MATH 256 – Midterm 2 – March 20, 2014.

Last name: ______________ First name: ______________ Student #: ______________

I attend the tutorial in room: MATH 105 MATH 203 (circle one)

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. **[3 pts]** Calculate the Laplace transform of the function $g(t) = 3\delta(t-2) + 3u_3(t) + te^{4t}$.

$$\mathcal{L}\{g(t)\} = 3e^{-2s} + \frac{3e^{-3s}}{s} + \frac{1}{(s-4)^2}$$

1 1 1

2. **[3 pts]** Find the inverse Laplace transform of $Y(s) = e^{-2s}\dfrac{s}{s^2+5s+6}$.

$$H(s) = \frac{s}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$s = A(s+2) + B(s+3)$$

$$B = -2 \quad 1$$

$$A = 3$$

$$h(t) = 3e^{-3t} - 2e^{-2t} \quad 1$$

$$y(t) = u_2(t)\,h(t-2) \quad 1$$

3. **[3 pts]** Find the inverse Laplace transform of $Y(s) = \dfrac{12}{s^2+4s+40}$.

$$Y(s) = \frac{12}{s^2+4s+40} = \frac{12}{(s+2)^2+36} = 2\cdot\frac{6}{(s+2)^2+6^2}$$

1 1

$$y(t) = 2e^{-2t}\sin 6t \quad 1$$

4. **[3 pts]** Find the general solution to the equation

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

Hint: one of the eigenvalues is $2 + 3i$.

$\lambda = 2+3i$

$$(A-\lambda I)\bar{v} = \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \bar{v} = 0$$

$$\bar{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \quad 1$$

$$x(t) = e^{2t}\left[c_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 3t \right) + c_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 3t \right) \right]$$

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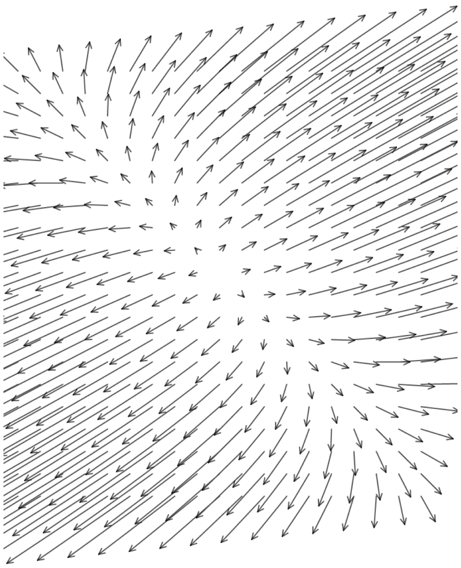
5. **[4 pts]** Morphine is cleared from the body at a rate proportional to the amount present with a rate constant $k = 1/3$ hour^{-1}. A patient is given an injection of 20 mg morphine immediately after surgery ($t = 0$) and again at 6 and 12 hours after surgery. Treating each injection as an instantaneous event, write down a differential equation to model the quantity of morphine in the patient's body as a function of time.

$$m' = -km + 20\left(\delta(t) + \delta(t-6) + \delta(t-12)\right)$$

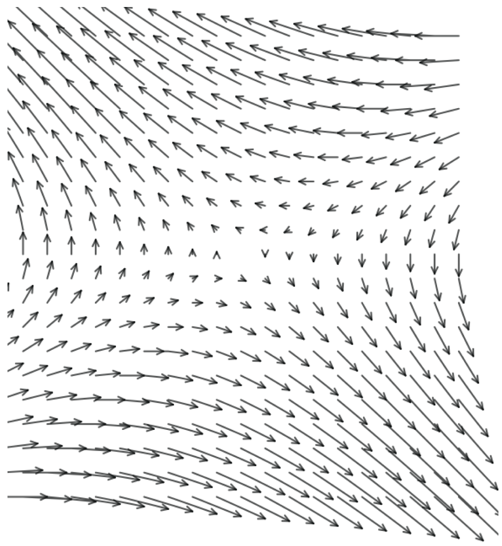
2 1 1

6. **[4 pts]** Match each solution to one of the vector fields.

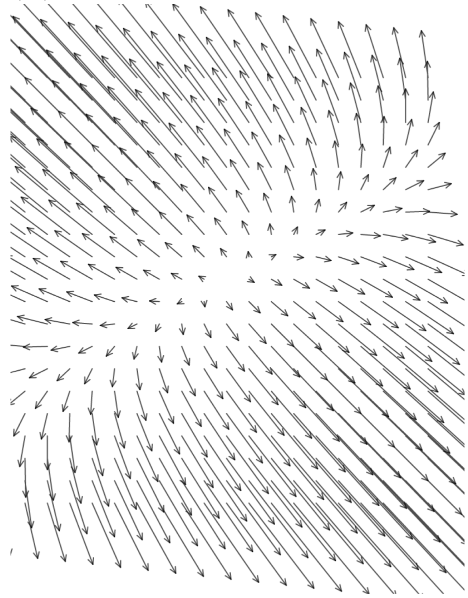
(A)



