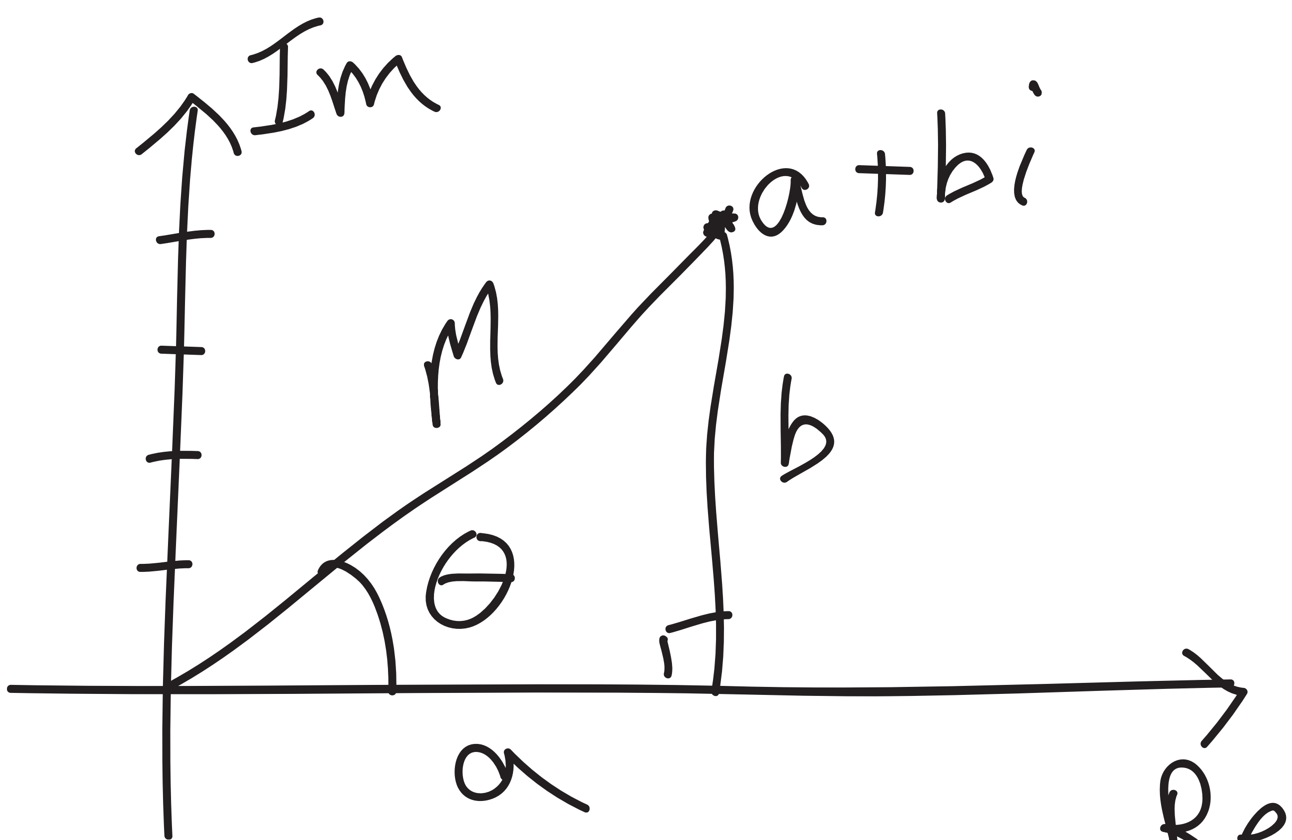
Today

- Midterm 1 postponed to Feb 10 (not updated on calendar yet).
- Solving a second order linear homogeneous equation with constant coefficients
  - complex roots to the characteristic equation,
  - repeated roots to the characteristic equation (Reduction of Order).
- Connections to matrix algebra.
- Solving a second order linear **non**homogeneous equation.

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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$$a = M\cos\theta$$

