Today

- Office hour poll
- Finish up with integrating factors
- The structure of solutions
- Separable equations

Method of integrating factors (Section 2.1)

- What's the integrating factor?

$$t\frac{dy}{dt} + 2y(t) = 1 \qquad \longrightarrow f(t) = t$$

$$t^2\frac{dy}{dt} + 4ty(t) = \frac{1}{t} \qquad \longrightarrow f(t) = t^2$$

$$\frac{dy}{dt} + y(t) = 0 \qquad \longrightarrow f(t) = e^t$$

$$\frac{dy}{dt} + \cos(t)y(t) = 0 \qquad \longrightarrow f(t) = e^{\sin(t)}$$

$$\frac{dy}{dt} + g'(t)y(t) = 0 \qquad \longrightarrow f(t) = e^{g(t)}$$

Method of integrating factors (Section 2.1)

- General case - all first order linear ODEs can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

- The appropriate integrating factor is $e^{\int p(t)dt}$.
- The equation can be rewritten $\frac{d}{dt}\left(e^{\int p(t)dt}y\right) = e^{\int p(t)dt}q(t)$ which is solvable provided you can find the antiderivative of the right hand side.

$$e^{\int p(t)dt}y(t) = \int e^{\int p(t)dt}q(t)dt + C$$

$$y(t) = e^{-\int p(t)dt}\int e^{\int p(t)dt}q(t)dt + Ce^{-\int p(t)dt}$$

The structure of solutions

- When the equation is of the form (called homogeneous)

$$\frac{dy}{dt} + p(t)y = 0$$

- the solution is

$$y(t) = C\mu(t)^{-1}$$

- where

$$\mu(t) = \exp\left(\int p(t)dt\right)$$

- is the integrating factor.

The structure of solutions

- When the equation is of the form (called nonhomogeneous)

$$\frac{dy}{dt} + p(t)y = q(t)$$

- the solution is $y(t) = k(t) + C\mu(t)^{-1}$

- where k(t) involves no arbitrary constants.

- Think about this expression as $y(t) = y_p(t) + y_h(t)$

- Directly analogous to solving the vector equations $A\overline{x} = 0$ and $A\overline{x} = \overline{b}$.

