Today

- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Homog. eq. with constant coeff. (Section 3.1)

Last class, we found that if $y_1(t)$ is a solution to

$$ay'' + by' + cy = 0$$

then so is $y(t) = C_1 y_1(t)$.

Homog. eq. with constant coeff. (Section 3.1)

- Which of the following functions are also solutions?

(A) $y(t) = y_1(t)^2$

(B) $y(t) = y_1(t)+y_2(t)$

(C) $y(t) = y_1(t)\, y_2(t)$

(D) $y(t) = y_1(t) / y_2(t)$

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- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

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$$y'' + y' = 0$$

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(C) $y(t) = \frac{7}{4} e^{4t} + \frac{5}{4} e^{-4t}$

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- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

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- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0,\ y'(0) = v_0$ with these two solutions?

- Solve this system for C_1, C_2...

- Can't do it. Why?

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$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

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- This quantity is called the Wronskian.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

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  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

  Find values of $C_1 \neq 0$ and $C_2 \neq 0$ so that $C_1y_1(t) + C_2y_2(t) = 0$.

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$$y(t) = C_1y_1(t) + C_2y_2(t)$$

  is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

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So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.

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- Example: Consider the equation $y'' + 9y = 0$. Find the roots of the characteristic equation (i.e. the r values).

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Challenge: come up with an initial condition for (iii) that has a bounded solution.

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Complex roots (Section 3.3)

- Complex number review (Euler's formula)
- Complex roots of the characteristic equation
- From complex solutions to real solutions

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- For any equation, $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$, the solutions have the form $x = \alpha \pm \beta i$ where α and β are both real numbers.
- For α+βi, we call α the real part and β the imaginary part.

Complex number review

- Adding two complex numbers:

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- What is the inverse of c+di written in the usual complex form p+qi?

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(B) $\dfrac{c + di}{c^2 + d^2}$