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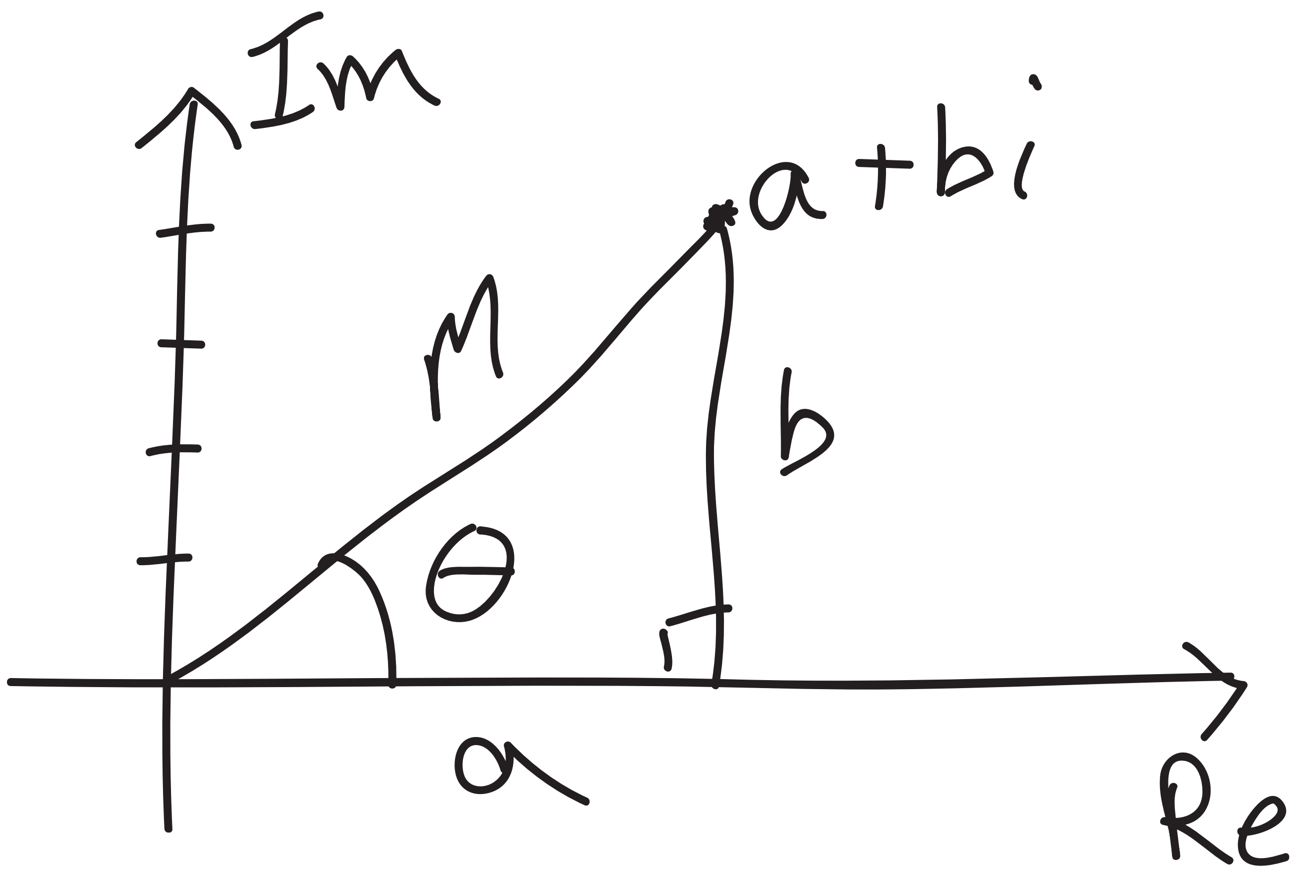

