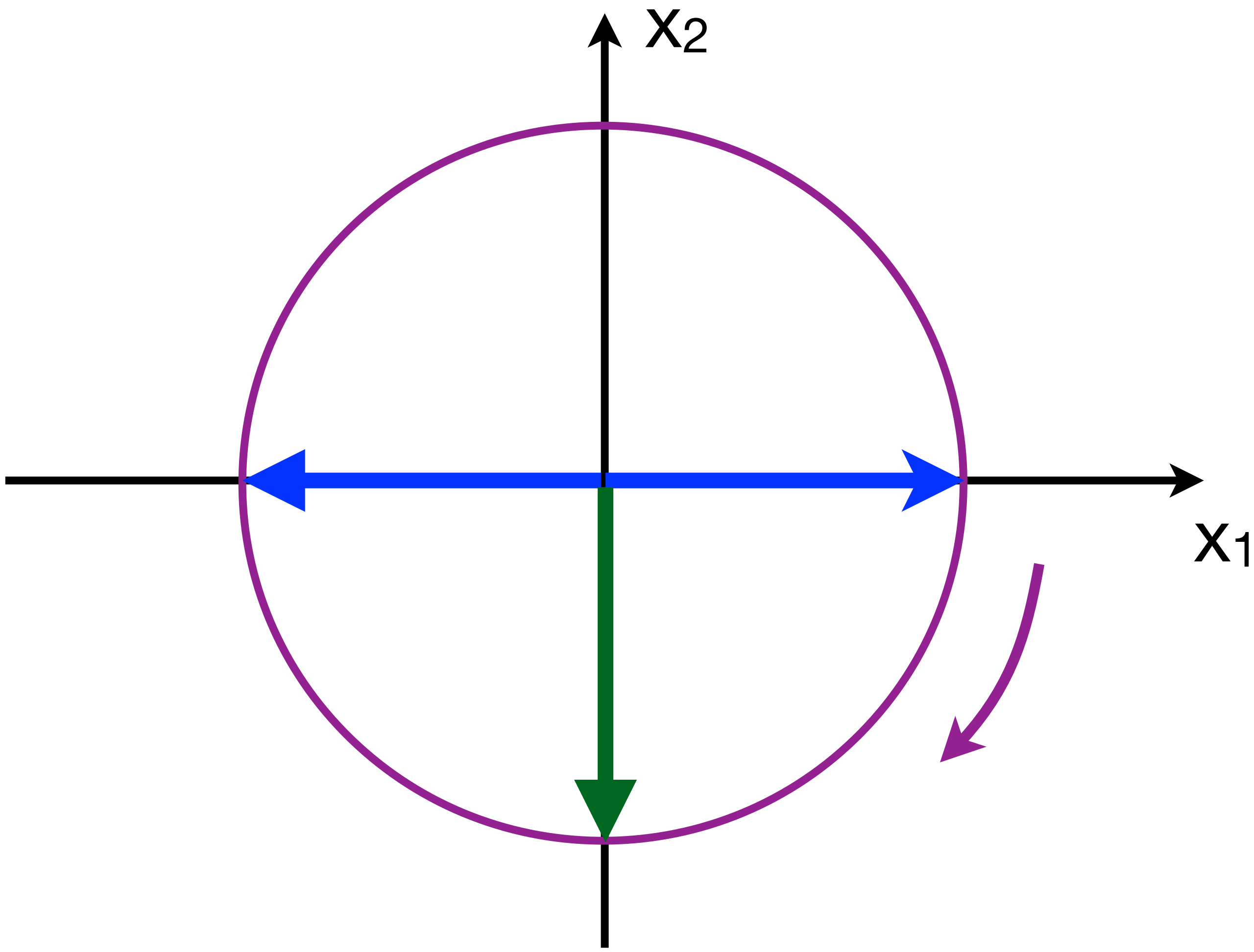


Complex eigenvalues - example

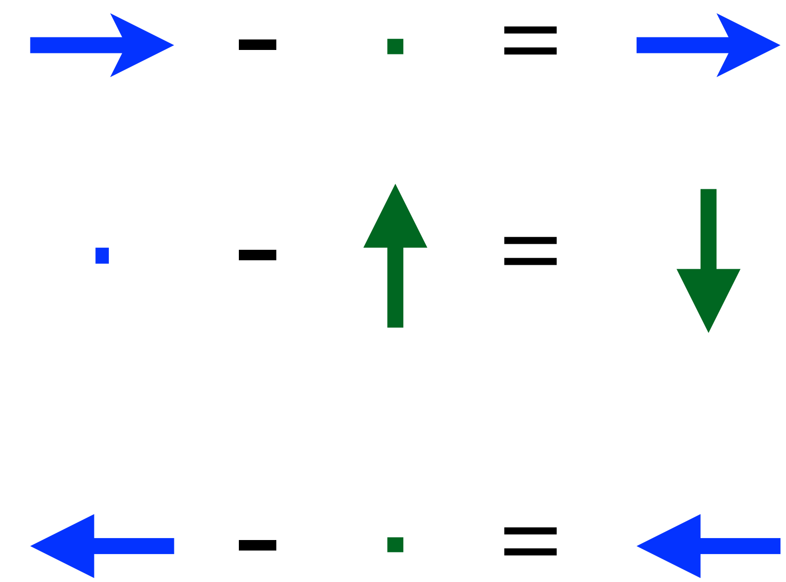
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



$$\mathbf{x}(\mathbf{0}) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

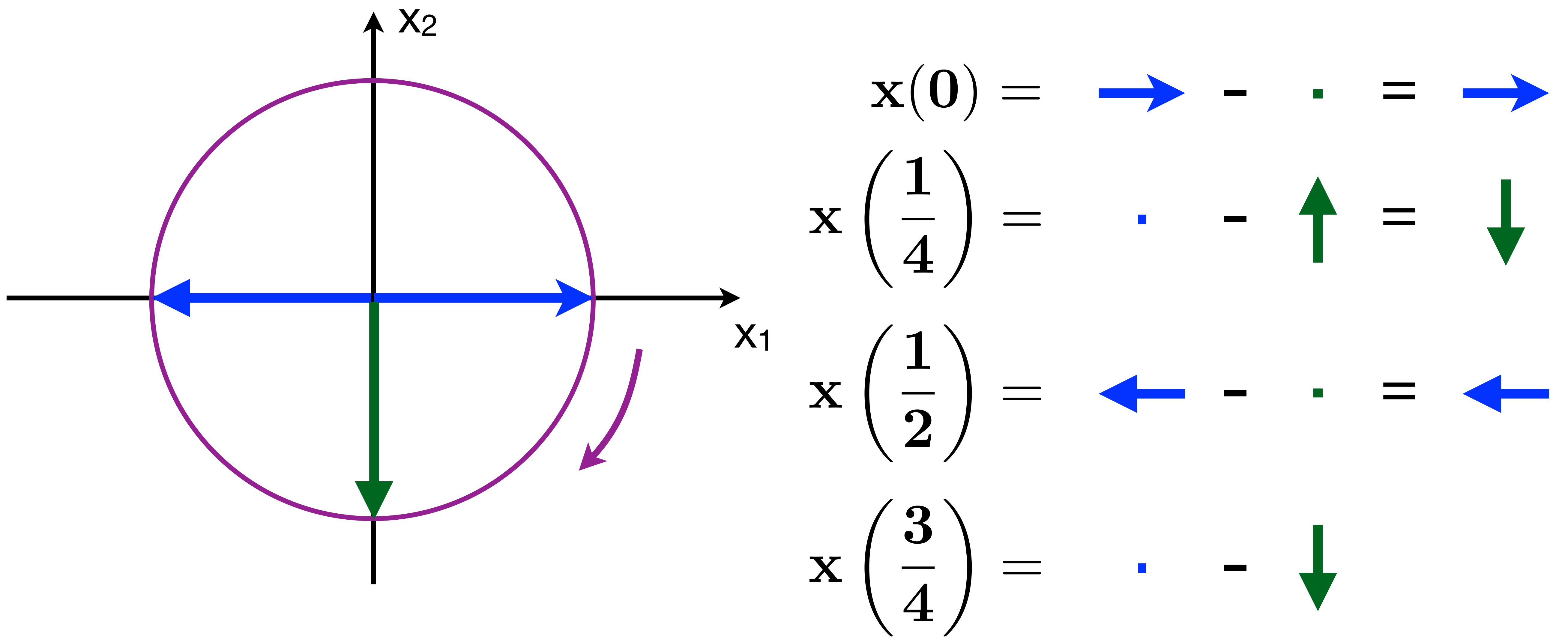
$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) =$$