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Complex number review

- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

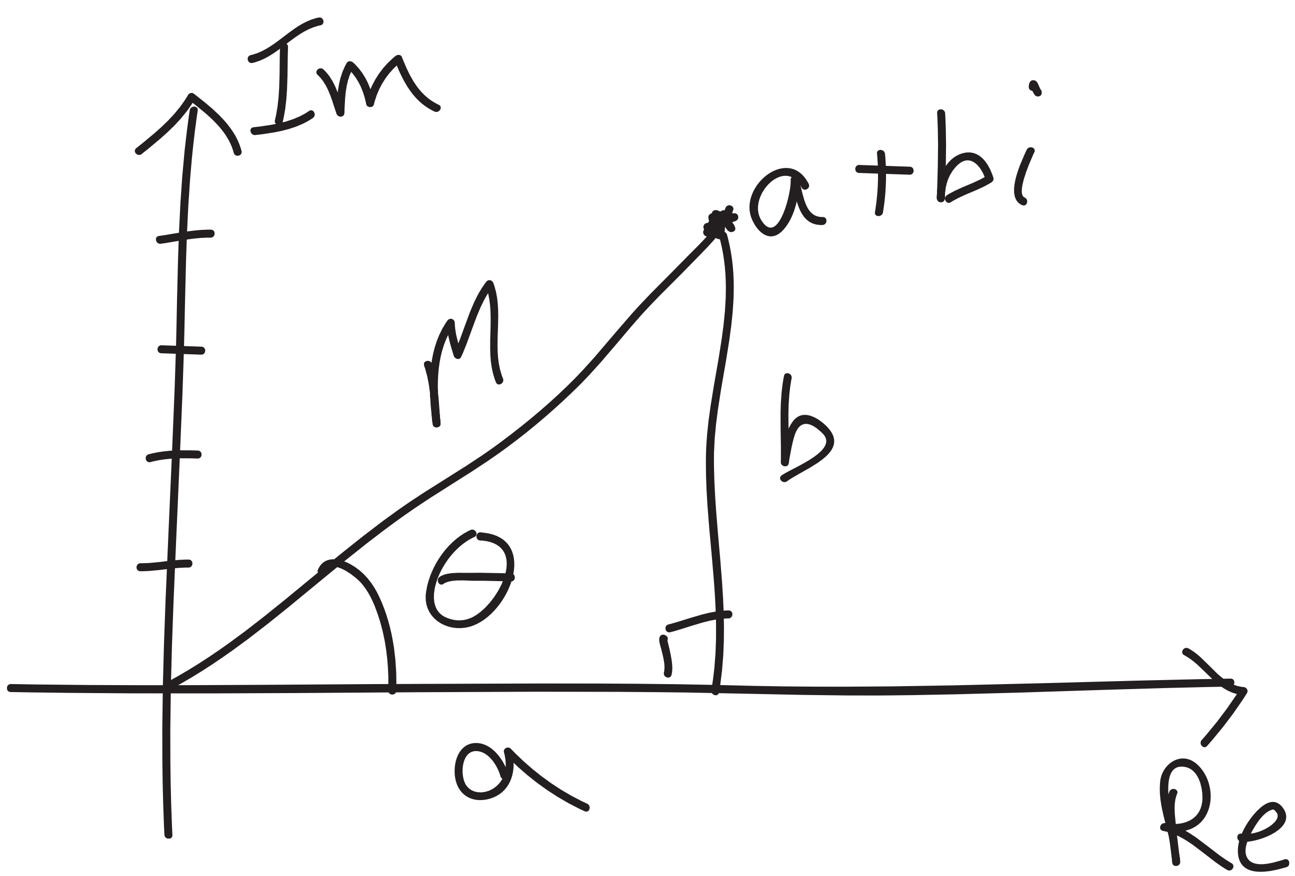
$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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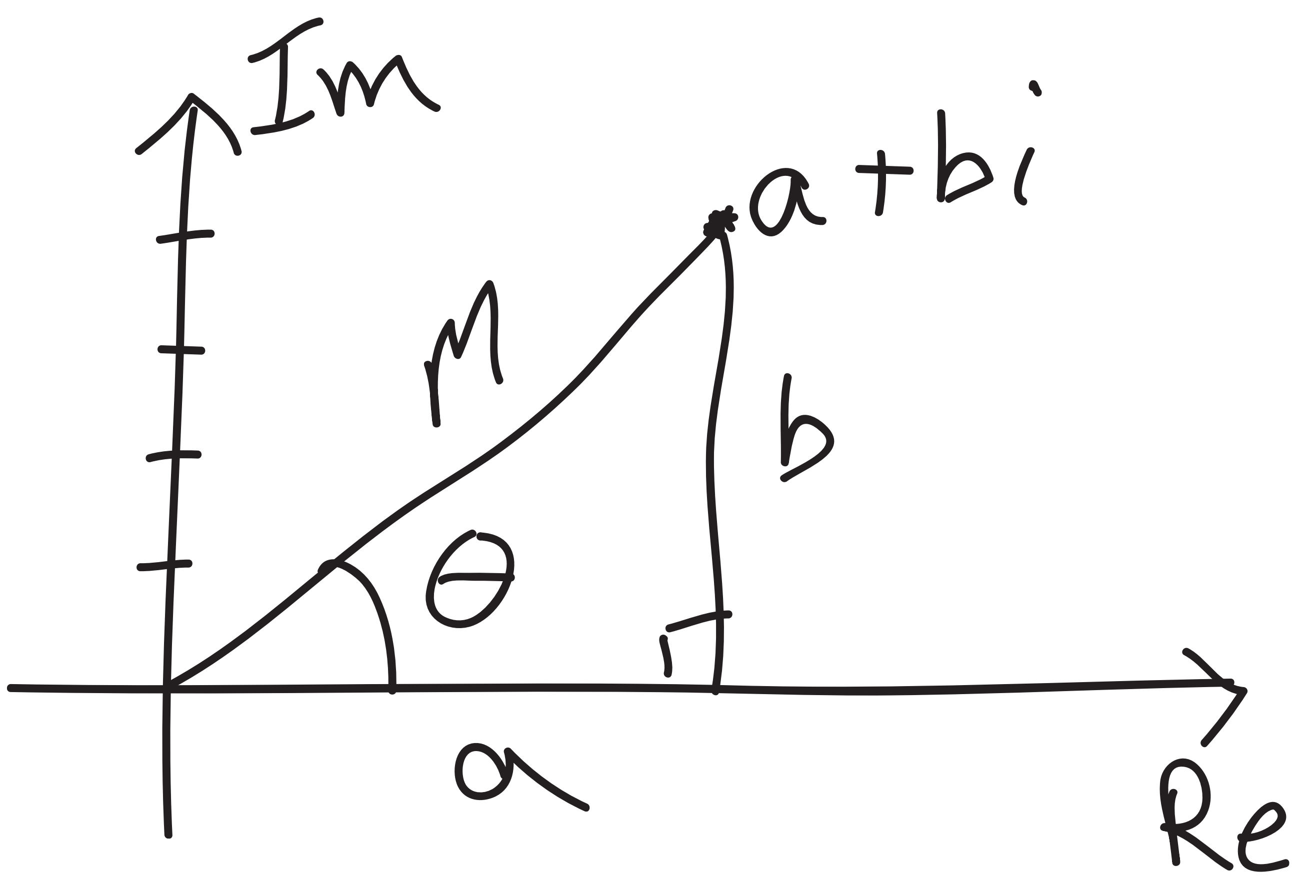