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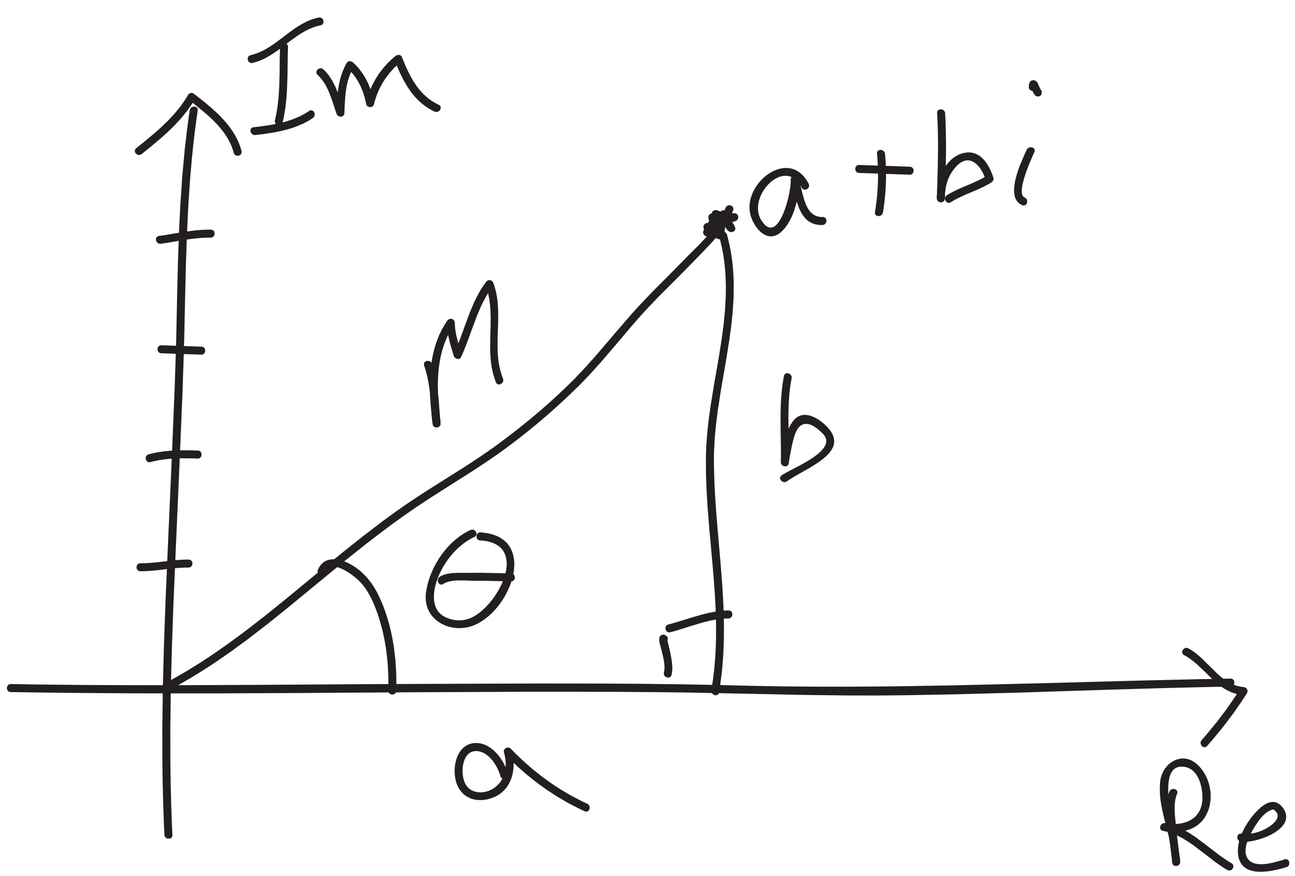

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$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

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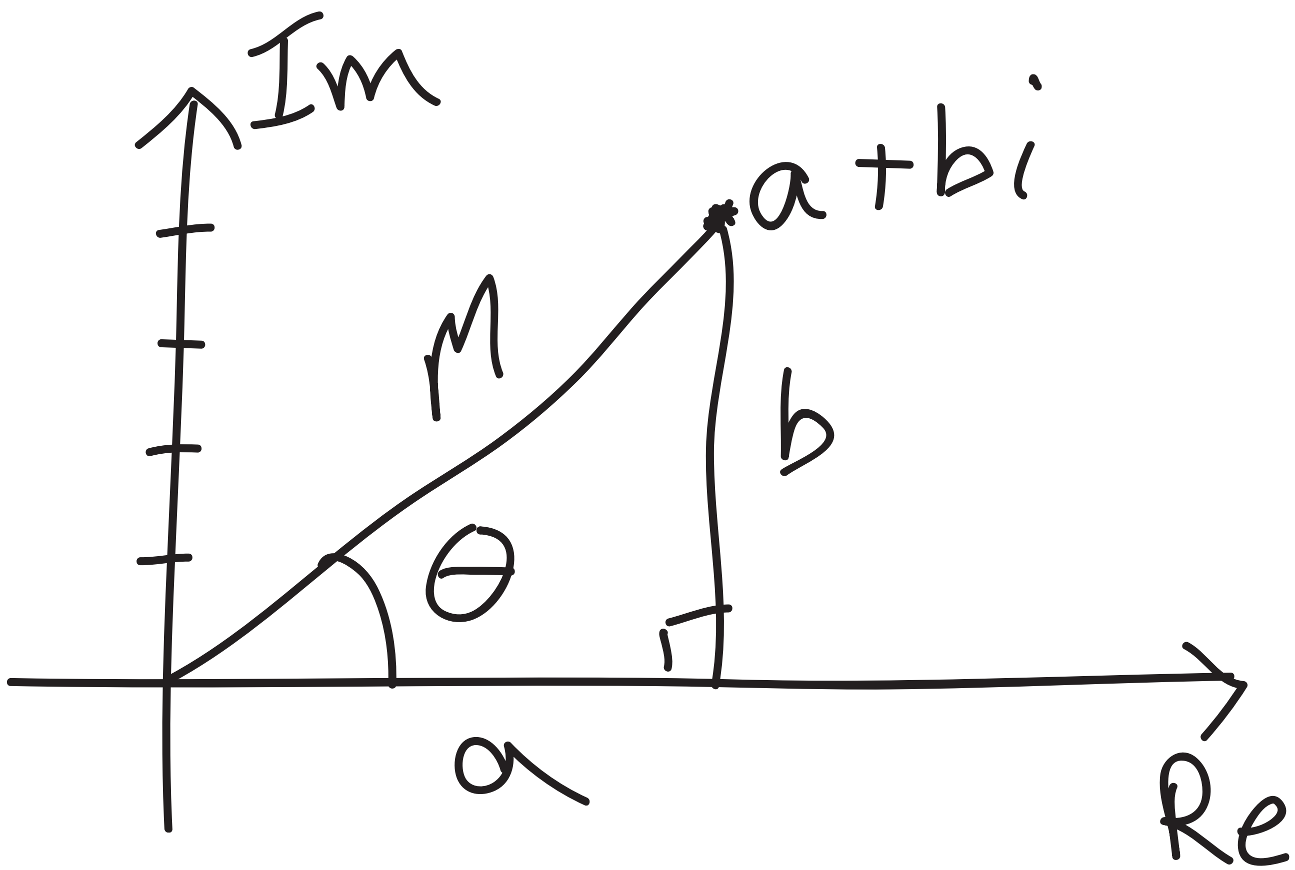