Examples

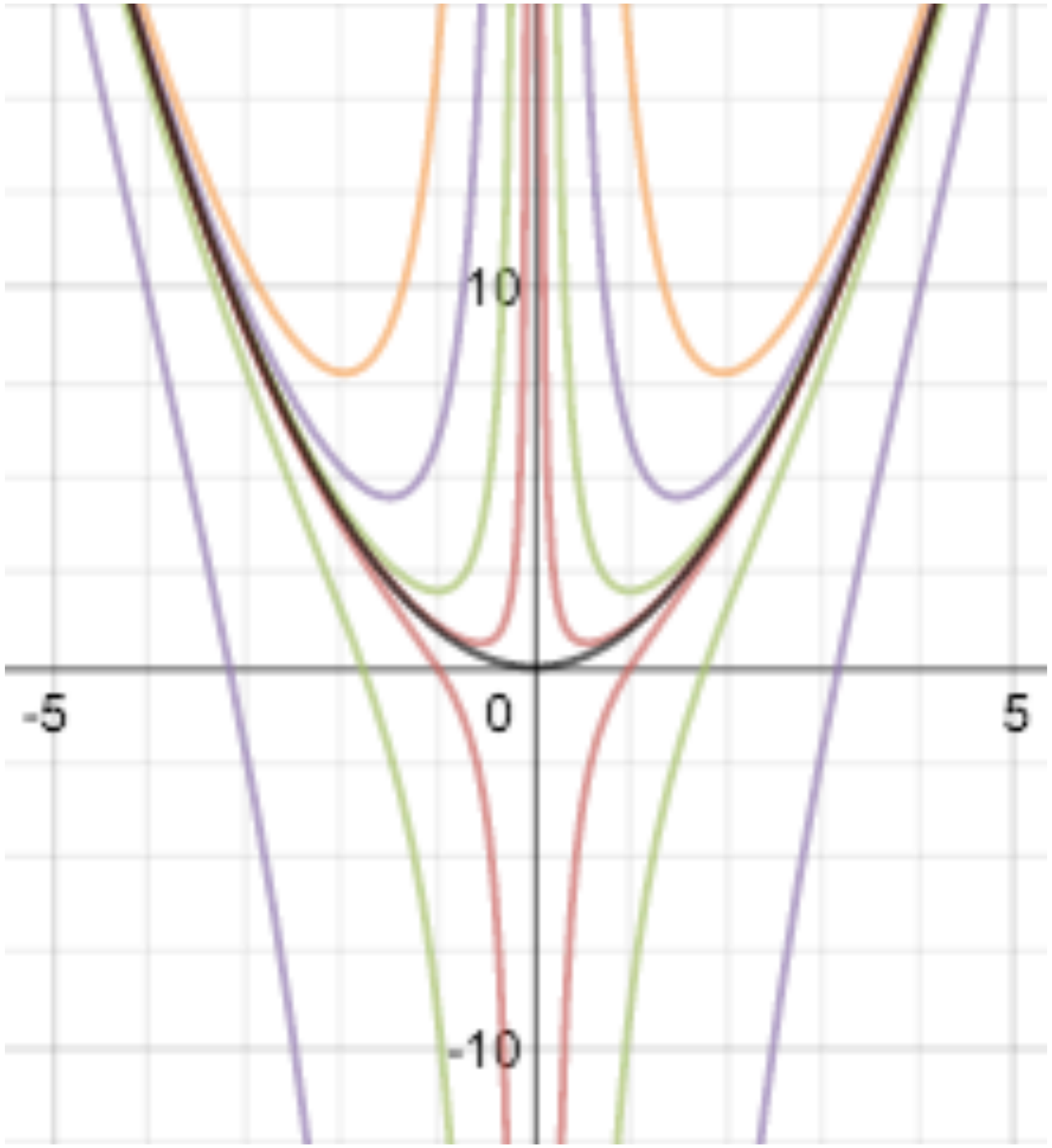
- Find the general solution to

$$t\frac{dy}{dt} + 2y = 4t^2$$

- and plot a few of the integral curves.

- Integral curve - the graph of a solution to an ODE.