Complex number review

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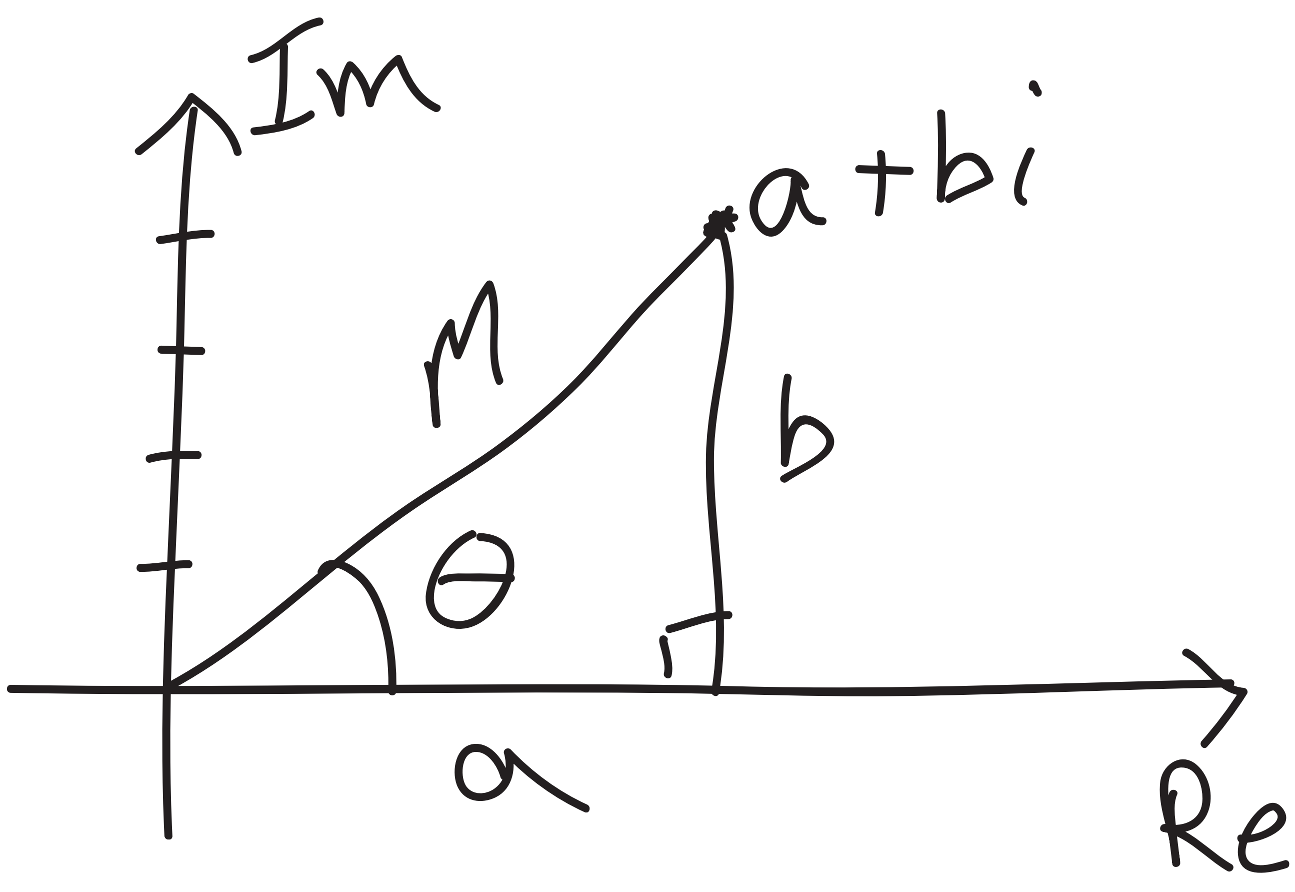