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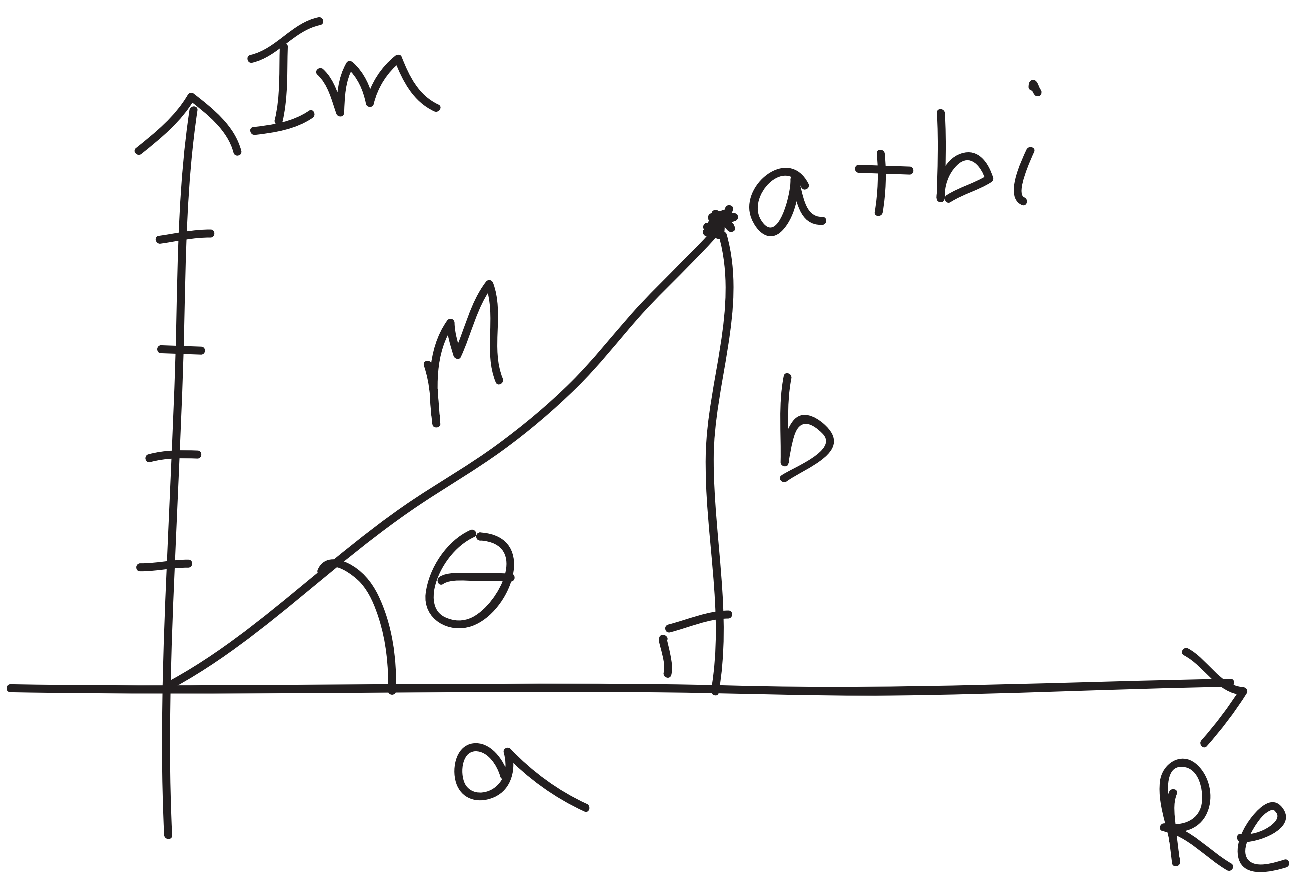

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θ is sometimes called the argument or phase of $a + bi$.