$$y(t) = t^2 + C\frac{1}{t^2}$$



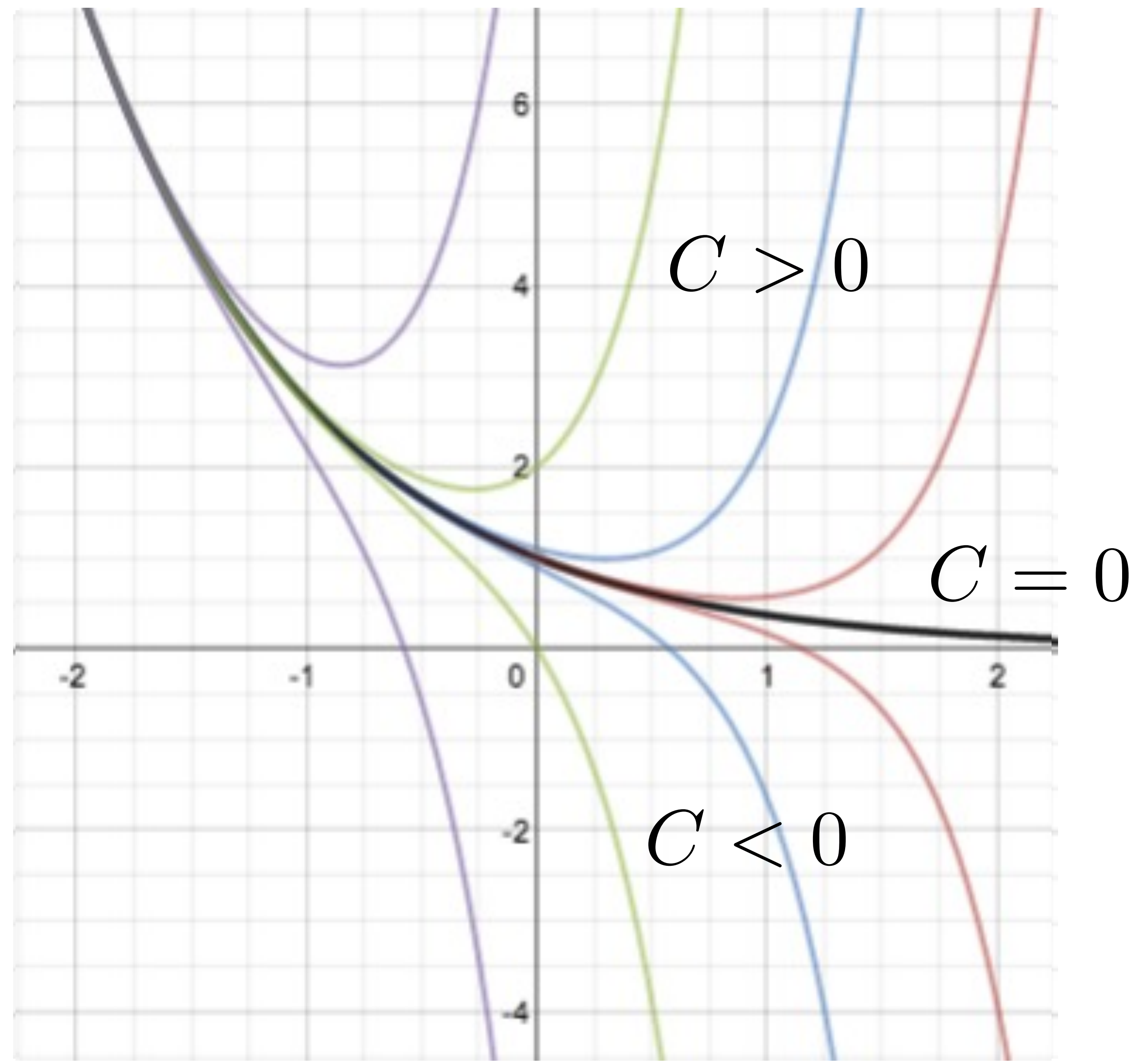
Examples

- Find the general solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- and plot a few of the integral curves.

$$y(t) = e^{-t} + Ce^{3t}$$



Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- depending on C, how many different results are possible for