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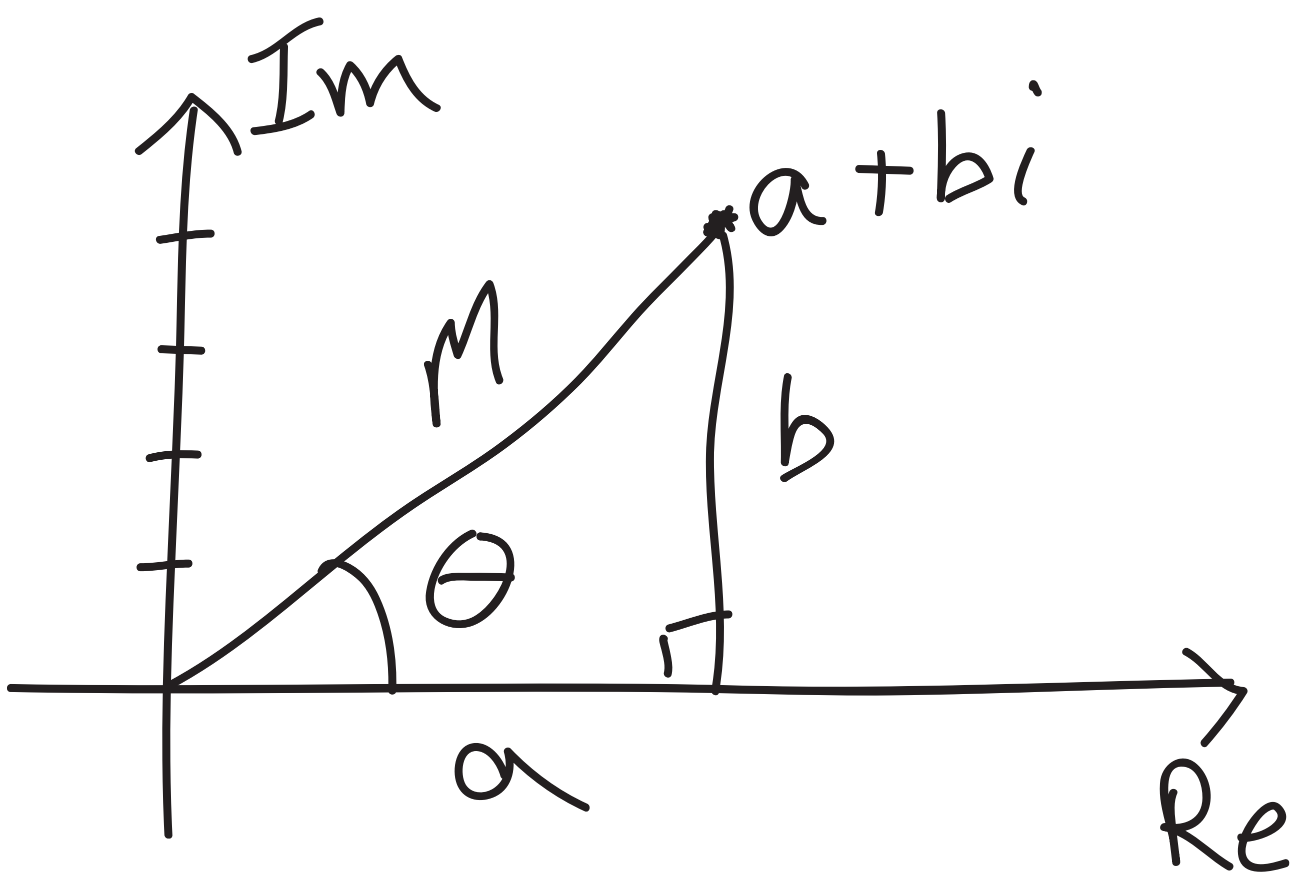