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(A) cos x

(C) e^x

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(D) ln x

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

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$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

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$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

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$$\cos\theta + i\sin\theta$$

$$= e^{i\theta}$$

Complex number review

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Euler's formula:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

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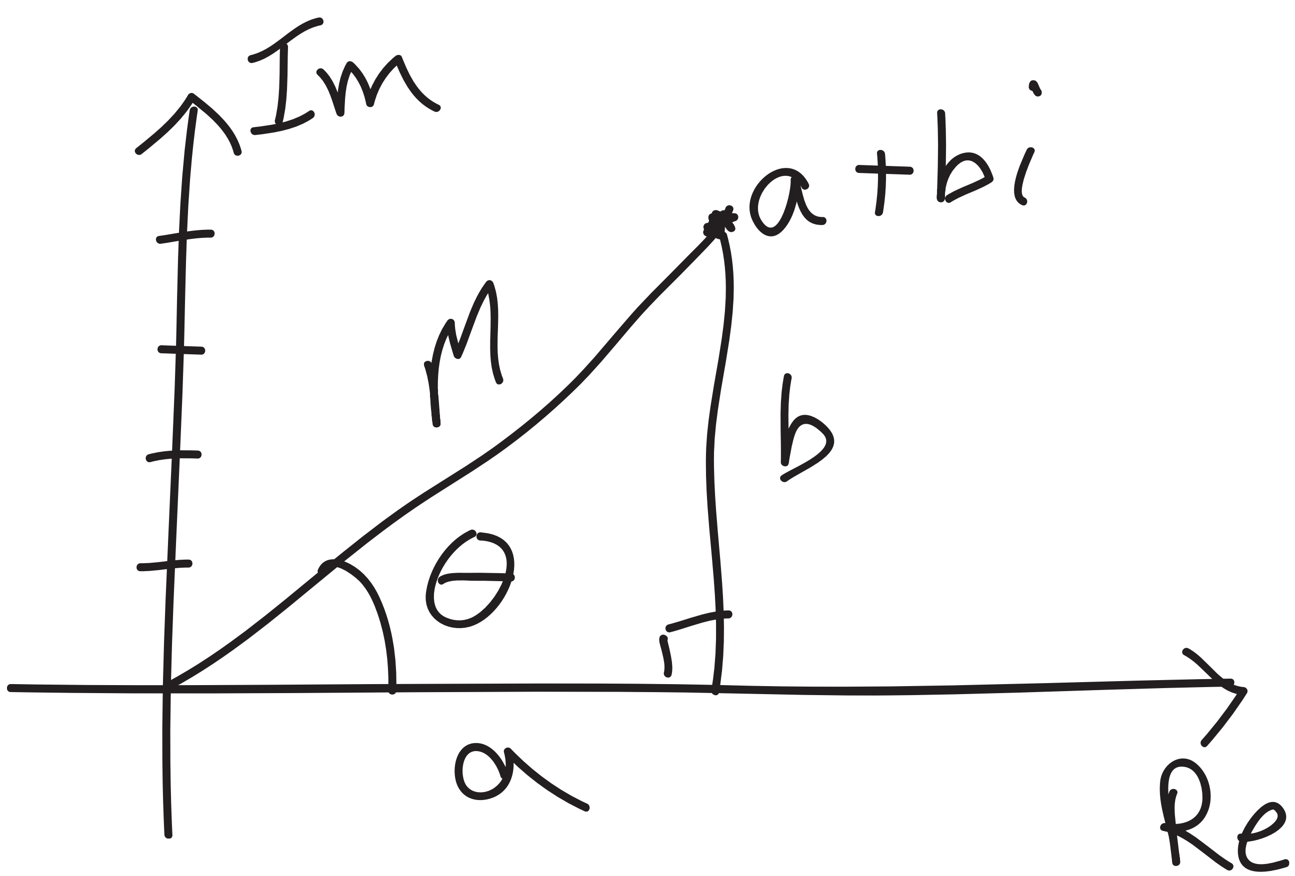

