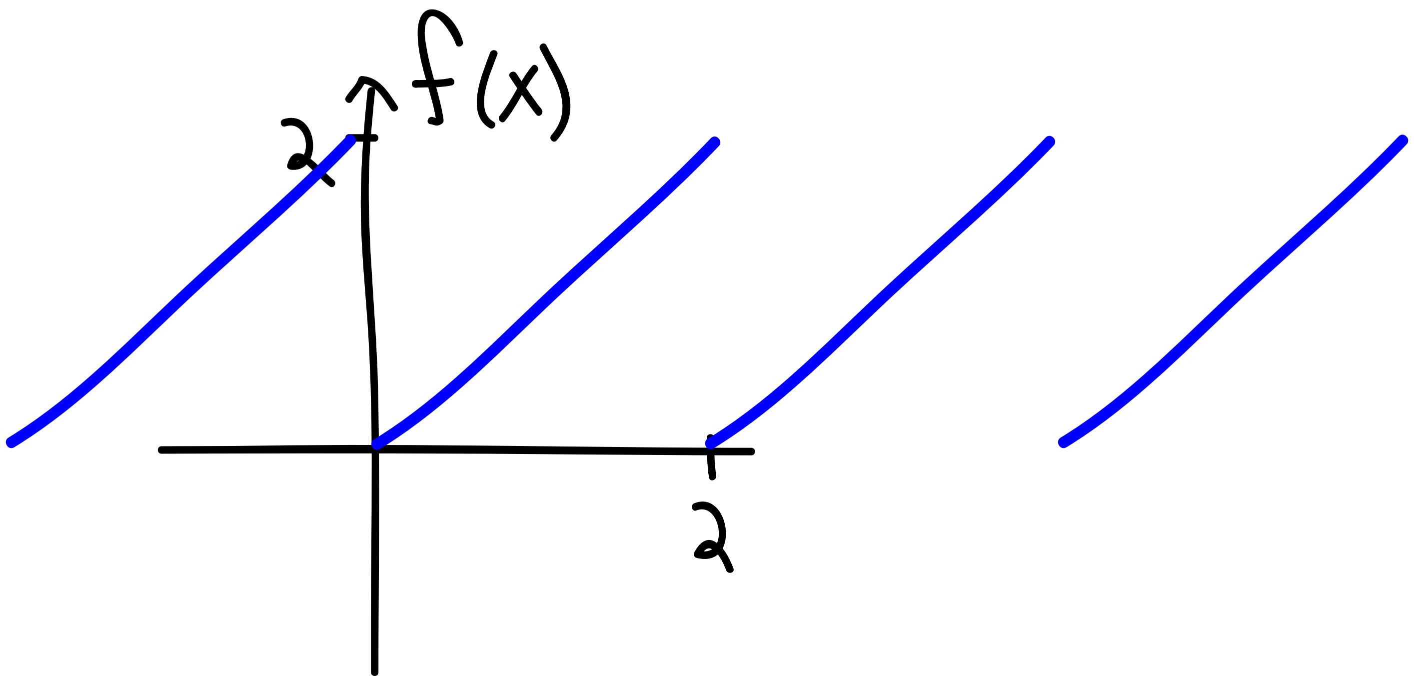
Today

- Fourier Series examples - even and odd extensions, other symmetries
- Using Fourier Series to solve the Diffusion Equation