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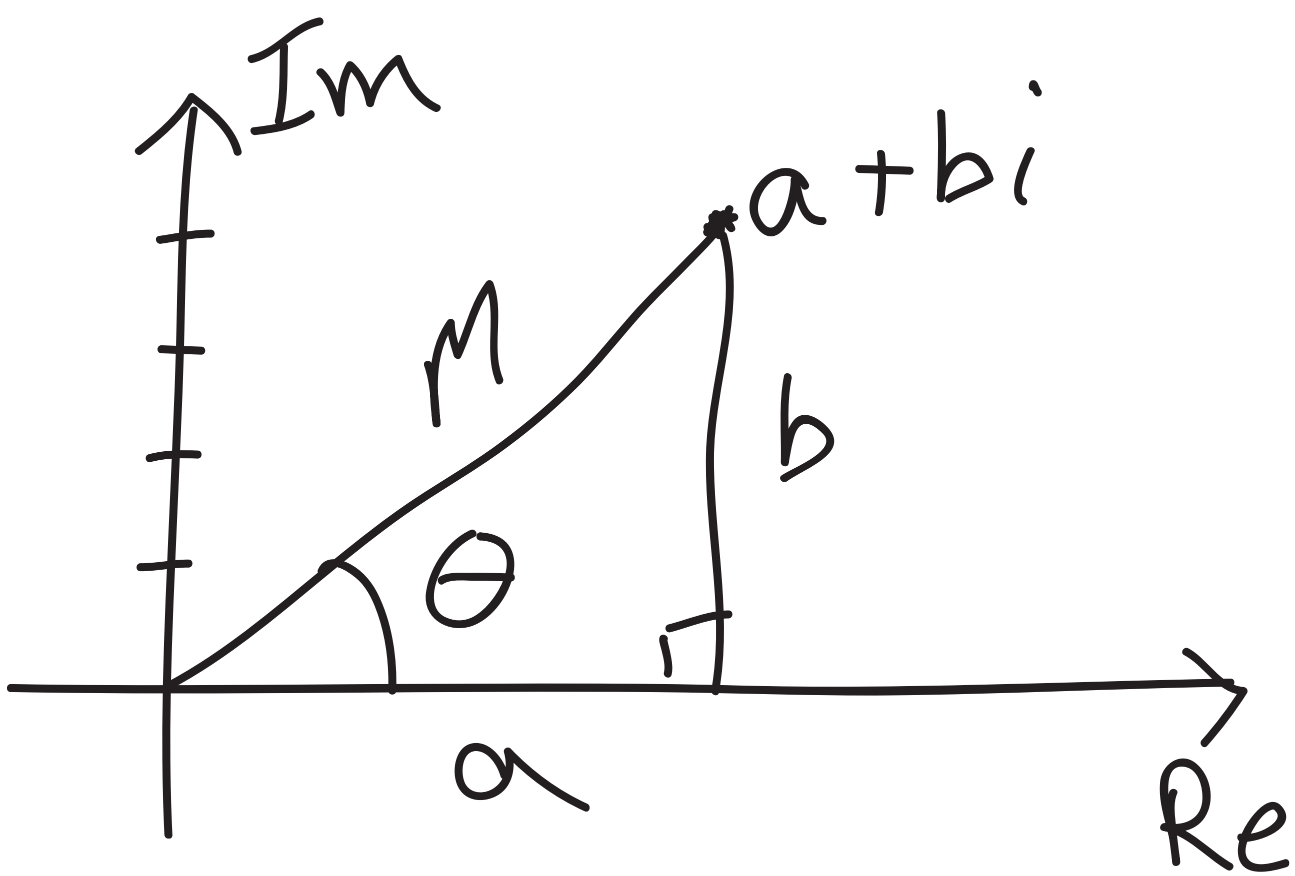

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θ is sometimes called the argument or phase of $a + bi$.