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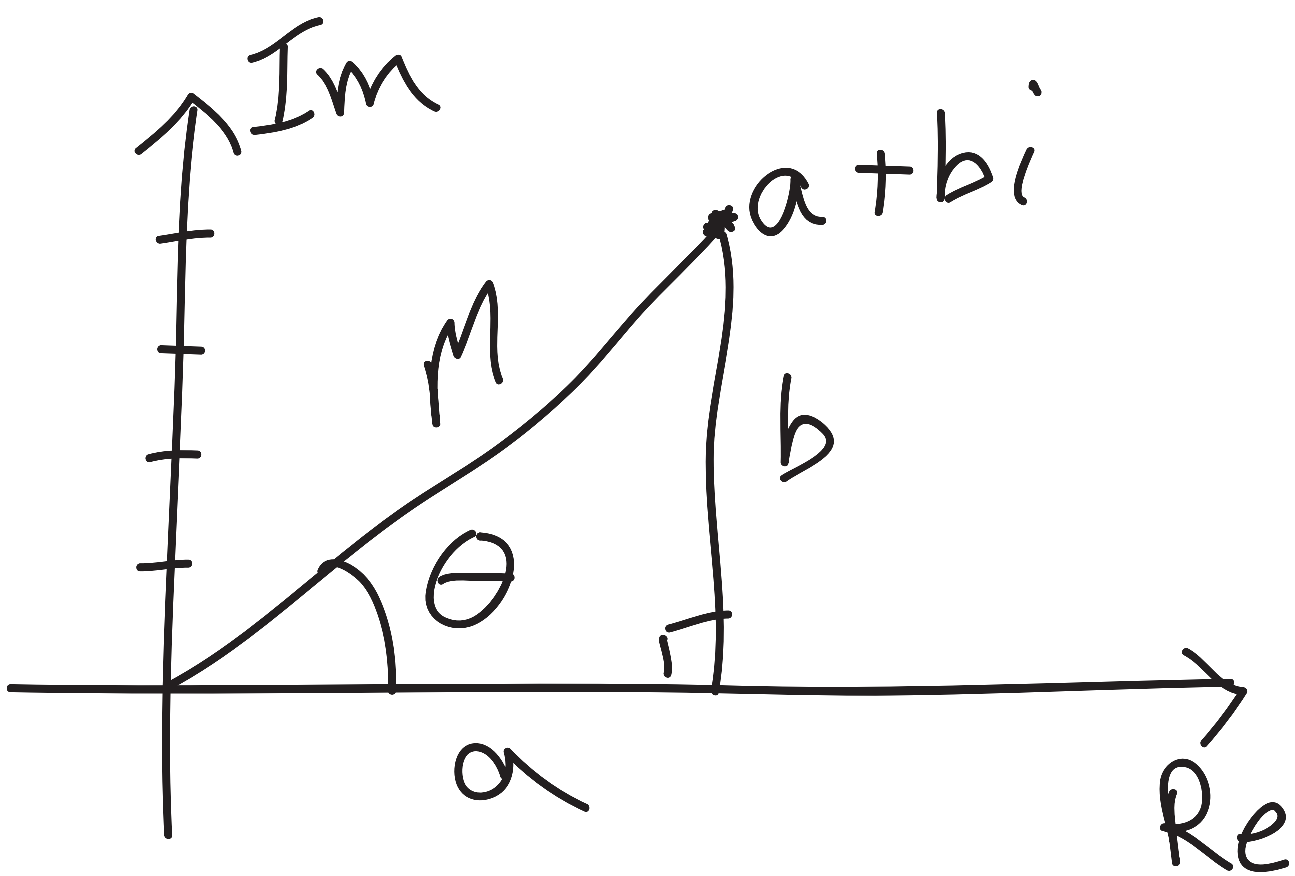