Examples - calculate the Fourier Series



$$a_0 = \frac{1}{2}$$

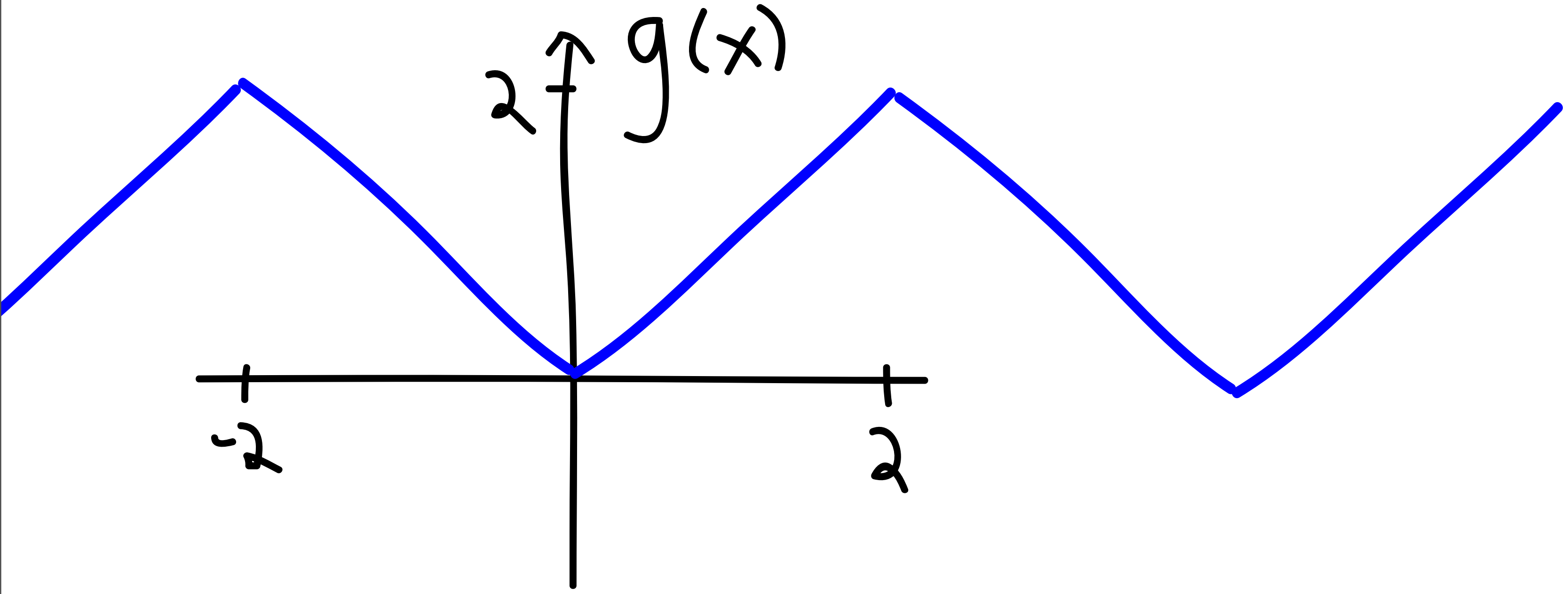
$$a_n = \frac{1}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{(-1)^{n+1}}{n\pi}$$

What is L? $L = 1$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi}\sin n\pi x \quad \text{for } x \neq 0, 2.$$

Examples - calculate the Fourier Series



$$a_n = \begin{cases} 0 & n \text{ even} \\ -\dfrac{8}{n^2\pi^2} & n \text{ odd} \end{cases}$$