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- Toward Euler's formula

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  - Taylor series - recall that a function can be represented as

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(A) cos x (C) e^x

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$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

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$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta$$

$$= e^{i\theta}$$

Complex number review

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Euler's formula:

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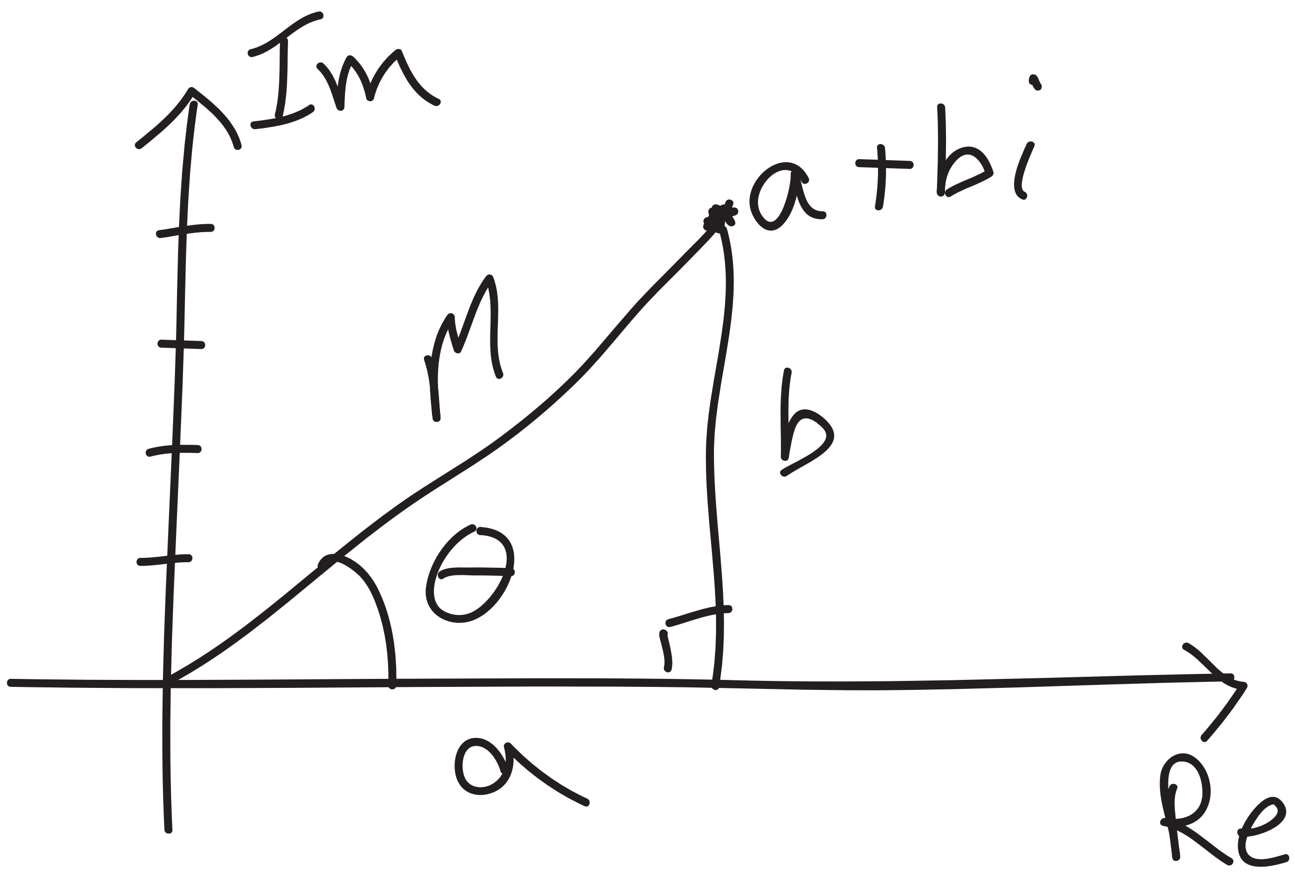