$$\mathbf{x}\left(\frac{\mathbf{3}}{\mathbf{4}}\right) =$$

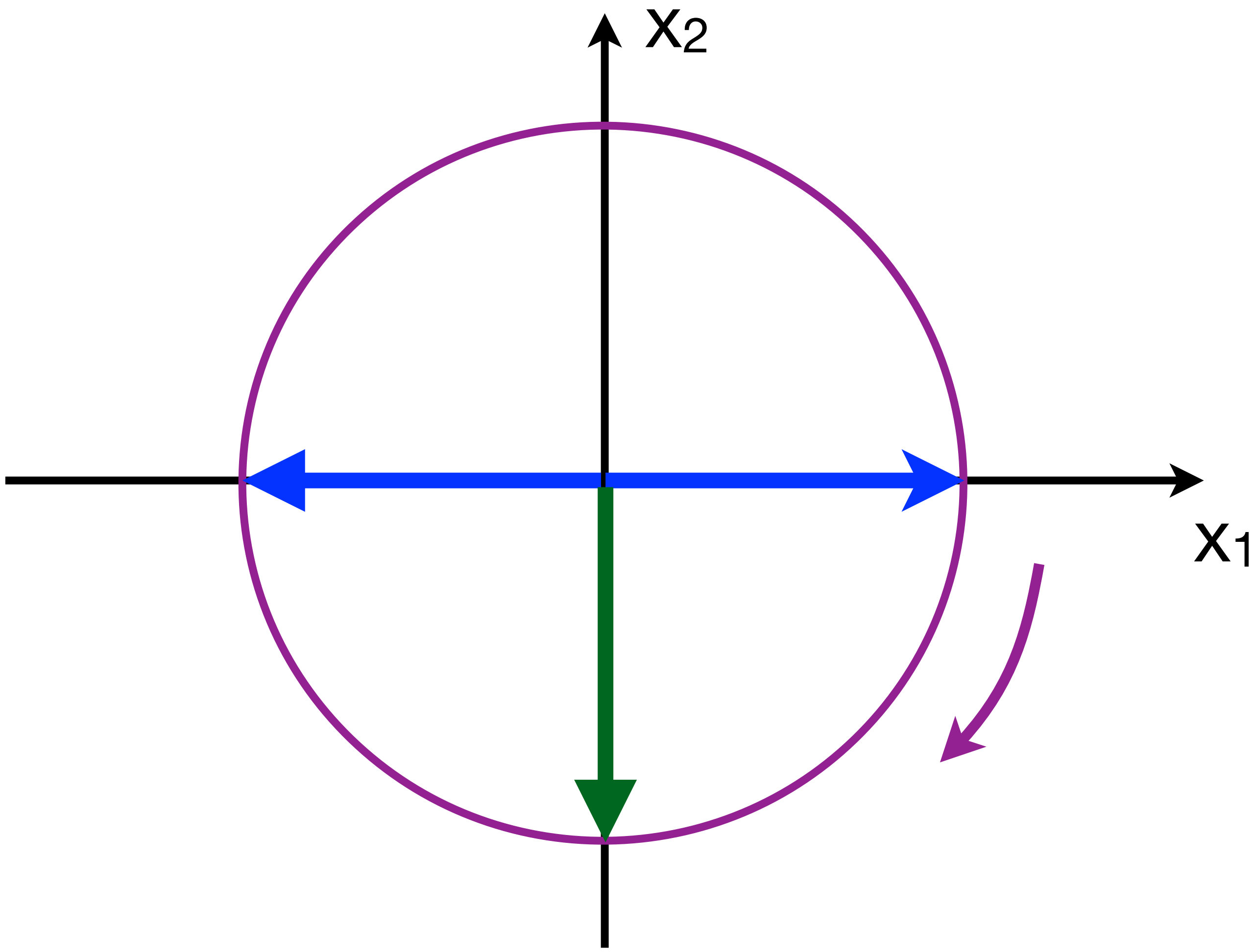
Complex eigenvalues - example

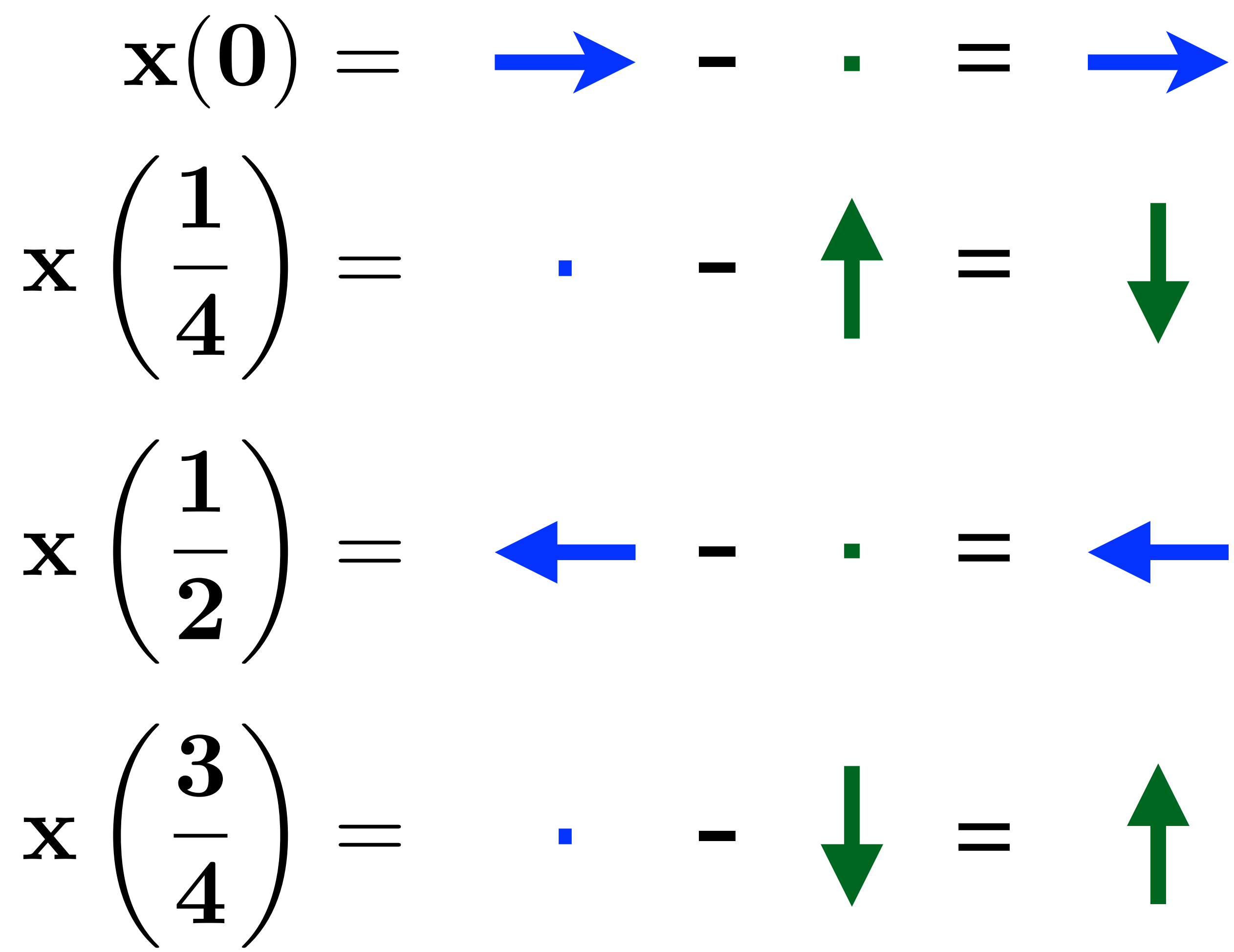
$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



Complex eigenvalues - example

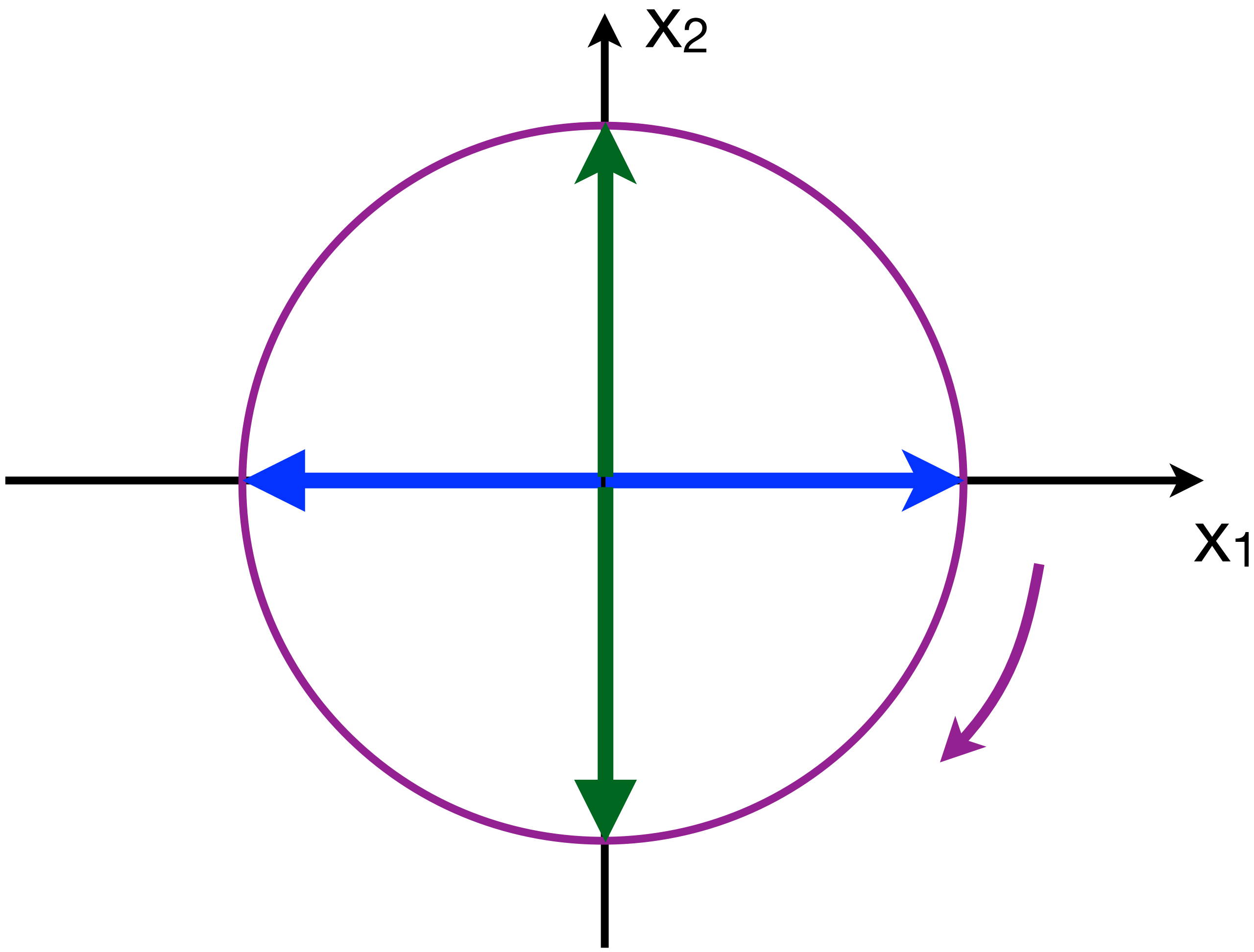
$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