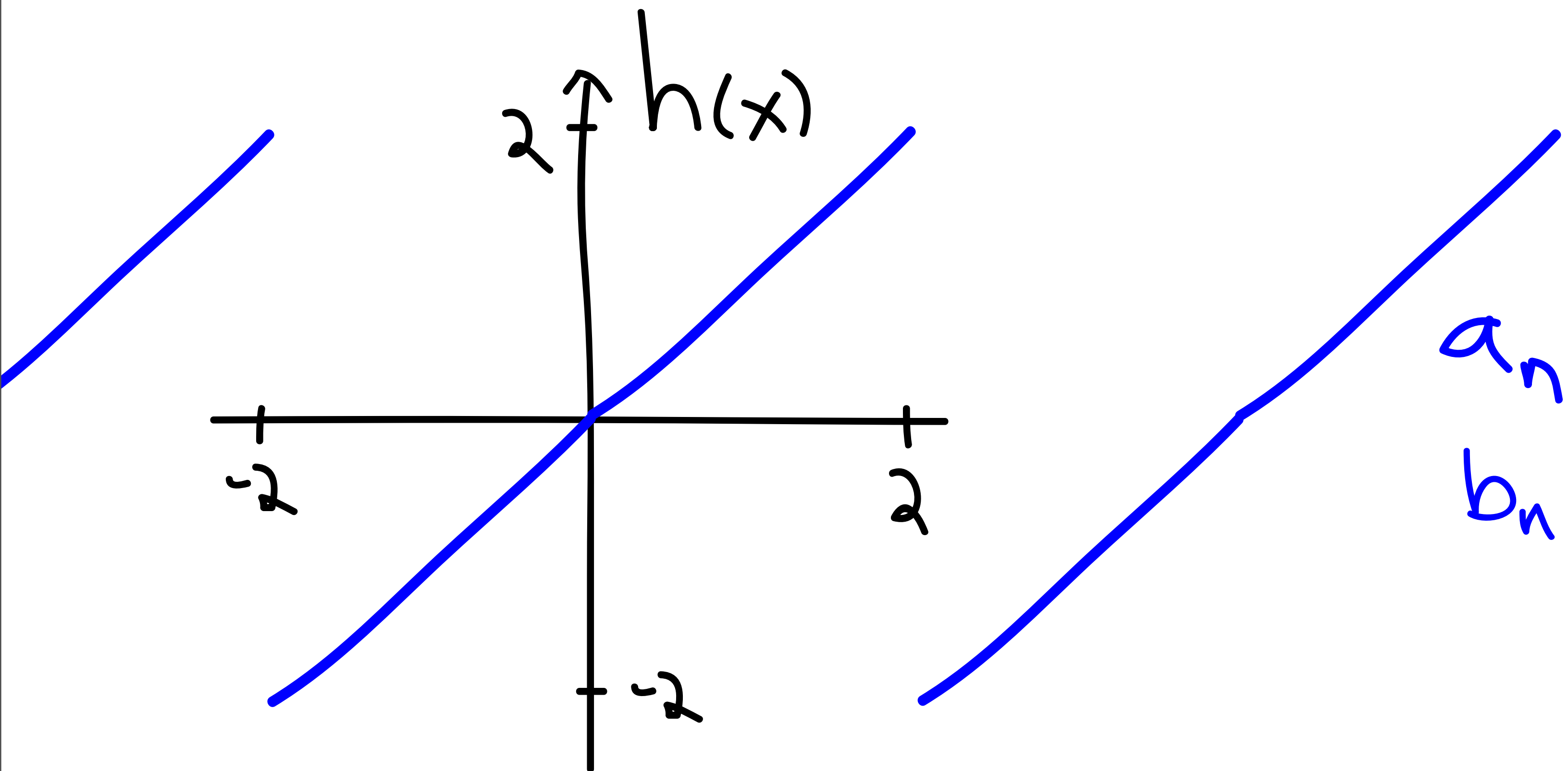
$$b_n = 0$$

What is L? $L = 2$

$$g(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \sum_{k=1}^{\infty} a_{2k-1} \cos\frac{(2k-1)\pi x}{L}$$

$$= 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\frac{(2k-1)\pi x}{2} \quad \text{for all } x.$$