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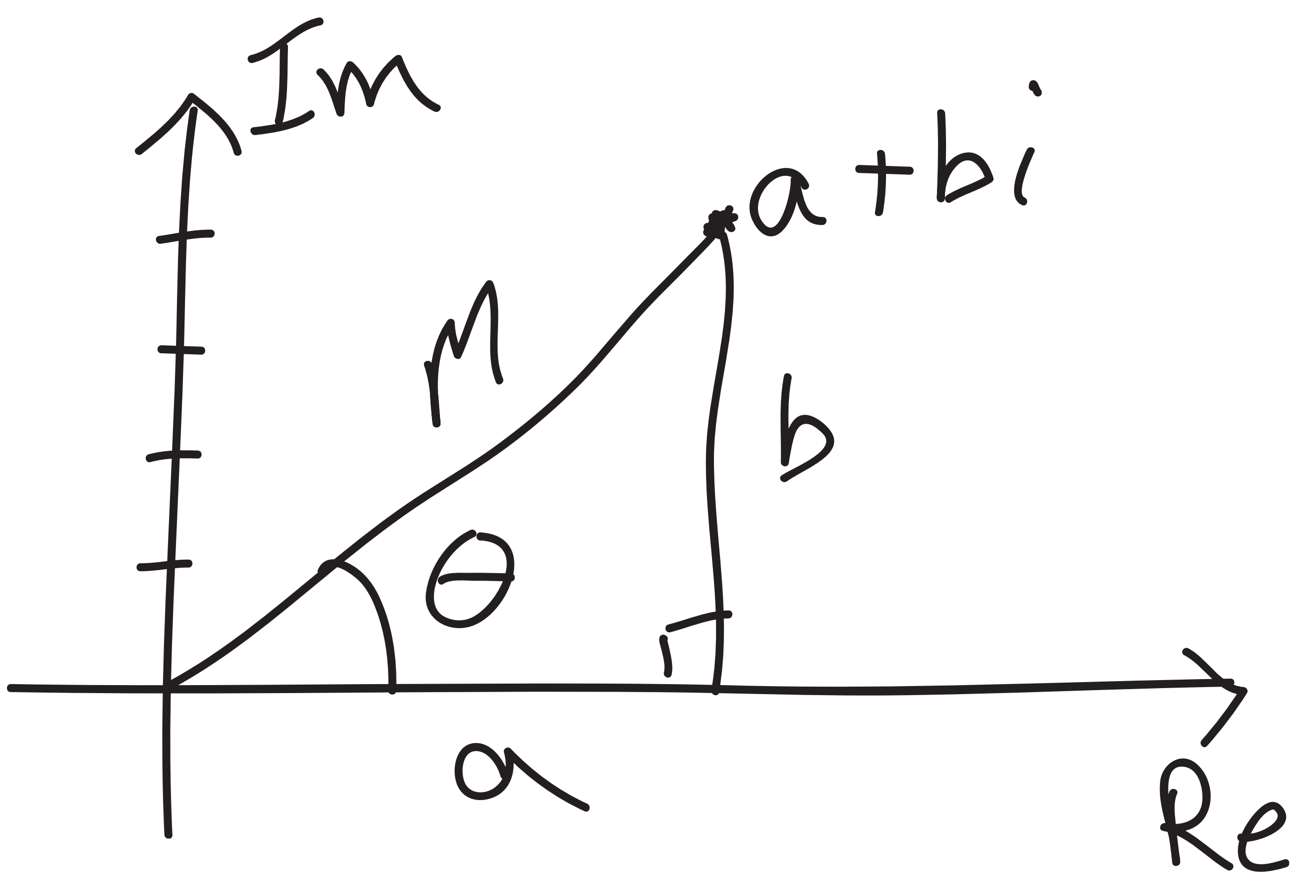