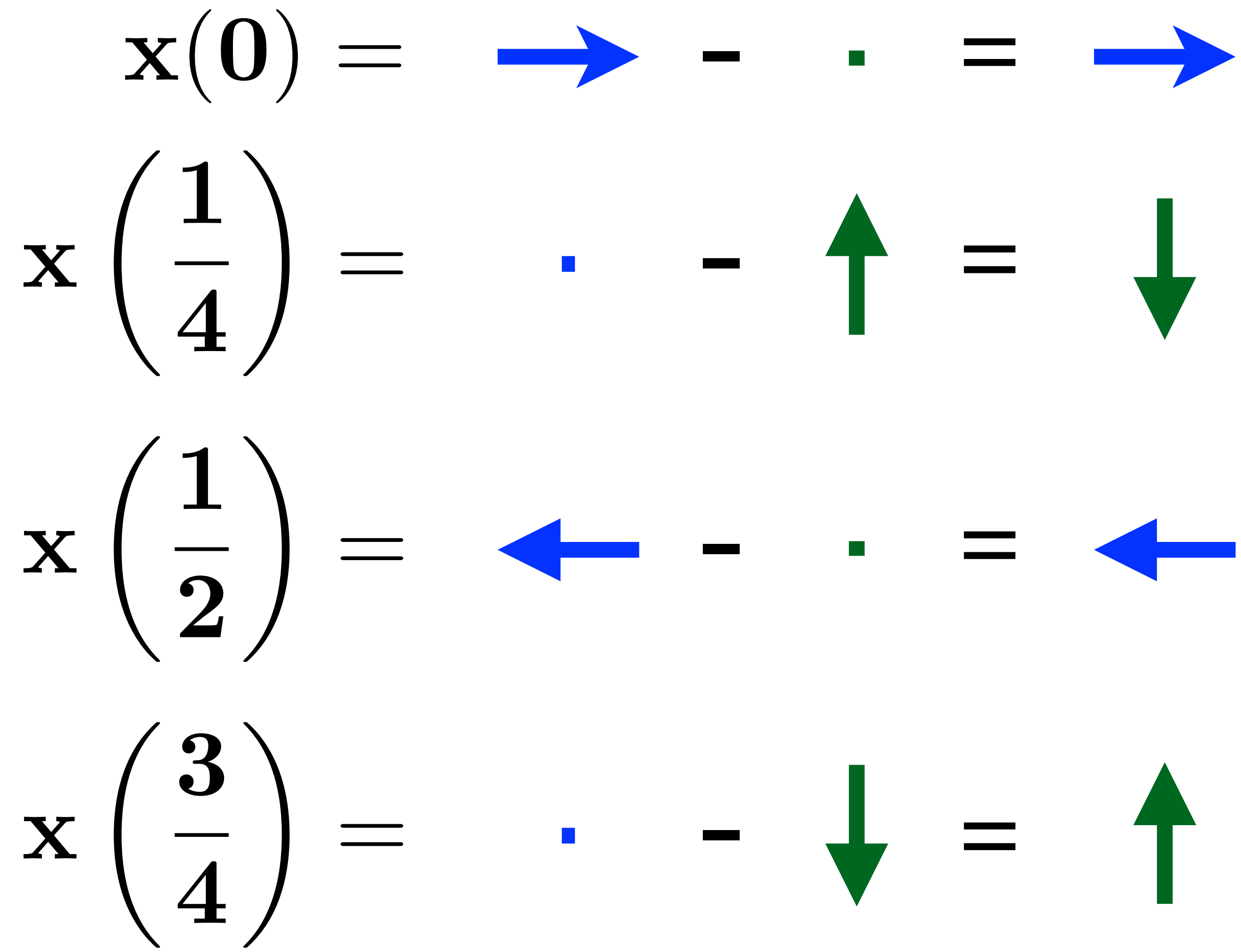




Complex eigenvalues - example

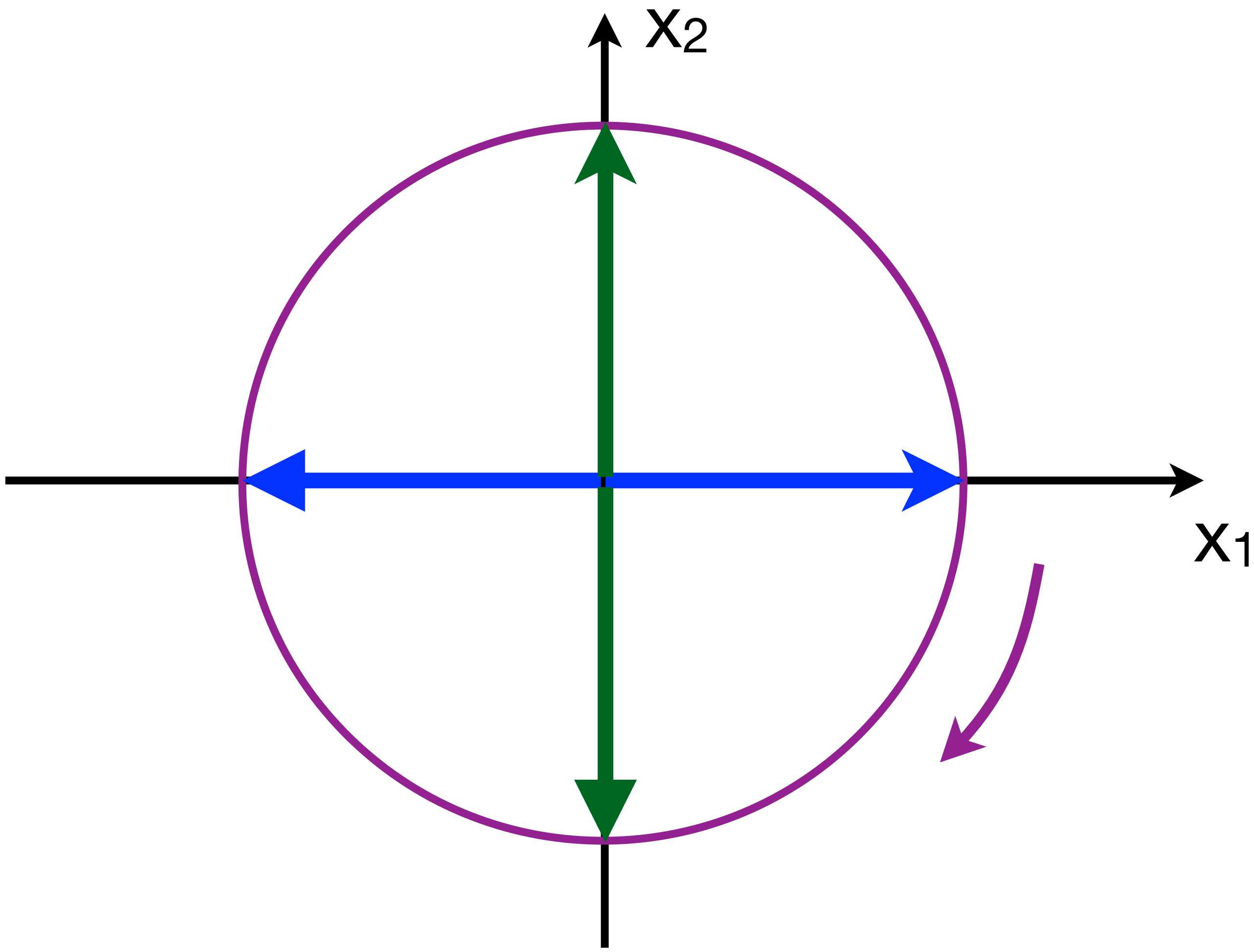
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$





Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



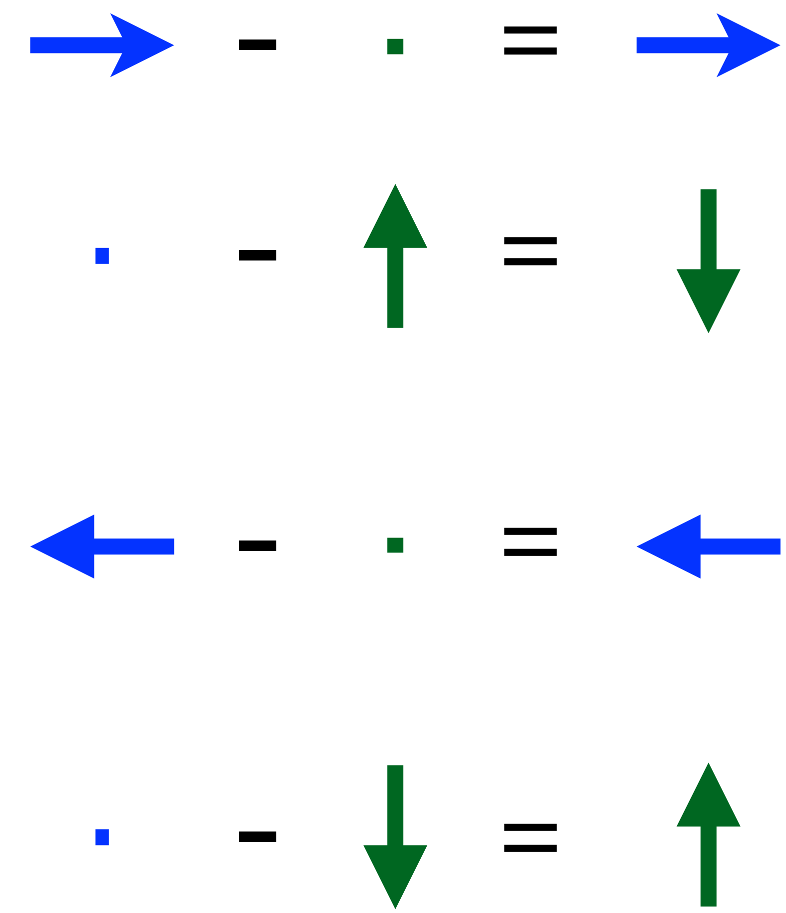
$$\mathbf{x}(\mathbf{0}) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{4}}\right) =$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) =$$

