Find the solution to the equation

$$u_t = 4u_{xx}$$

with IC $u(x,0) = 2x$

and BCs $u(0,t) = 1$

$$u_x(5,t) = 2$$

$u_{ss}(x) = ax + b$

$u_{ss}(0) = b = 1$

$\frac{\partial u_{ss}}{\partial x}(5) = a = 2$

$U_{ss}(x) = 2x + 1$

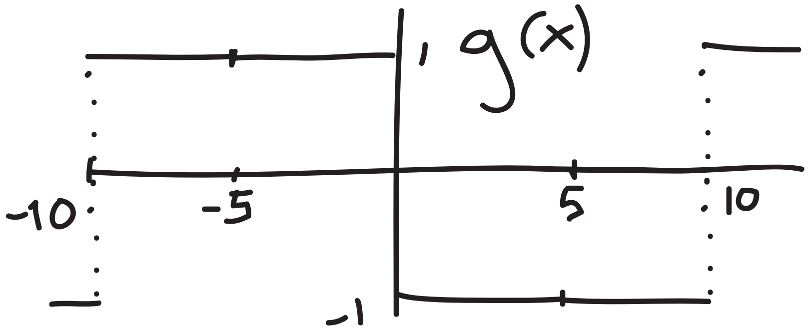
$$u(x,t) = 2x + 1 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{100}\cdot 4t} \sin \frac{n\pi x}{10}$$

$$u(x,0) = 2x + 1 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = 2x$$

Choose b_n so that

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = -1 \quad \text{on } (0,5).$$

Extend -1 as an even function about $x=5$ up to $x=10$ and as an odd function about $x=0$ down to $x=-10$. Then extend that as a periodic function with period 20.



$$b_n = \frac{1}{10}\int_{-10}^{10} g(x)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{2}{10}\int_0^{10} (-1)\sin\frac{n\pi x}{10}\,dx \quad *$$

$$= \frac{1}{5}\,\frac{10}{n\pi}\cos\frac{n\pi x}{10}\Big|_0^{10} = \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$u(x,t) = 2x+1 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{25}t}\sin\frac{n\pi x}{10}$$

Because $b_n = 0$ for n even,

$$u(x,t) = 2x+1 + \sum_{k=1}^{\infty} b_{2k-1} e^{-\frac{(2k-1)^2\pi^2}{25}t}\sin\frac{(2k-1)\pi x}{10}$$

* In general, because of the symmetries of $g(x)$, we could have simplified this integral a bit more:

$$b_n = \frac{4}{10}\int_0^{5} (-1)\sin\frac{n\pi x}{10}\,dx \quad \text{for } n \text{ odd}$$

and $b_n = 0$ for n even.