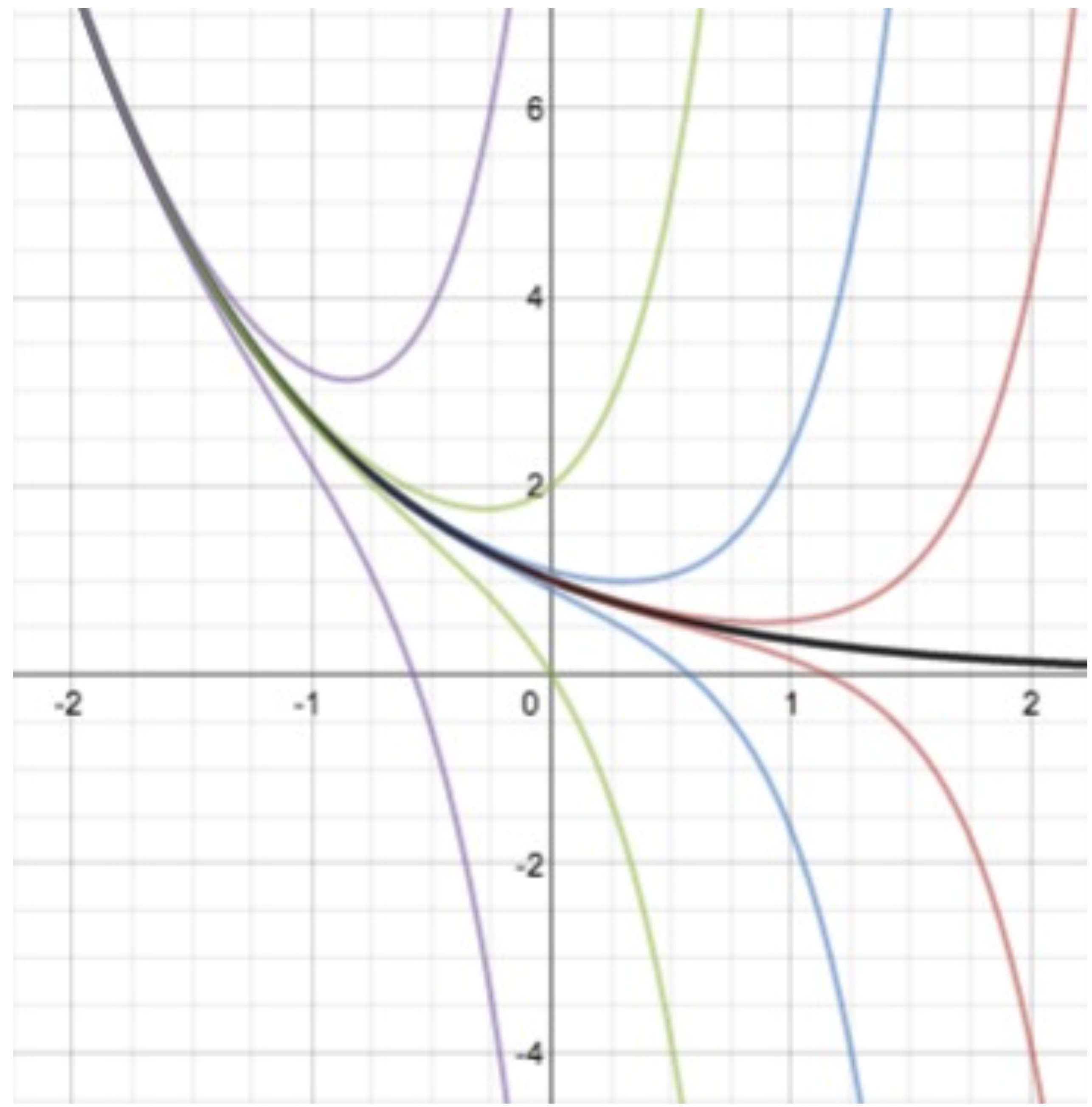
$$\lim_{t\to\infty} y(t)$$

(A) 0

(B) 1

(C) 2

(D) 3



Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -e^t$$

- depending on C, how many different results are possible for

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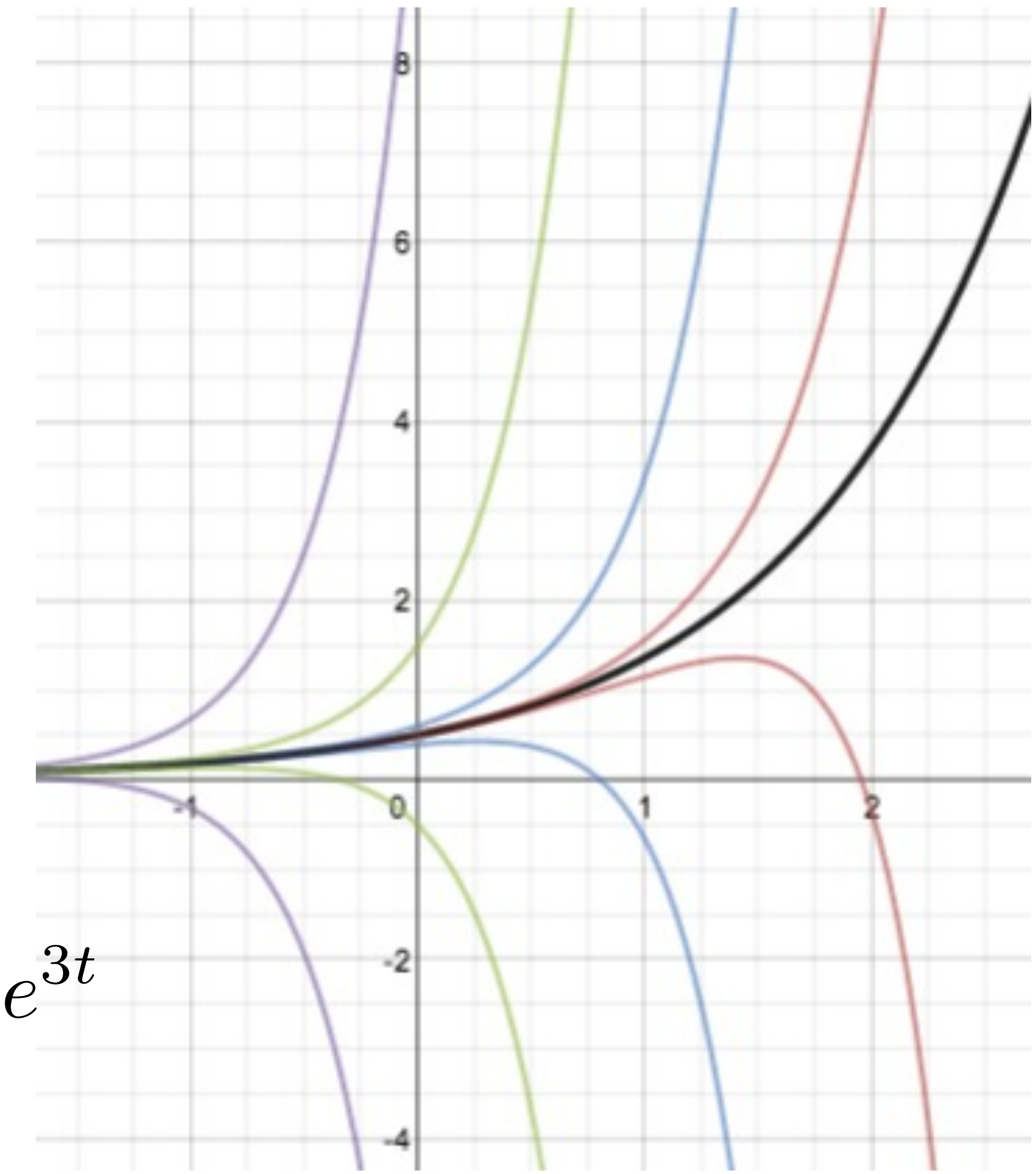
(B) 1