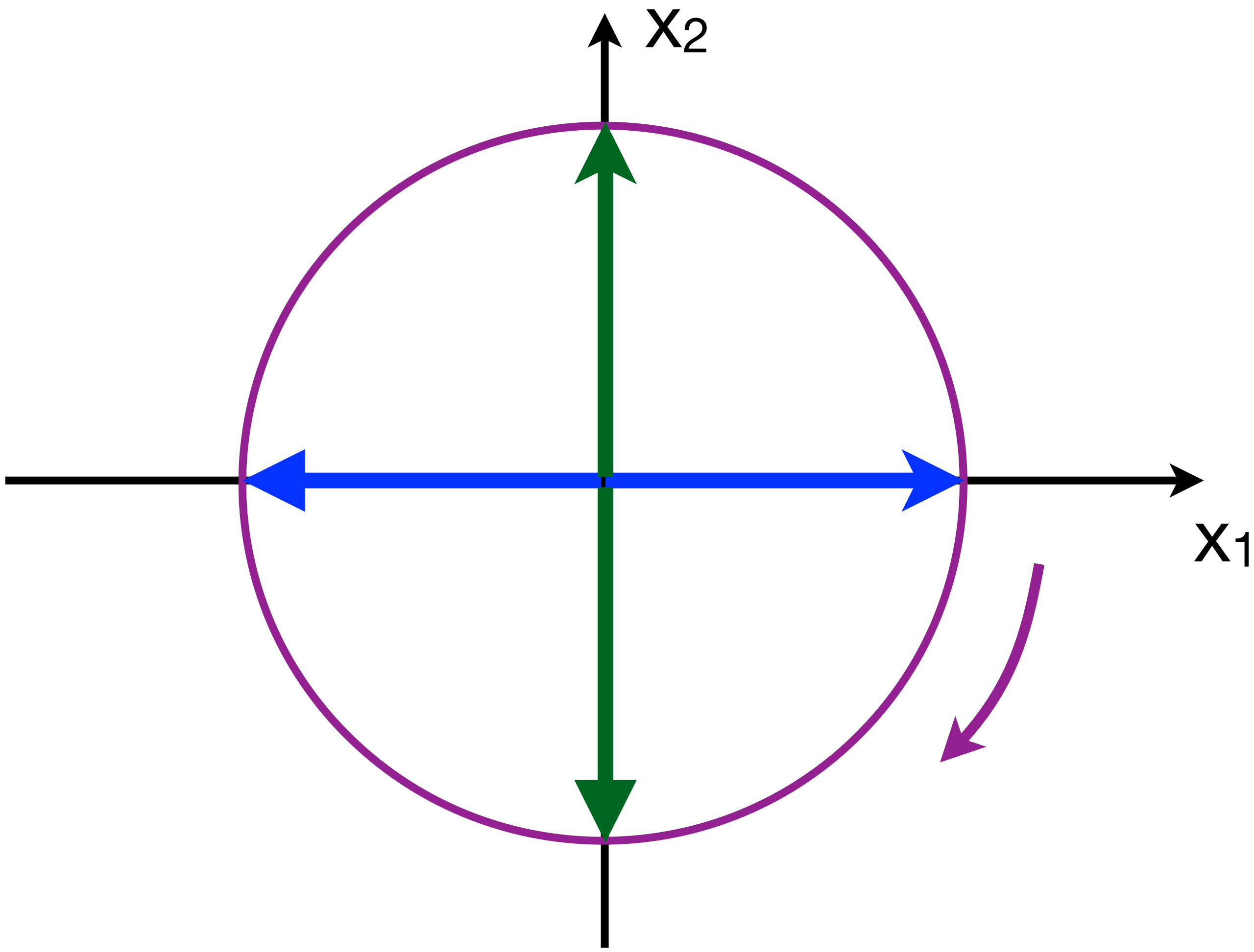
$$\mathbf{x}\left(\frac{\mathbf{3}}{\mathbf{4}}\right) =$$

$$\mathbf{x}\left(\mathbf{1}\right) =$$



Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



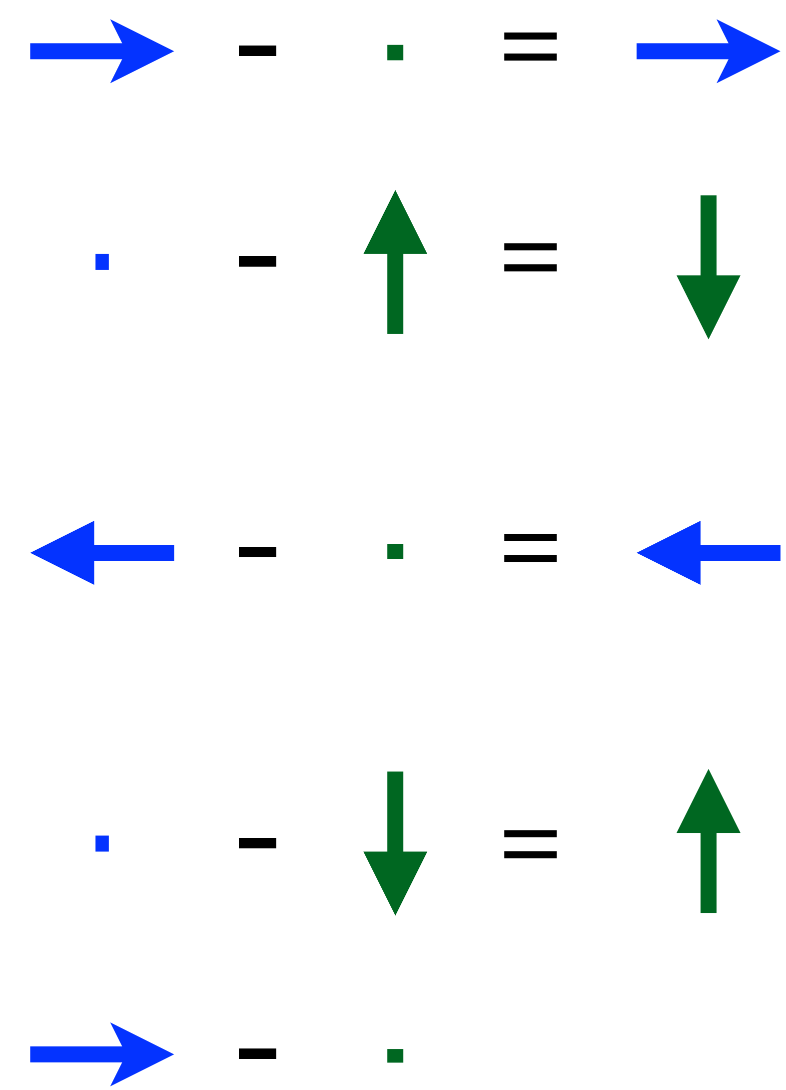
$$\mathbf{x(0)} =$$

$$\mathbf{x}\left(\mathbf{\frac{1}{4}}\right) =$$

$$\mathbf{x}\left(\mathbf{\frac{1}{2}}\right) =$$

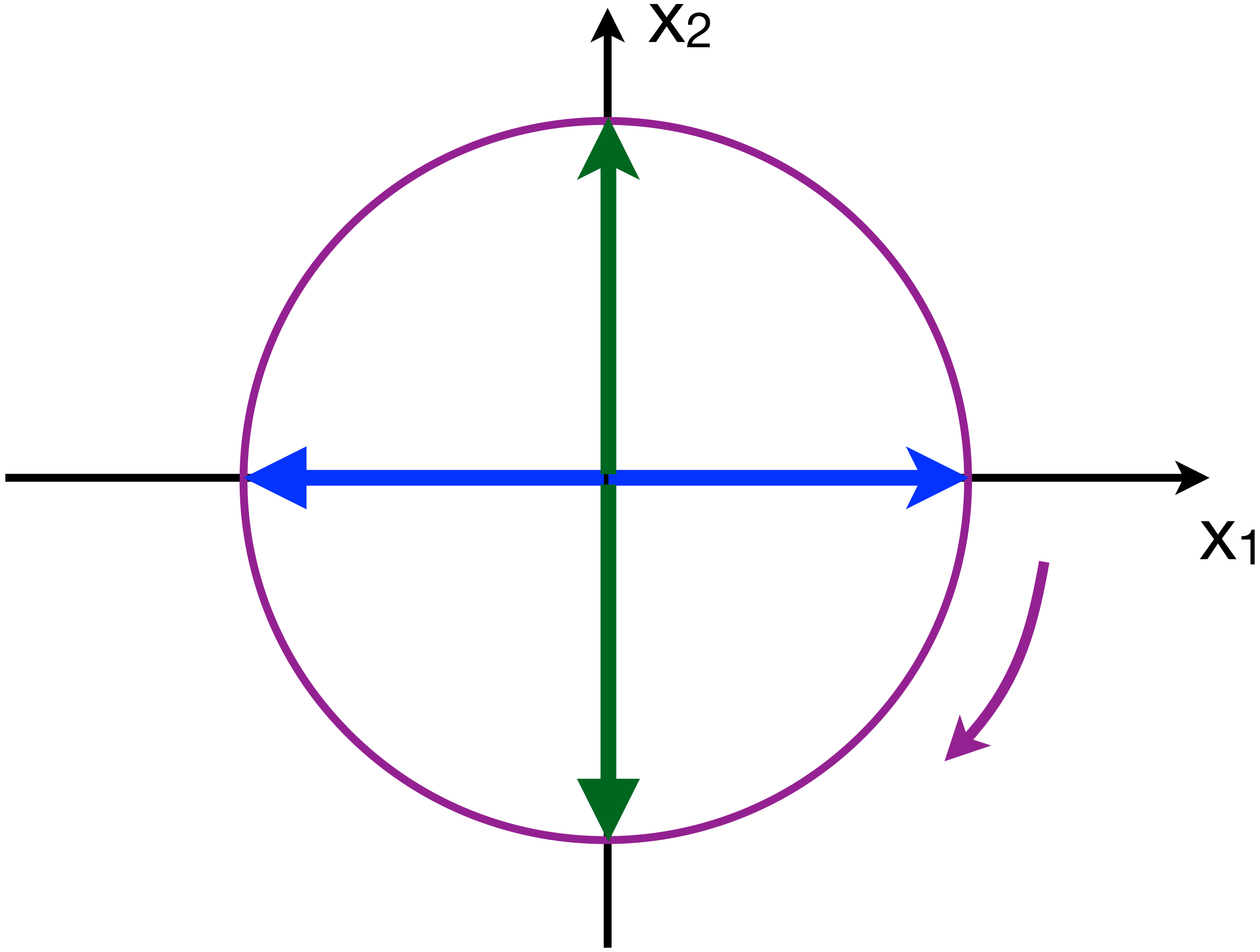
$$\mathbf{x}\left(\mathbf{\frac{3}{4}}\right) =$$