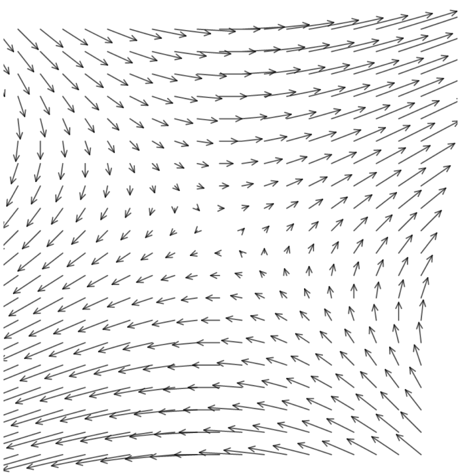
(B)



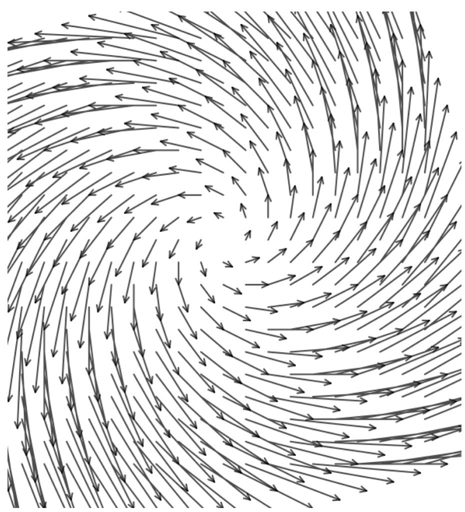
(C)



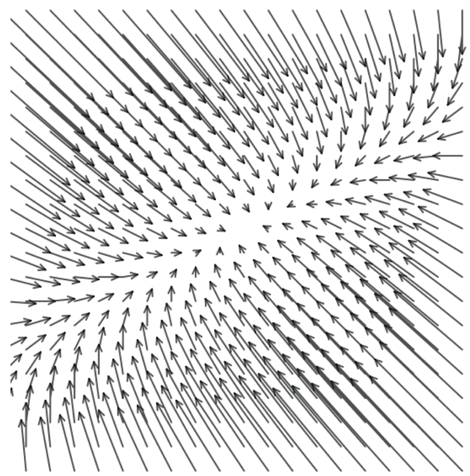
(D)



(E)

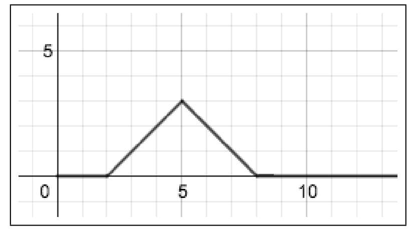


(F)



Solution	Vector field (enter A,B,C,D,E or F)
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$	F
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$	B
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$	C
$\mathbf{x}(\mathbf{t}) = e^{2t} \left[C_1 \begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix} \right]$	E

7. [**4 pts**] Write down an expression for the function $g(t)$ shown in the figure below using Heavside functions in the form $u_c(t)$.



$$g(t) = u_2(t)(t-2) - 2u_5(t)(t-5) + u_8(t)(t-8)$$

2 1 1

8. [**4 pts**] Consider the equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} \alpha & \beta \\ 1 & \alpha \end{pmatrix}.$$

In each row of the table below, give inequalities involving α and β which ensure that the steady state is of the given type. The first row provides an example.

Type	Condtion(s) on α and β
unstable node	$0 < \beta < \alpha^2$, $\alpha > 0$
stable node	$\alpha < 0$, $\beta > 0$, $\alpha^2 > \beta$
unstable spiral	$\alpha > 0$, $\beta < 0$
stable spiral	$\alpha < 0$, $\beta < 0$
saddle	$\alpha^2 < \beta$

$$\lambda = \frac{trA \pm \sqrt{(trA)^2 - 4\det A}}{2}$$

$tr\,A = 2\alpha$

$\det A = \alpha^2 - \beta$

S.N. $\to$ $\det A > 0 \Rightarrow \boxed{\alpha^2 > \beta}$

1 $(trA)^2 - 4\det A > 0 \Rightarrow 4\alpha^2 - 4\alpha^2 + 4\beta > 0$ $\boxed{\beta > 0}$

u.s. $\to$ $trA > 0 \Rightarrow \boxed{\alpha > 0}$

1 $(trA)^2 - 4\det A < 0 \Rightarrow \boxed{\beta < 0}$

s.s $\to$ $trA < 0 \Rightarrow \boxed{\alpha < 0}$

1 $(trA)^2 - 4\det A < 0 \Rightarrow \boxed{\beta < 0}$

saddle $\to$ $\det A < 0 \Rightarrow \boxed{\alpha^2 < \beta}$

1

Anything on this page will not be marked. It is for rough work.

Laplace transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$u_c(t)f(t-c)$	$e^{-sc}F(s)$
$\delta(t-c)$	e^{-sc}
$\int_0^t f(t-w)g(w)\ dw$	$F(s)G(s)$

Equations

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)]$$