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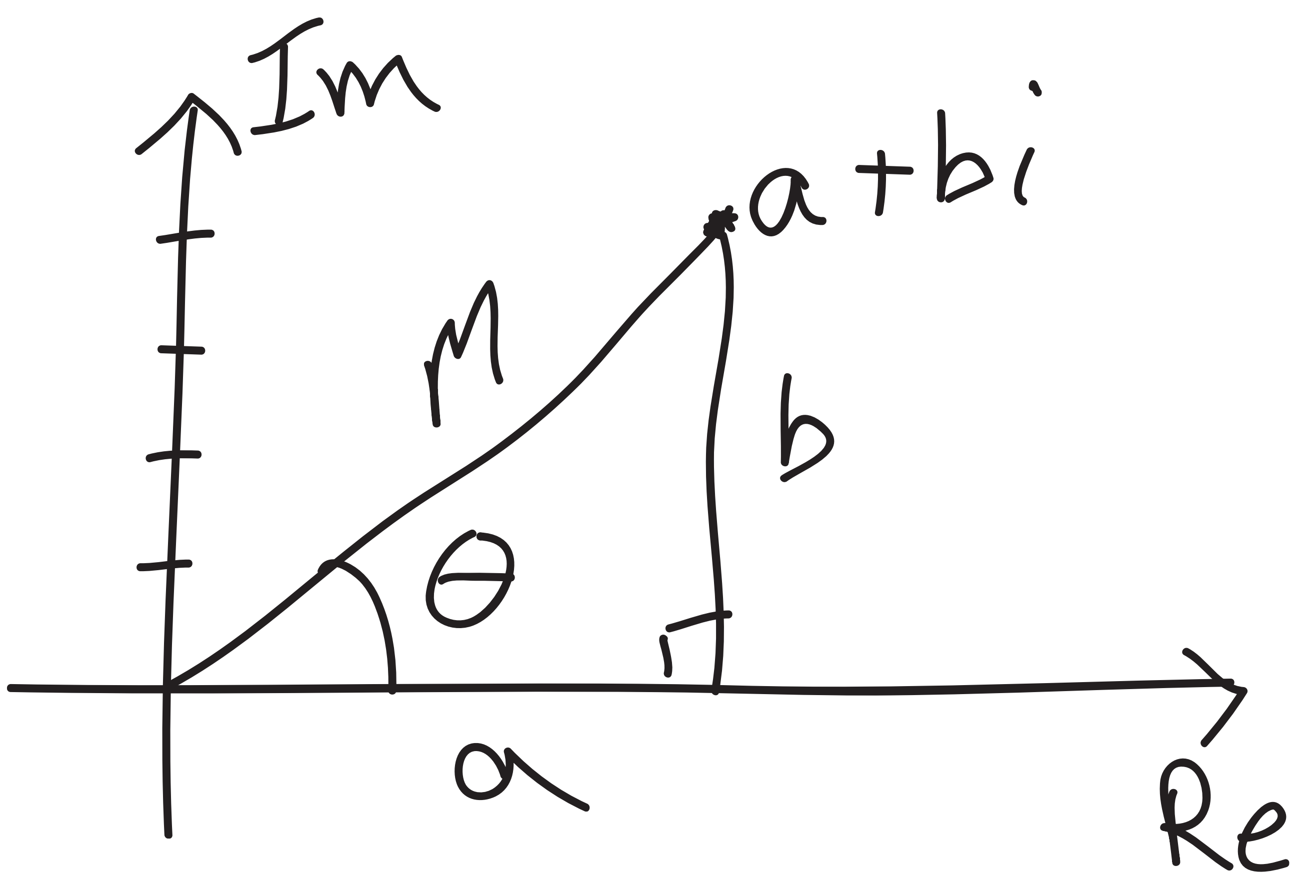

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(Polar form makes multiplication much cleaner)