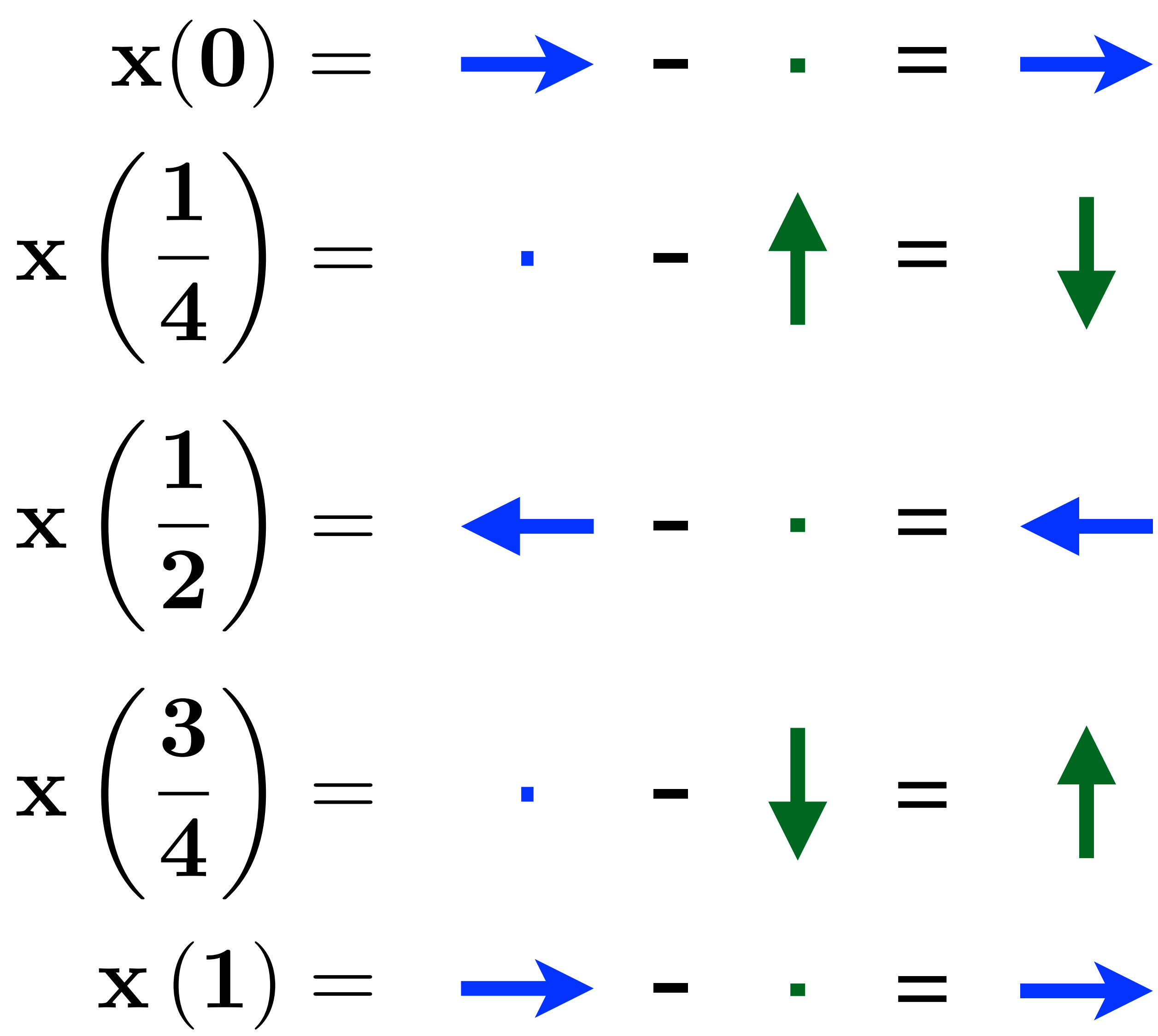
$$\mathbf{x}\left(\mathbf{1}\right) =$$



Complex eigenvalues - example

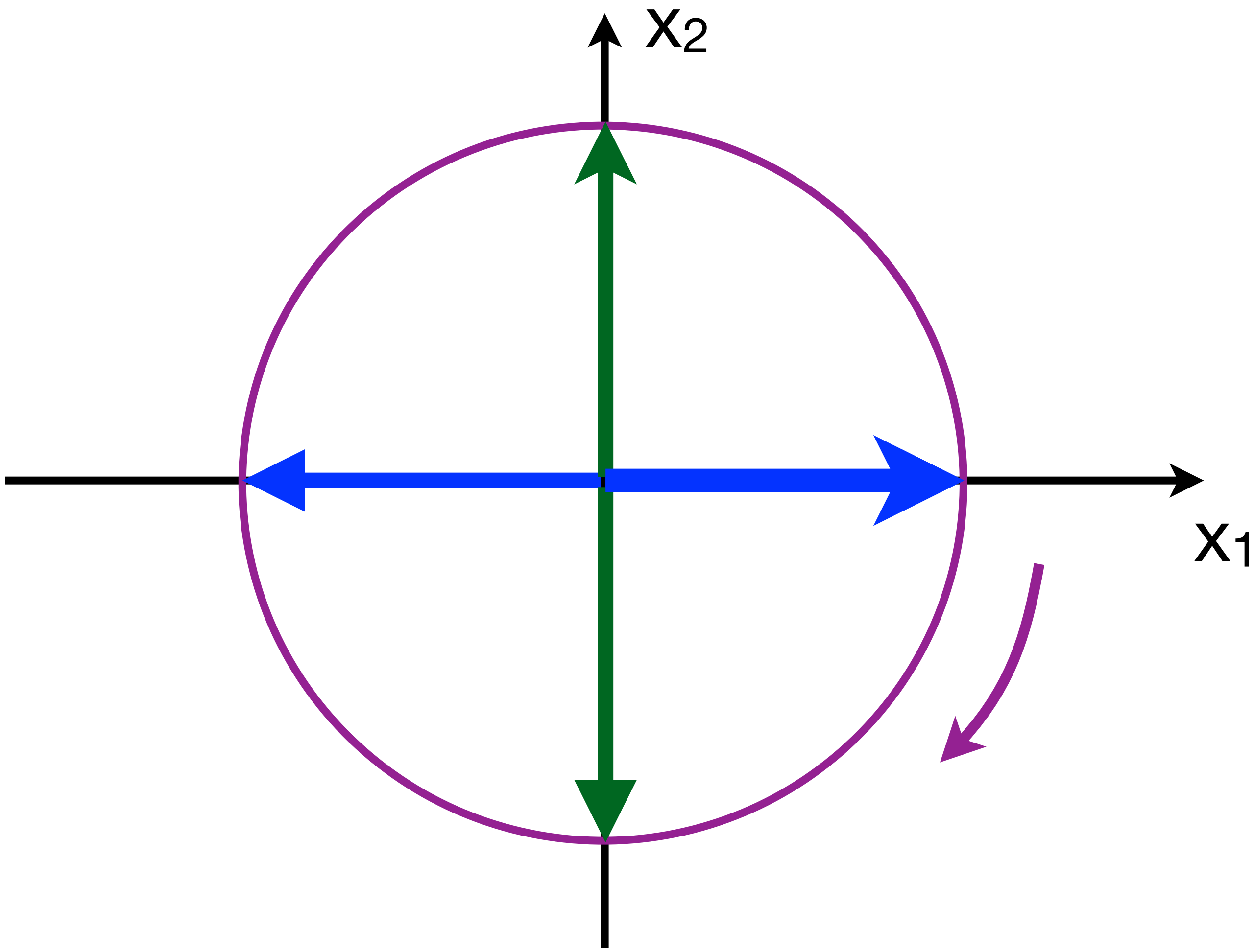
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix}1\\0\end{pmatrix}\cos(2\pi t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(2\pi t)$$