Examples - calculate the Fourier Series



What is L? L=2

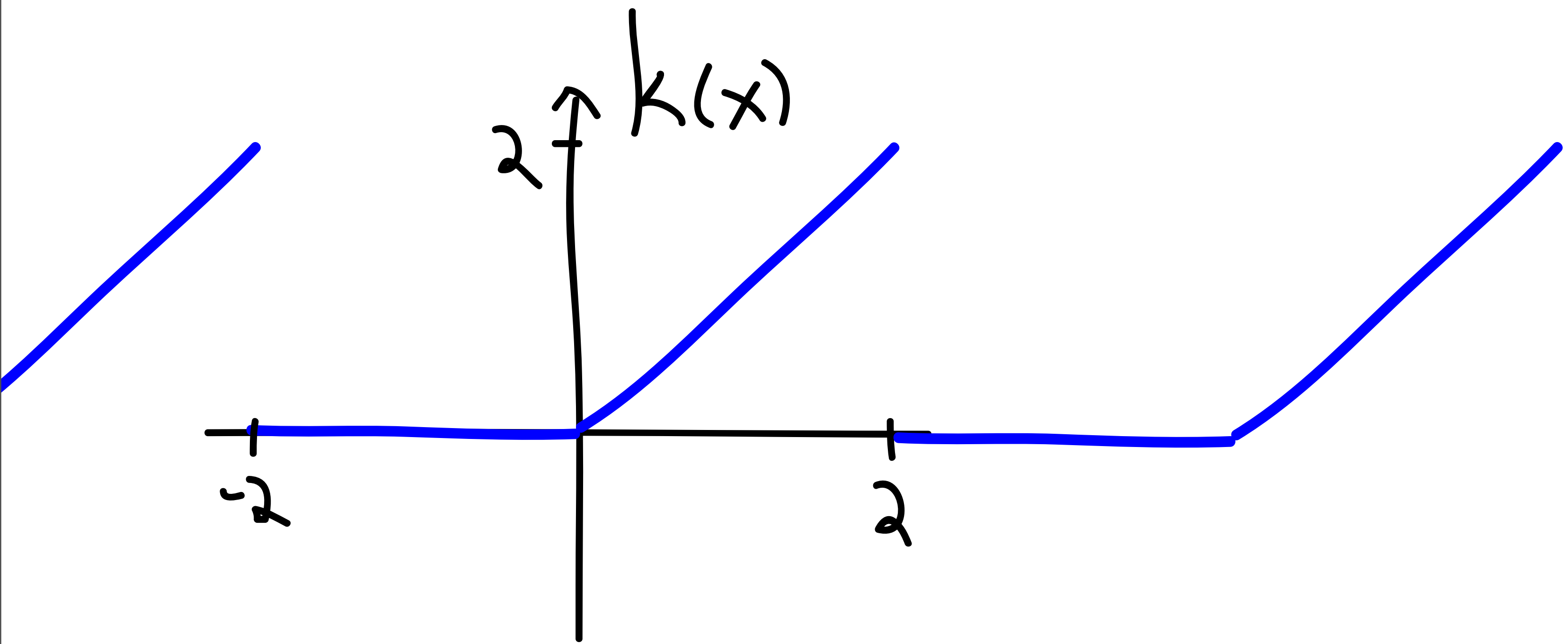
$a_n = 0$

$b_n = \frac{(-1)^{n+1} 4}{n\pi}$

$h(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$

for $x \neq -2, 2$.

Examples - calculate the Fourier Series



$a_0 = 1$

$a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$

$b_n = -\frac{2}{n\pi}(-1)^n$

What is L? L = 2

$$k(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos\frac{n\pi x}{2}$$

$$+ \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{2} \quad \text{for } x \neq -2, 2.$$

Even and odd extensions

- For a function *f(x)* defined on [0,L], the even extension of *f(x)* is the function

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ f(-x) & \text{for } -L \leq x < 0. \end{cases}$$

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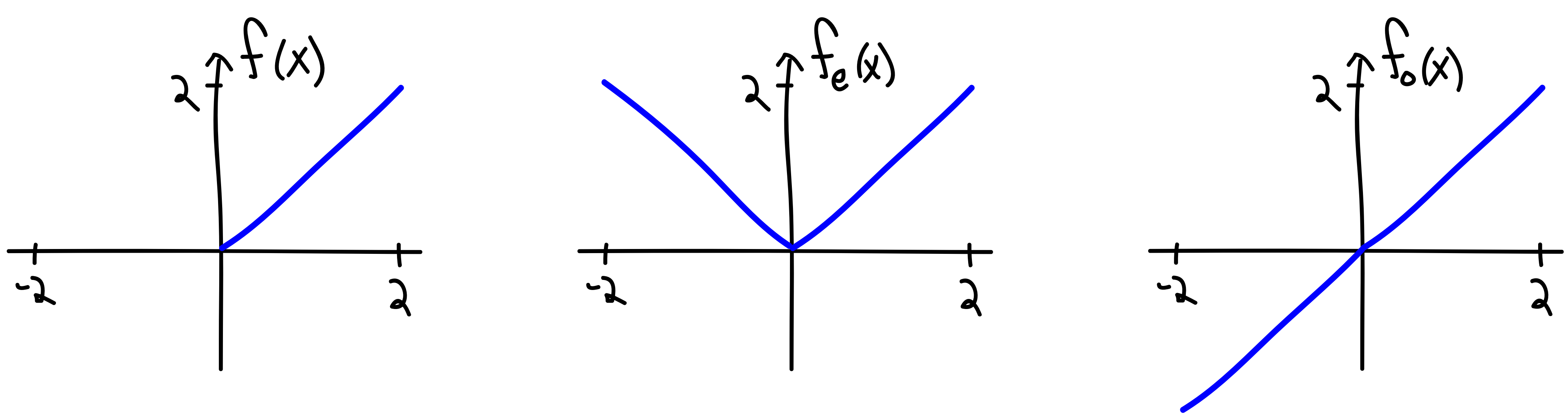
$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ -f(-x) & \text{for } -L \leq x < 0. \end{cases}$$



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- Because these functions are even/odd, their Fourier Series have a couple simplifying features:

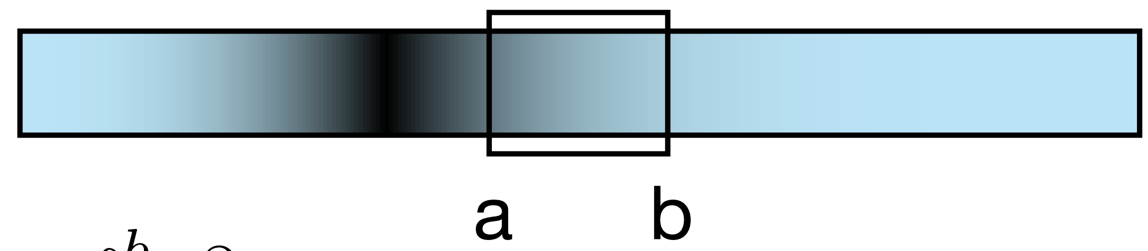
$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