(C) 2

(D) 3

$$y(t) = \frac{1}{2}e^t + Ce^{3t}$$



Separable equations (Section 2.2)

- What is $\dfrac{d}{dt}e^y$?

(A) e^y

(B) $e^y\dfrac{dy}{dt}$

(C) ye^{y-1}

(D) $ye^{y-1}\dfrac{dy}{dt}$

- Solve $\dfrac{dy}{dt} = e^{-y}$.

(A) $y(t) = 0$

(B) $y(t) = \ln(t) + C$

(C) $y(t) = \ln(t + C)$

(D) $y(t) = e^{t+C}$

Separable equations (Section 2.2)

- First order ODEs of the form: $\dfrac{dy}{dx} = f(x)h(y)$
- Rename h(y)=1/g(y): $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$
- Find antiderivatives for f and g and rewrite as $\dfrac{dy}{dx} = \dfrac{F'(x)}{G'(y)}$
- Rewrite as $G'(y)\dfrac{dy}{dx} = F'(x)$
- Recognize a chain rule: $\dfrac{d}{dx}G(y) = G'(y)\dfrac{dy}{dx}$
- and take antiderivatives to get $G(y) = F(x) + C$
- Finally, solve for y if possible: $y(x) = G^{-1}(F(x) + C)$

Separable equations (Section 2.2)

- Solve: $$\frac{dy}{dx} = -\frac{x}{y}$$

(A) $y(x) = x$

(B) $y(x) = -x$

(C) $y(x) = \sqrt{C - x^2}$

(D) $y(x) = \sqrt{x^2 + C}$

(E) $y(x) = C - x^2$

$$y\frac{dy}{dx} = -x$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + D$$

$$y^2 = -x^2 + C$$

Does (C) cover all possible initial conditions?

Separable equations (Section 2.2)

$$y(x) = \sqrt{C - x^2}$$

- y(0)=2 ----> C=4
- y(1)=1 ----> C=2
- y(1)=-2 ----> C=?
- General solution: $y = \pm\sqrt{C - x^2}$
- Or express implicitly: $y^2 = -x^2 + C$

Separable equations (Section 2.2)

- Solve: $$\frac{dy}{dt} = \frac{1}{\cos(y)}$$

(A) $y(t) = \sin(t)$

(B) $y(t) = \arcsin(t + C)$

(C) $\sin(y) = t + C$

(D) $y(t) = \arcsin(t) + C$

(E) $y(t) = \arccos(t + C)$

Separable equations (Section 2.2)