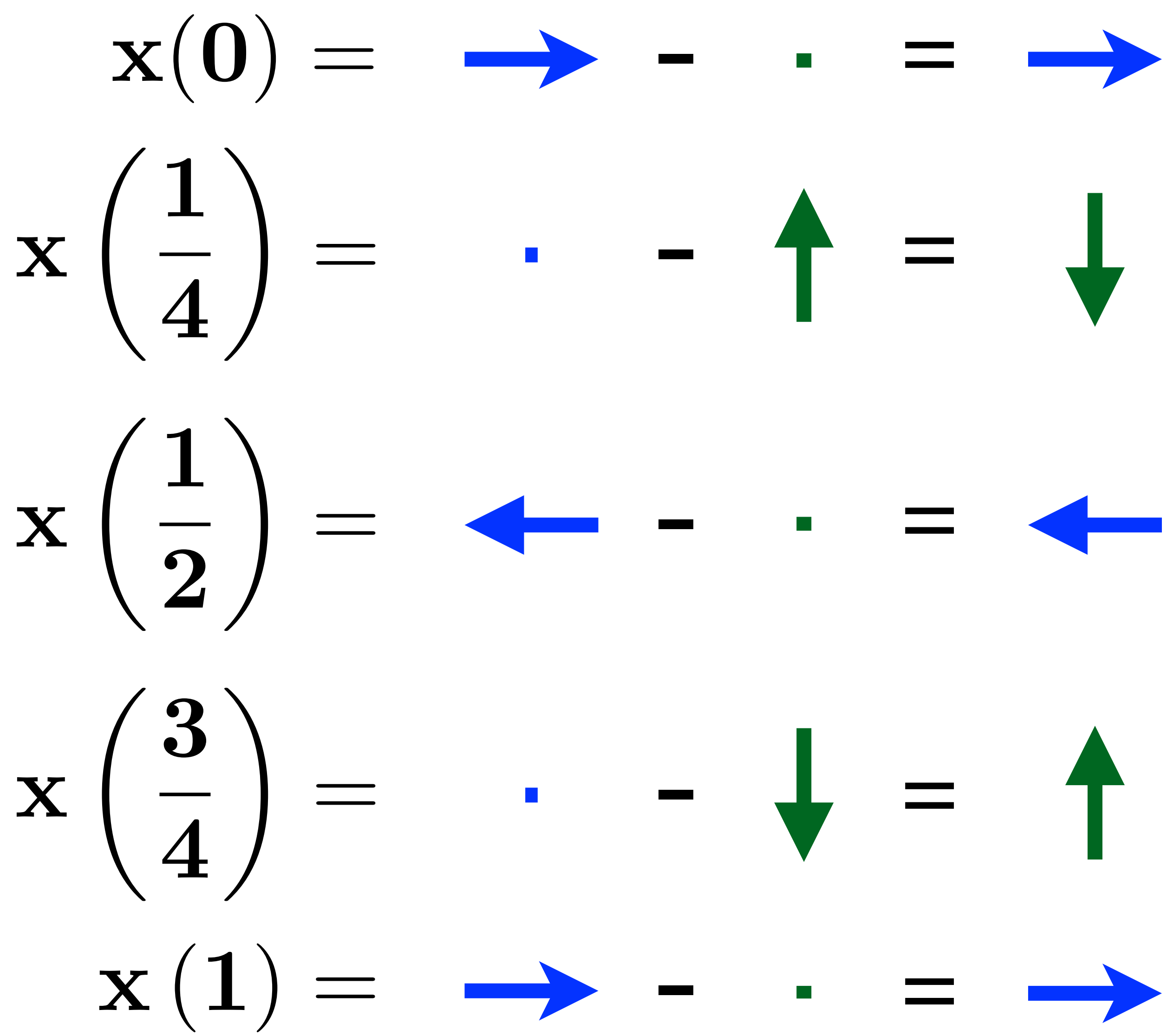




Complex eigenvalues - example

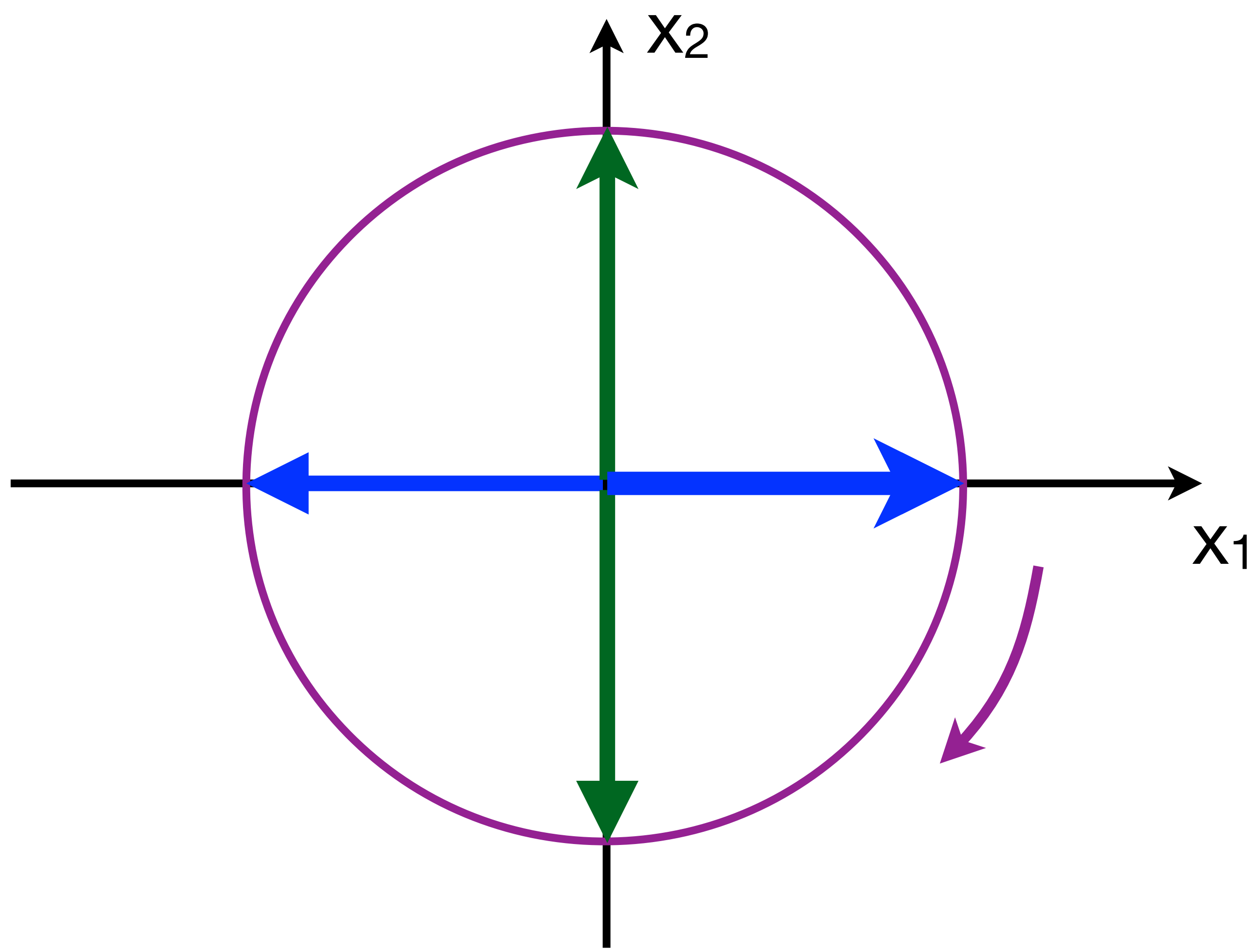
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

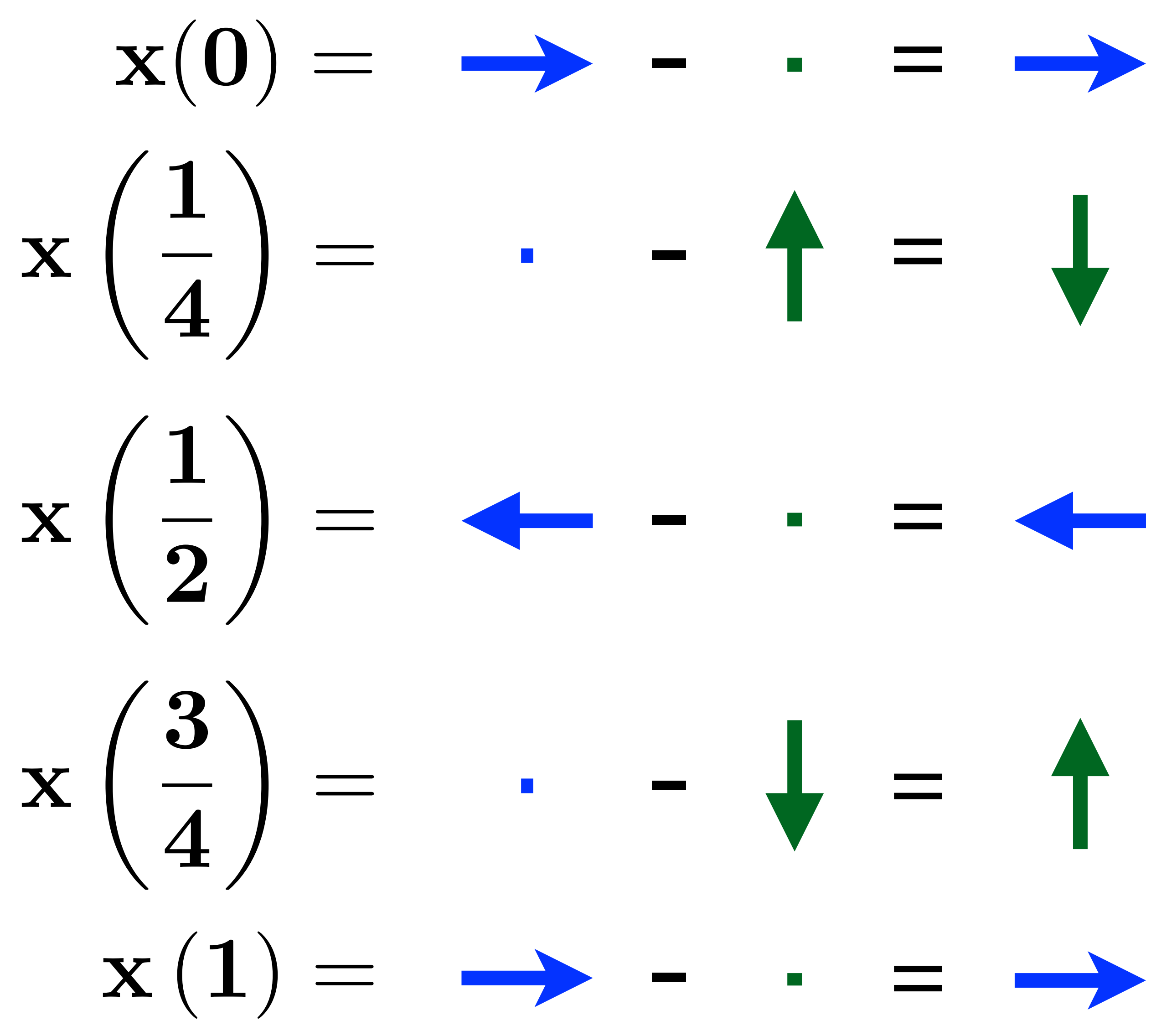




Complex eigenvalues - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$





Fastest method: at t=0, cos(t)=1 and decreasing, -sin(t)=0 and increasing.

