Today

- Step and ramp functions (continued)
- The Dirac Delta function and impulse force
- Modeling with delta-function forcing

Step function forcing (6.3, 6.4)

- Solve using Laplace transforms:

$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

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- So we just need h(t) and we're done.

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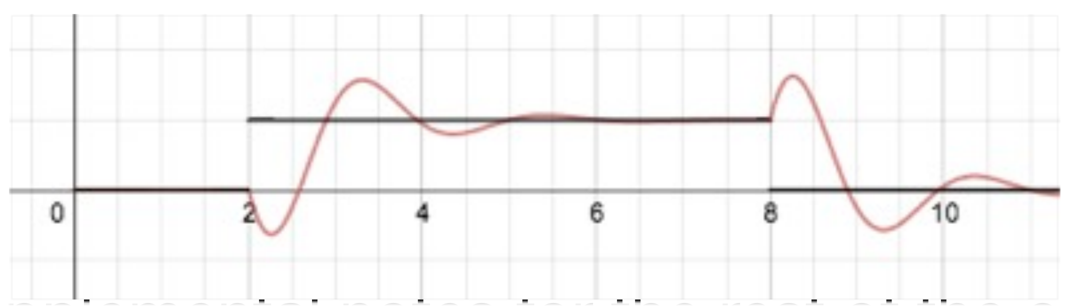

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- An example with a ramped forcing function:

$$y'' + 4y = \begin{cases} 0 & \text{for } t < 5, \\ \frac{t-5}{5} & \text{for } 5 \leq t < 10, \\ 1 & \text{for } t \geq 10. \end{cases}$$

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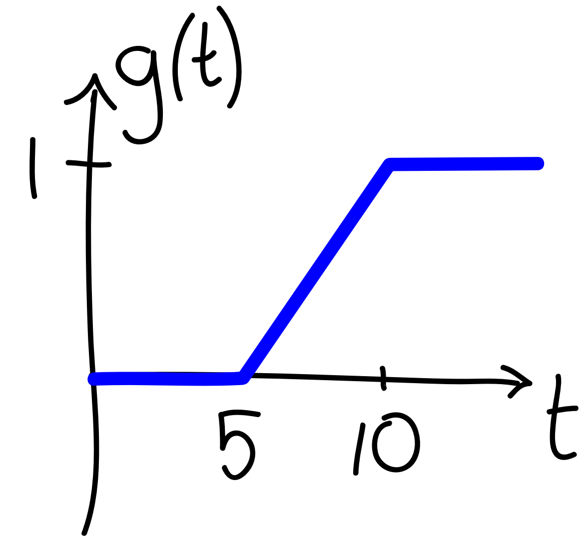

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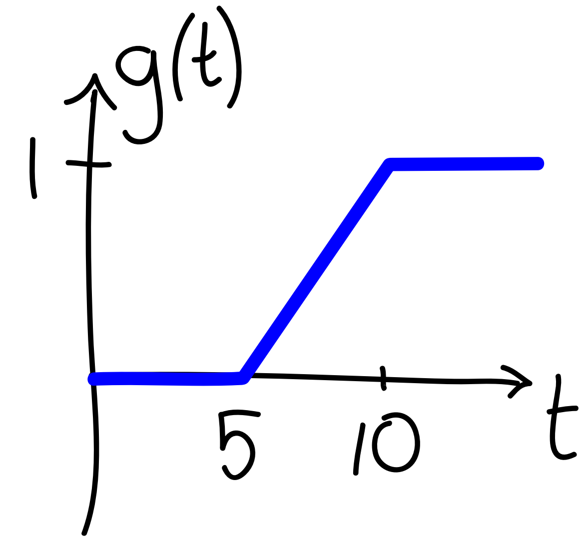

- Write g(t) is terms of $u_c(t)$:

(A) $g(t) = u_5(t) - u_{10}(t)$

(B) $g(t) = u_5(t)(t-5) - u_{10}(t)(t-5)$

(C) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/5$

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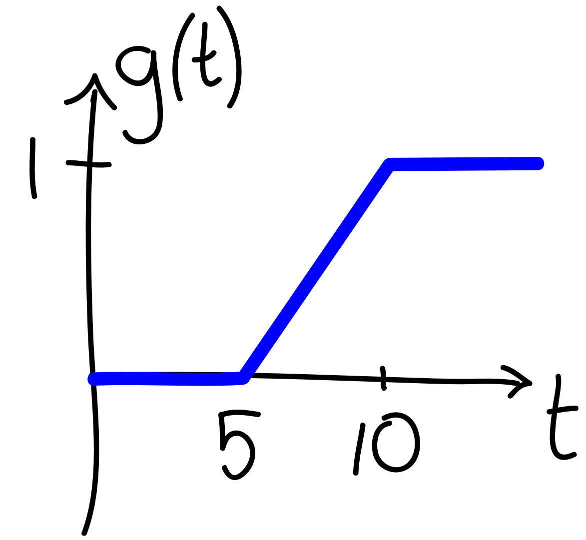

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Step function forcing (6.3, 6.4)

- An example with a ramped forcing function:

Two methods:

1. Build from left to right, adding/subtracting what you need to make the next section:

$$g(t) = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

2. Build each section independently:

$$g(t) = \left(u_5(t) - u_{10}(t)\right)\frac{1}{5}(t-5) + u_{10}(t)\cdot 1$$

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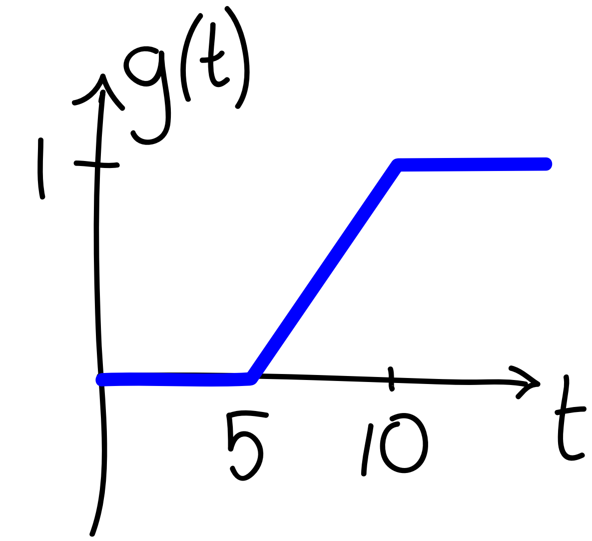

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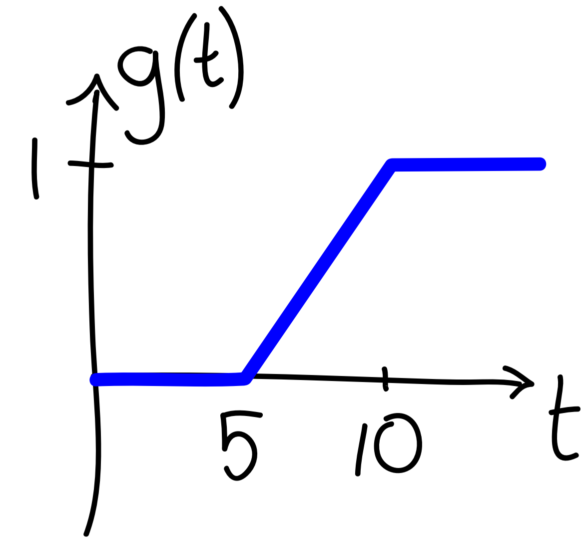

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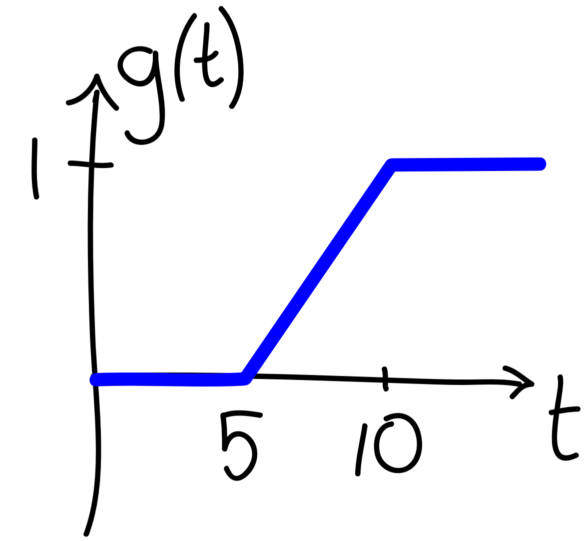

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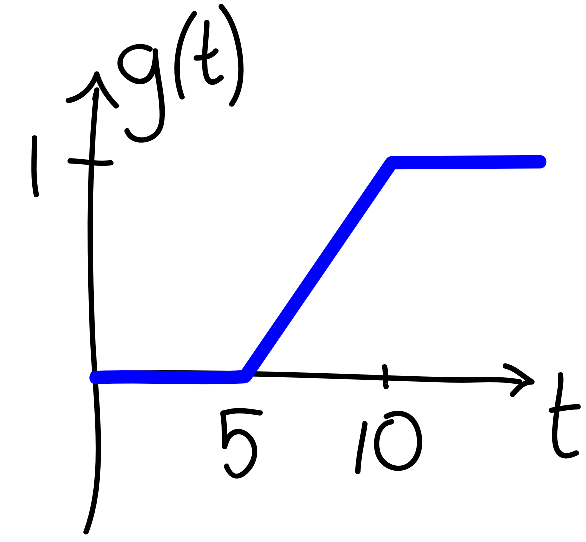

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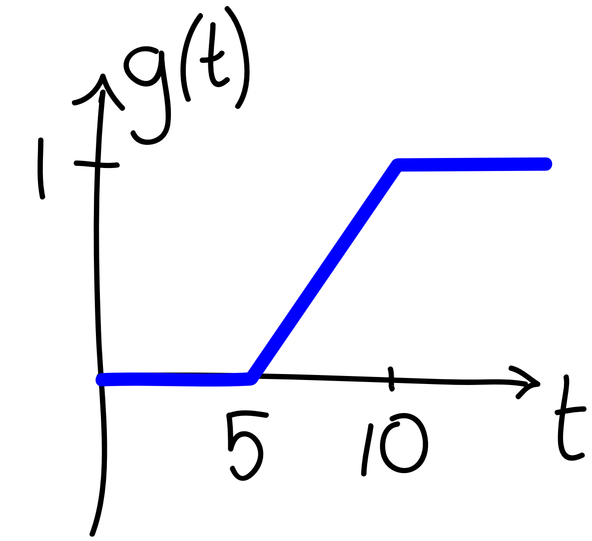

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$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

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- If the force is large and sudden (say a hammer hitting the mass), maybe we just need to get this integral correct and the details don't matter.

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$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) = \begin{cases} \text{``}\infty\text{''} & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

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Delta-function forcing (6.5)

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- I_0 can be replaced by any type of quantity
- e.g. m_0 mass added to tank suddenly

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