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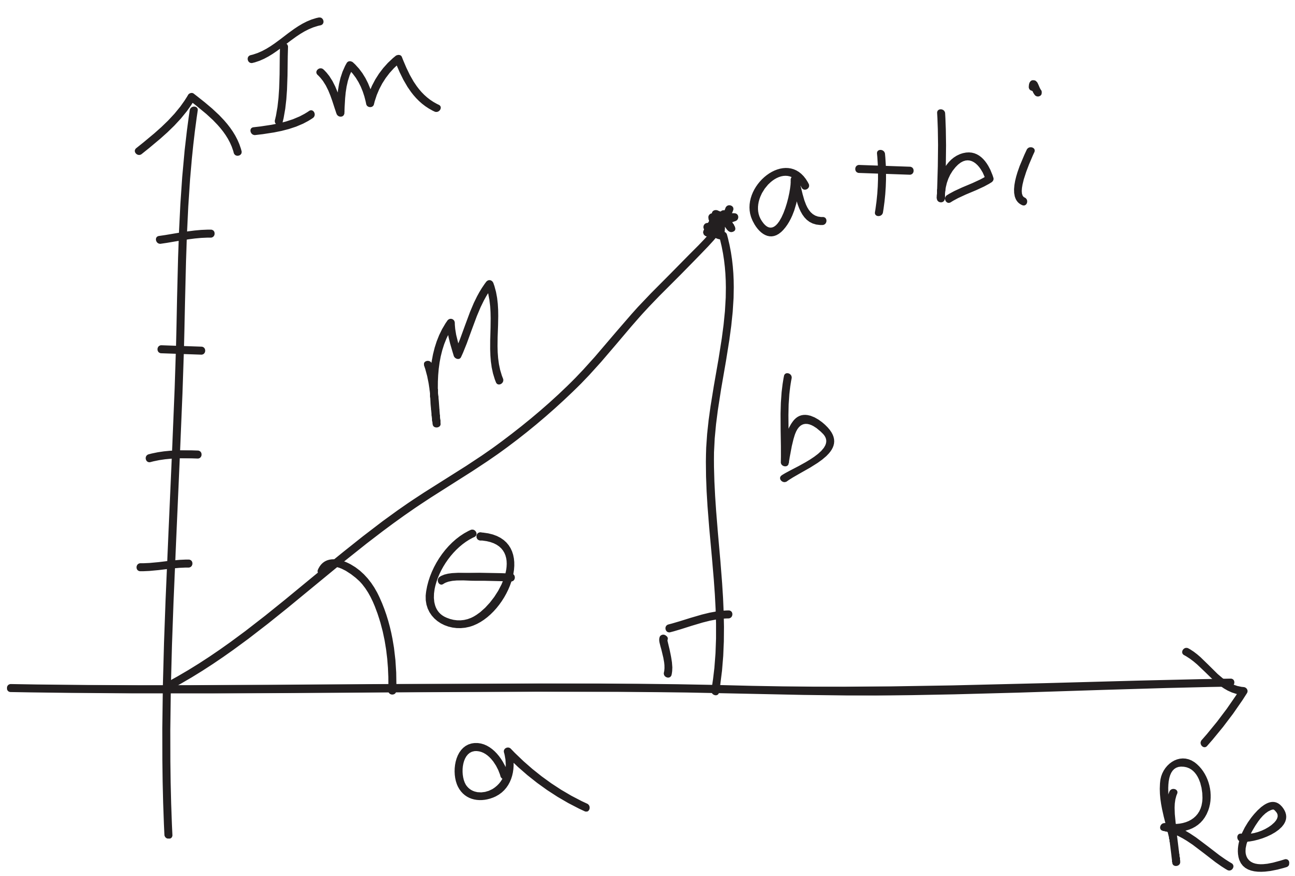

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(Polar form makes multiplication much cleaner)

Complex roots (Section 3.3)

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- General solution:

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)}$$

Complex roots (Section 3.3)

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i$, $r_2 = 1 - 2i$

(B) $r_1 = -1 + 2i$, $r_2 = -1 - 2i$

(C) $r_1 = 1 - 2i$, $r_2 = -1 + 2i$

(D) $r_1 = 2 + 4i$, $r_2 = 2 - 4i$

(E) $r_1 = -2 + 4i$, $r_2 = -2 - 4i$

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Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

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- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

(A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

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Complex roots (Section 3.3)

- Example: Find the solution to the IVP

$$y'' + 2y' + 5y = 0, \;\; y(0) = 1, \;\; y'(0) = 0$$

- General solution: $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

(B) $y(t) = e^{-t}\left(\cos(2t) - \dfrac{1}{2}\sin(2t)\right)$

(C) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

(D) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) + \sin(2t))$

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(A) $\quad y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

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- Heuristic explanation for exponential solutions and Reduction of order.

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$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

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So yes!

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Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

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(B) $y(t) = C_1 e^{2t} + C_2 e^{4t}$

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Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

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Second order, linear, constant coeff, homogeneous

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Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

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- This one is linear because

$$L[cy] = cL[y]$$

$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

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Systems of equations written in operator notation.

System of equations	Operator definition	Equation in operator notation

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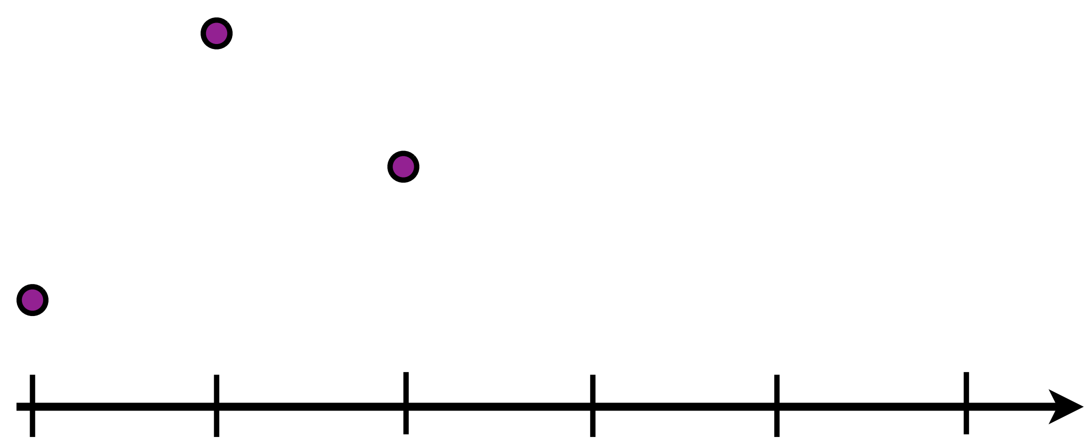