Complex eigenvalues - example

- Same equation, initial condition chosen so that C_1=0 and C_2=1.

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t)$$

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 \left(\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t) \right) \right. \\ \left. + C_2 \left(\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) \right) \right]$$

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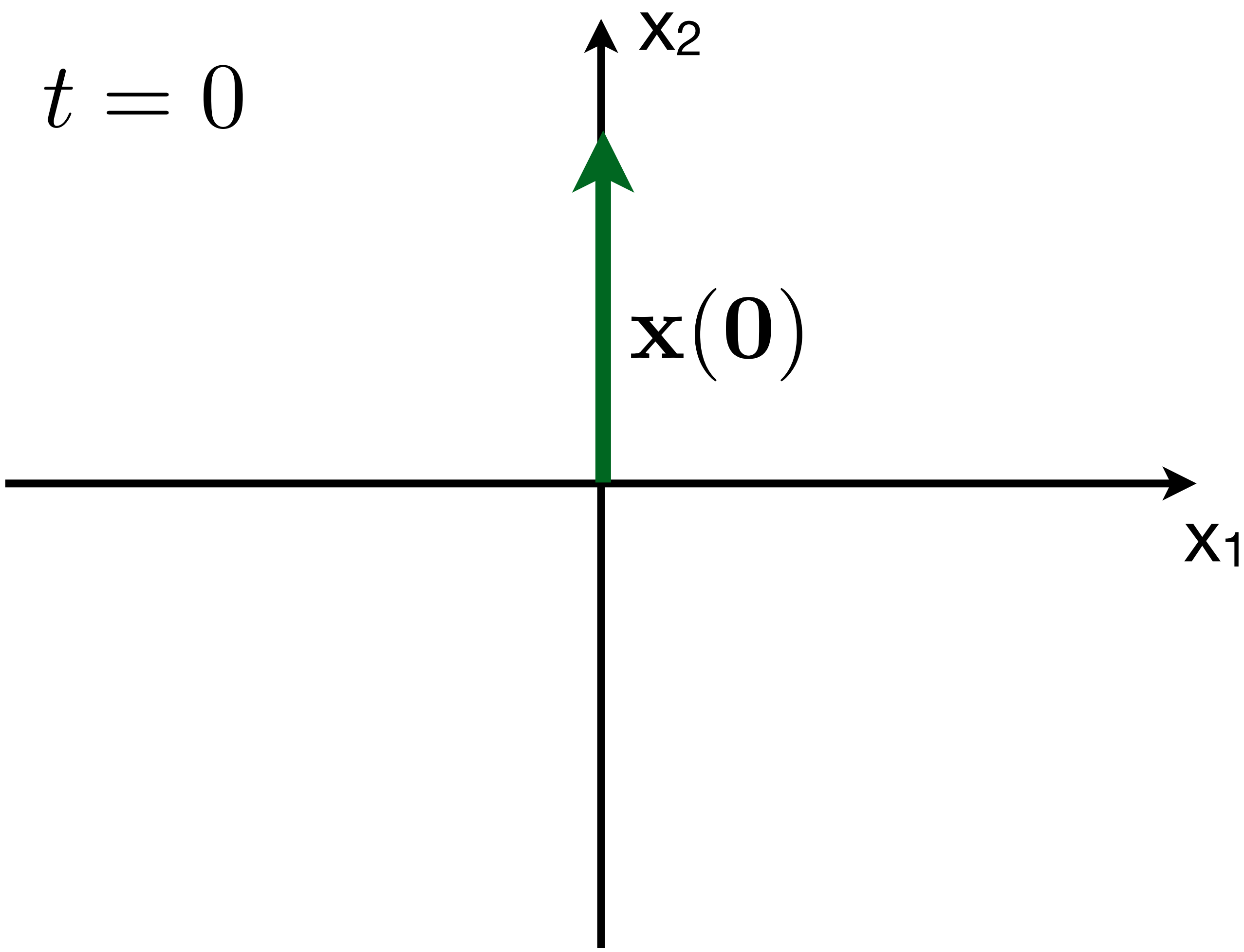

- What happens as t increases?
  - (A) The vector rotates clockwise.
  - (B) The vector rotates counter-clockwise.
  - (C) The tip of the vector maps out a circle in the first quadrant.
  - (D) The tip of the vector maps out a circle in the second quadrant.
  - (E) Explain please.

Complex eigenvalues - example

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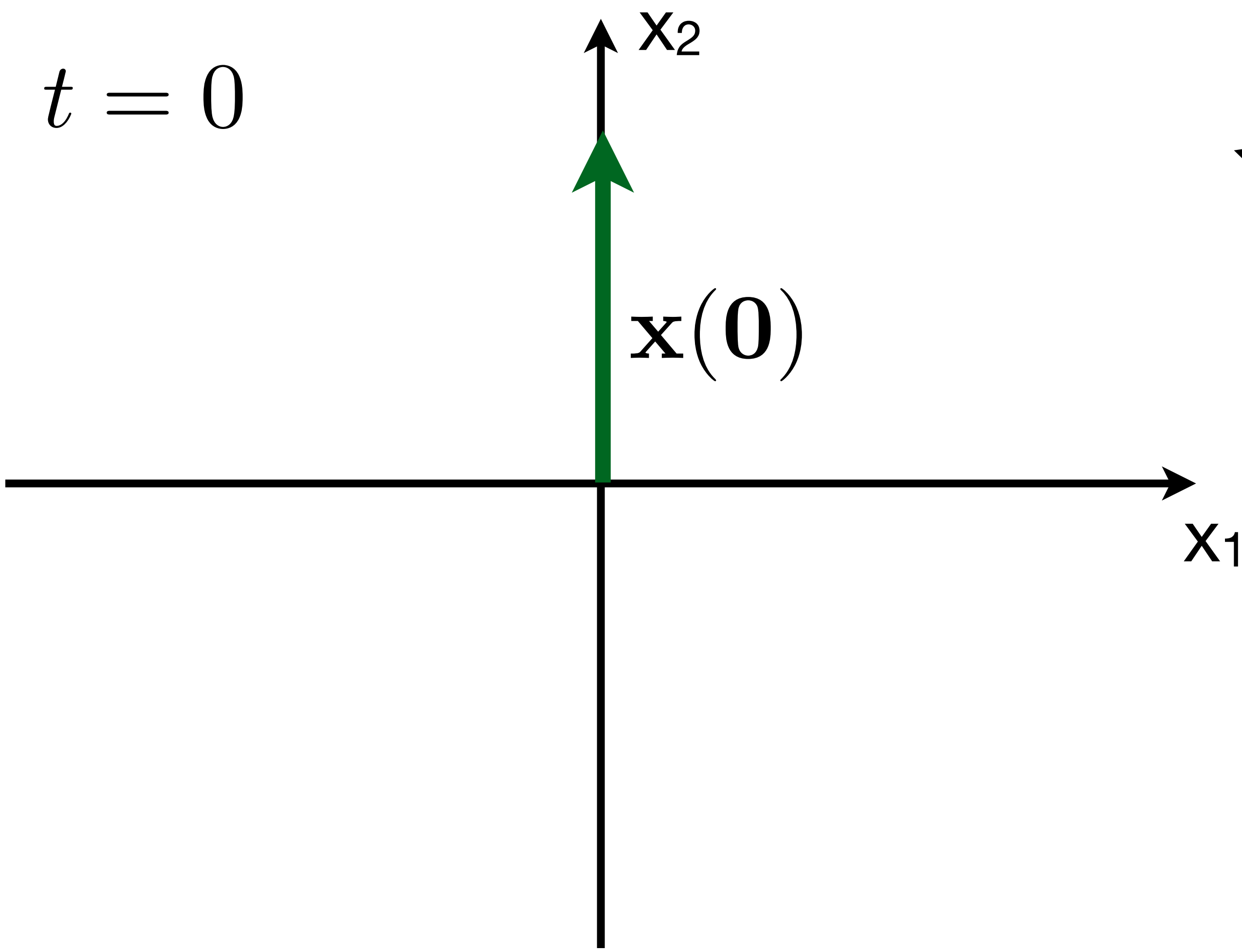
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t)$$

$t = 0$



- What happens as t increases?

(A) The vector rotates clockwise. ★

(B) The vector rotates counter-clockwise.

(C) The tip of the vector maps out a circle in the first quadrant.

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(E) Explain please.