The Diffusion Equation

c(x,t) is linear mass density of ink in a long narrow tube.

$$Q_{ab}(t) = \int_a^b c(x,t)\ dx$$



$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt}\int_a^b c(x,t)\ dx = \int^b \frac{\partial}{\partial t} c(x,t)\ dx$$

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2 c}{dx^2}$$

Define the flux J_a to … ossing the line x=a (+ -->) .

$$\frac{dQ_{ab}}{dt}(t) = -J_b + \ldots$$

Need a model for flu… : $J_a = -D\left.\frac{\partial c}{\partial x}\right|_{x=a}$

$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a = D\left.\frac{\partial c}{\partial x}\right|_{x=b} - D\left.\frac{\partial c}{\partial x}\right|_{x=a} = D\left.\frac{\partial c}{\partial x}\right|_a^b$$

$$\int_a^b \frac{\partial}{\partial t}c(x,t)\ dx = \int_a^b D\frac{\partial^2 c}{\partial x^2}\ dx \quad \Rightarrow \quad \frac{\partial}{\partial t}c(x,t) = D\frac{\partial^2}{\partial x^2}c(x,t)$$

The Diffusion Equation

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$c(x,t) = ae^{bt}\sin(wx)$$

$$\frac{\partial c}{\partial t} = abe^{bt}\sin(wx) \qquad D\frac{\partial^2 c}{\partial x^2} = -Daw^2e^{bt}\sin(wx)$$

$$c(x,t) = ae^{-w^2Dt}\sin(wx)$$

Still need to determine a and w. Need to impose other conditions:

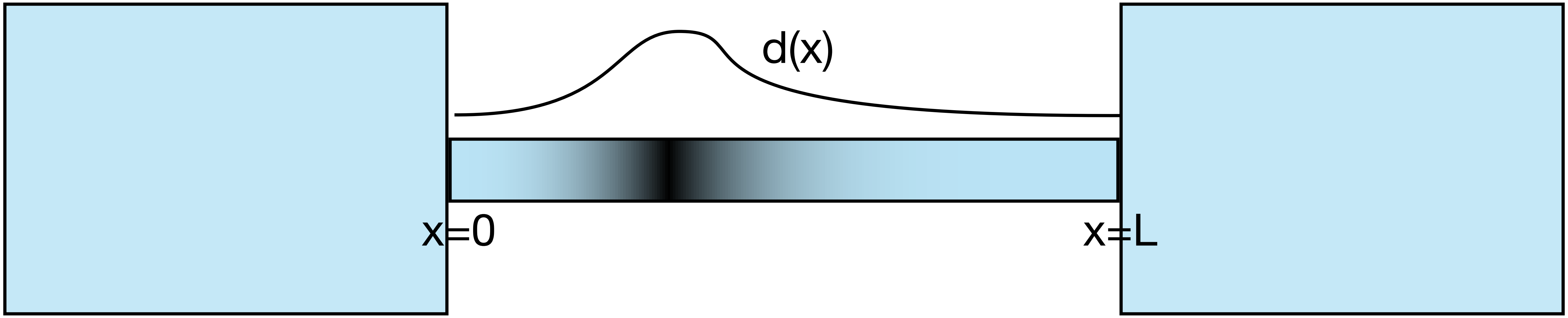
- A time derivative requires an initial condition c(x,0).
- Two space derivatives require two boundary conditions c(0,t) and c(L,t).

The Diffusion Equation

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

An initial condition specifies where all the mass is initially: c(x,0) = d(x).



A common boundary condition states that the concentration is forced to be zero at the end point(s) (infinite reservoir): c(0,t) = 0 = c(L,t).

The Diffusion Equation

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$c(x,t) = ae^{-w^2Dt}\sin(wx)$$

$$c(0,t) = 0,\ \ c(L,t) = 0$$

$$c(0,t) = a\sin(0) = 0$$

<-- would not have happened with cosine!

$$c(L,t) = a\sin(wL) = 0$$

$$wL = n\pi$$

$$w = \frac{n\pi}{L}$$

$$c_n(x,t) = ae^{-\frac{n^2\pi^2}{L^2}Dt}\sin\left(\frac{n\pi}{L}x\right)$$