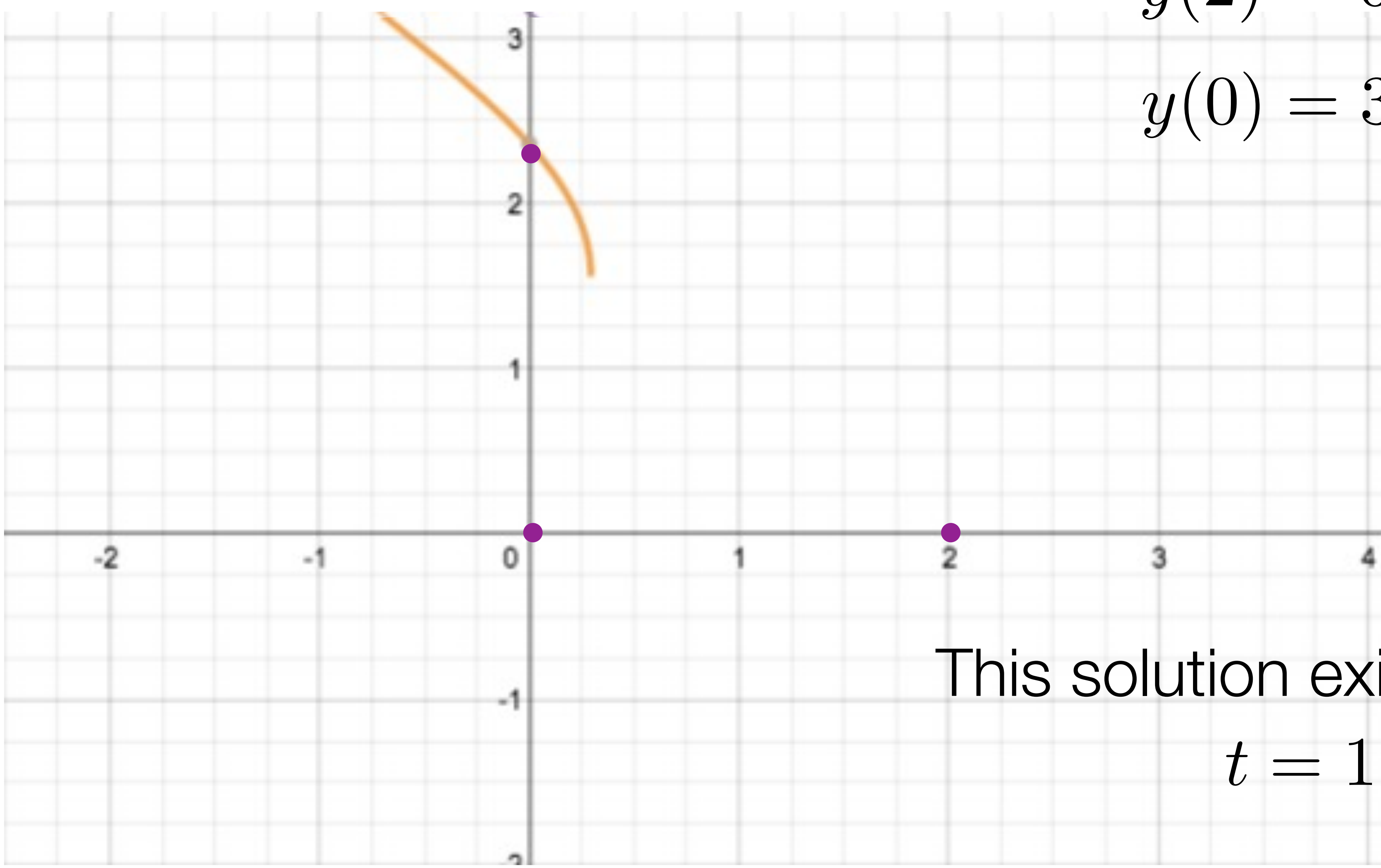
$\sin(y) = t + C$

$y(t) = \arcsin(t + C)$

with IC $y(0) = 0$ $\quad C = 0$

$y(2) = 0$ $\quad C = -2$

$y(0) = 3\pi/4$ $\quad C = \dfrac{1}{\sqrt{2}}$



This solution exists only up until

$t = 1 - 1/\sqrt{2}$