Complex eigenvalues - example

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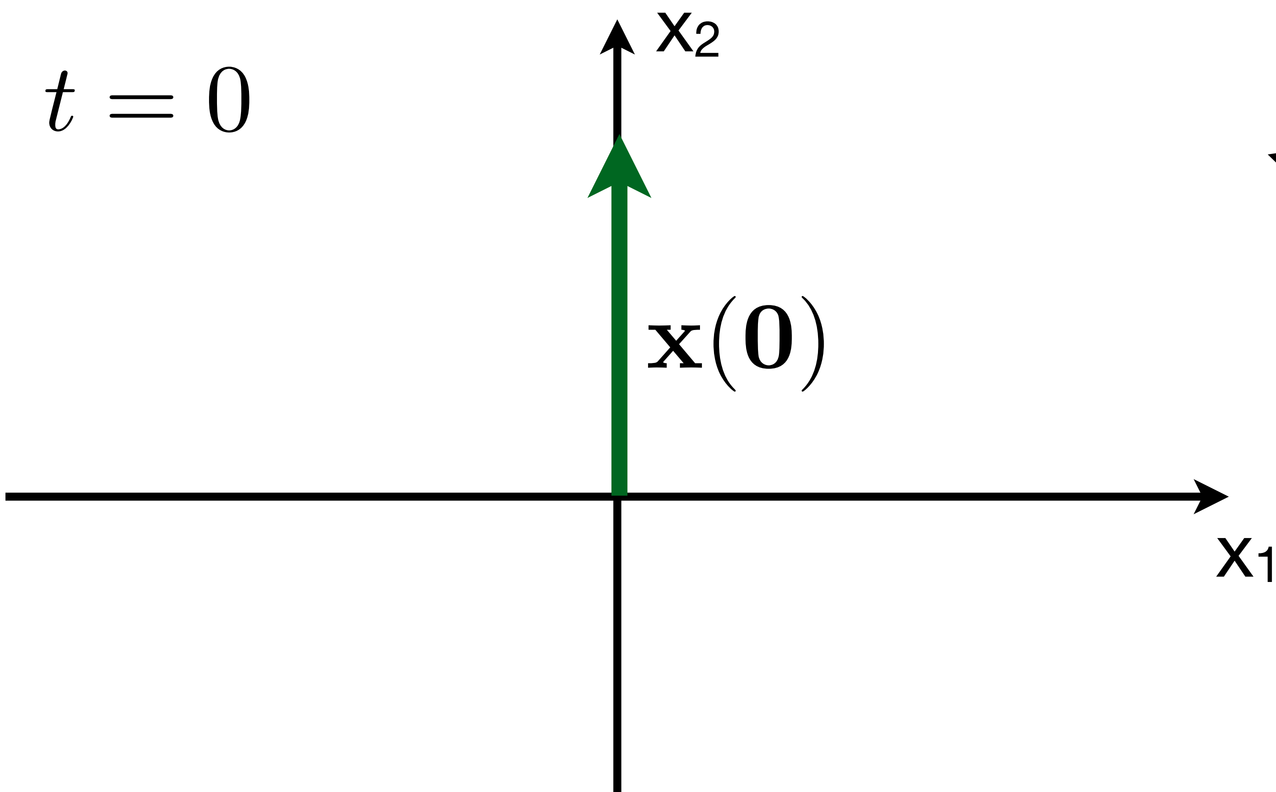

- "Same" solution as before, just π/2 delayed.

- What happens as t increases?

(A) The vector rotates clockwise.

(B) The vector rotates counter-clockwise.

(C) The tip of the vector maps out a circle in the first quadrant.

(D) The tip of the vector maps out a circle in the second quadrant.

(E) Explain please.

Complex eigenvalues - general case

- Looking at the general solution again...

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right)\right.$$

$$\left. + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\right]$$

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- Both parts rotate in the exact same way but the C_2 part is delayed by a quarter phase.

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- Both parts rotate in the exact same way but the C_2 part is delayed by a quarter phase.
- If an initial condition lies neither parallel to vector **a** nor to vector **b**, C_1 and C2 allow for intermediate phases to be achieved.
- **x(t)** can be rewritten (using trig identities) as

$$\mathbf{x}(\mathbf{t}) = Me^{\alpha t}\left(\mathbf{a}\cos(\beta t - \phi) - \mathbf{b}\sin(\beta t - \phi)\right)$$

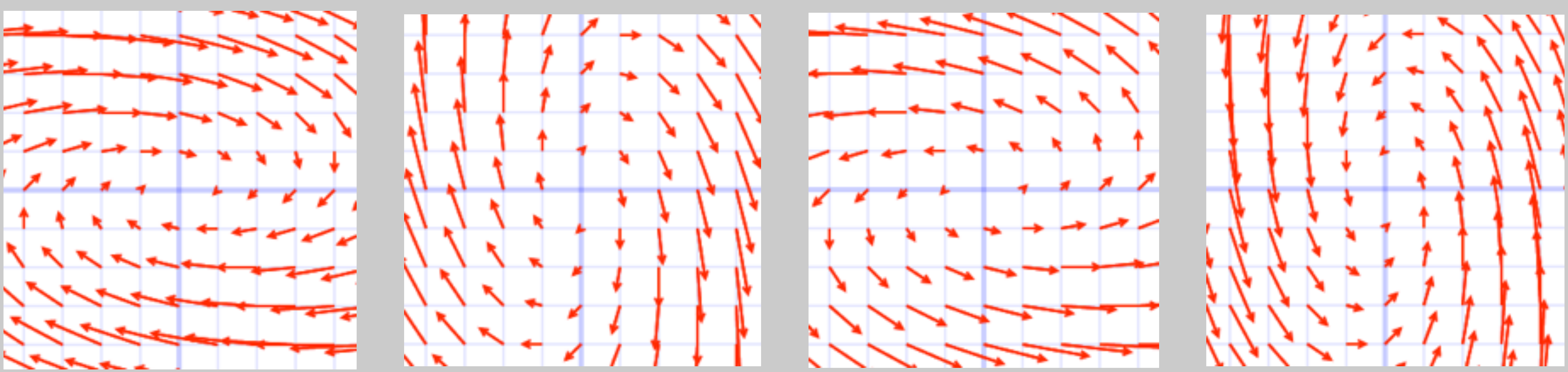
where M and ϕ are constants to replace C_1 and C_2.

Complex eigenvalues - example

- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$

(A) (B) (C) (D)



(E) Explain, please.

Complex eigenvalues - example

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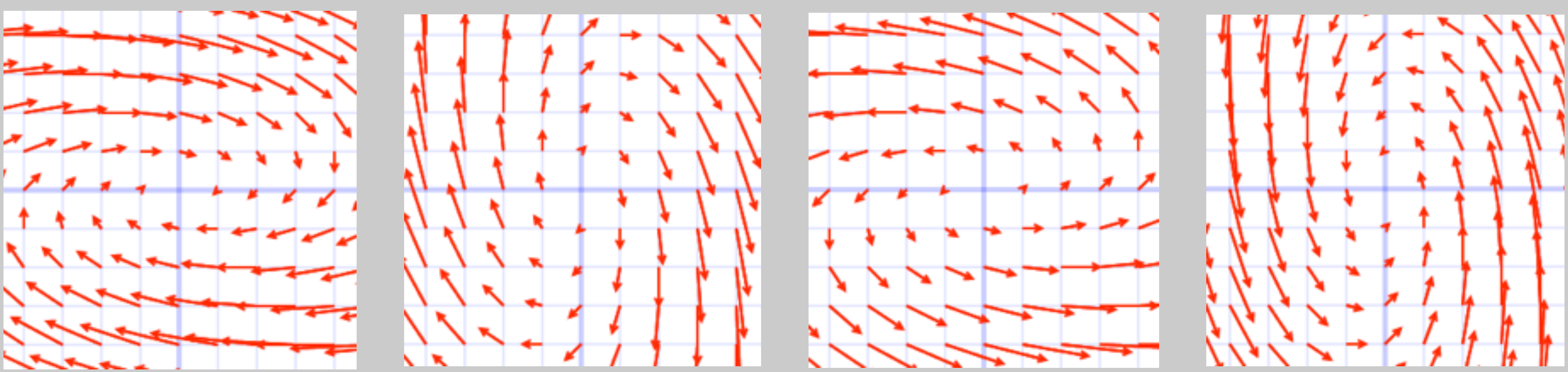

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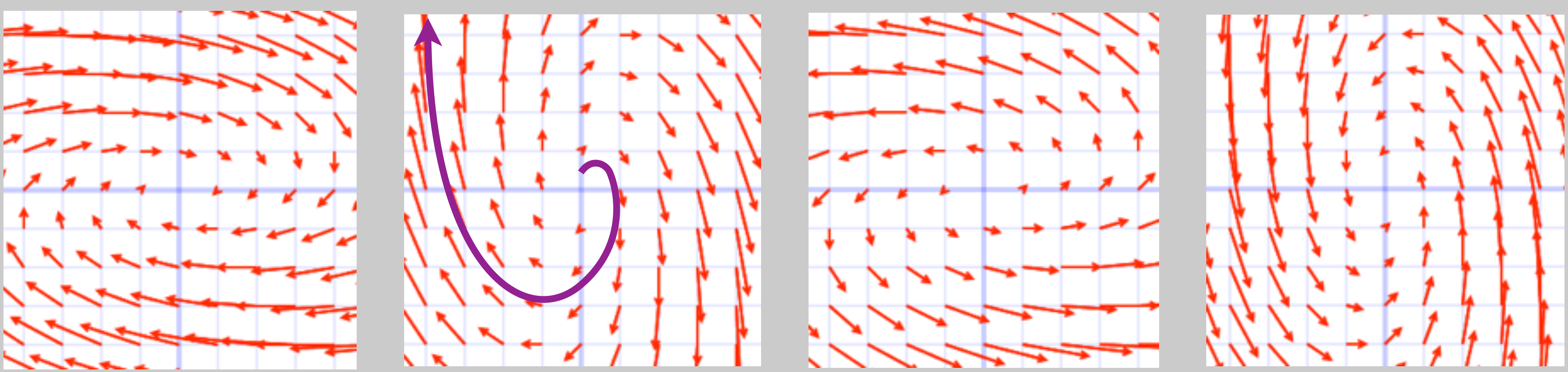

(E) Explain, please.