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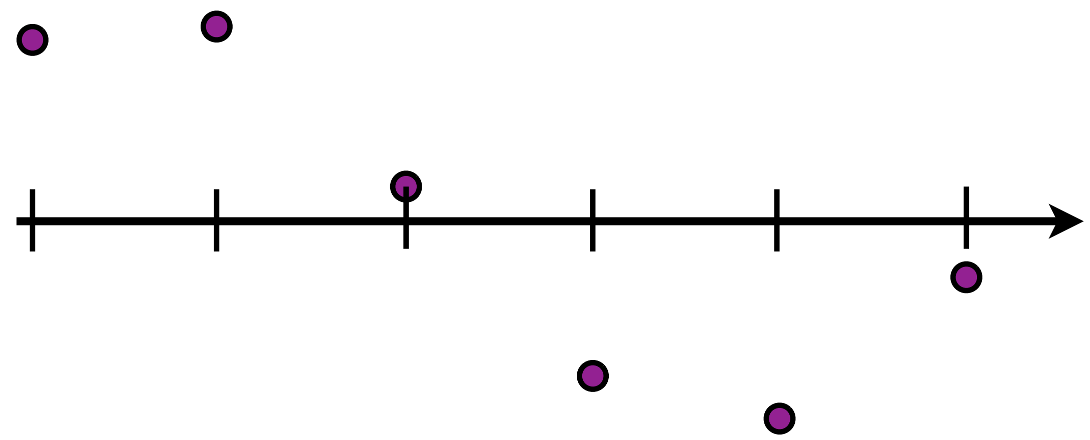

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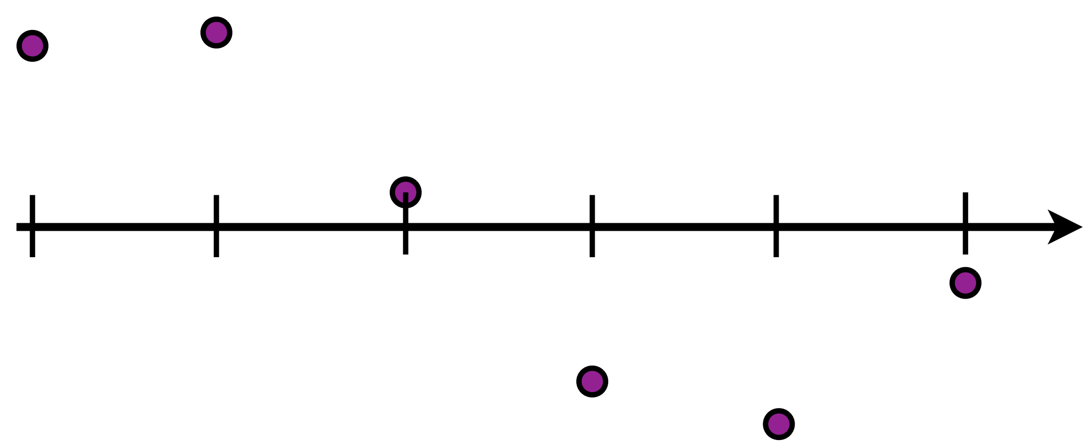

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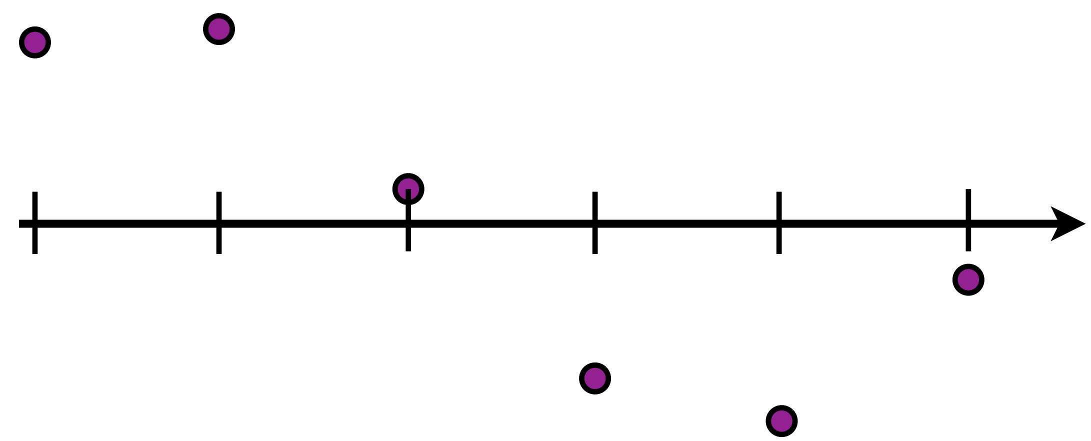

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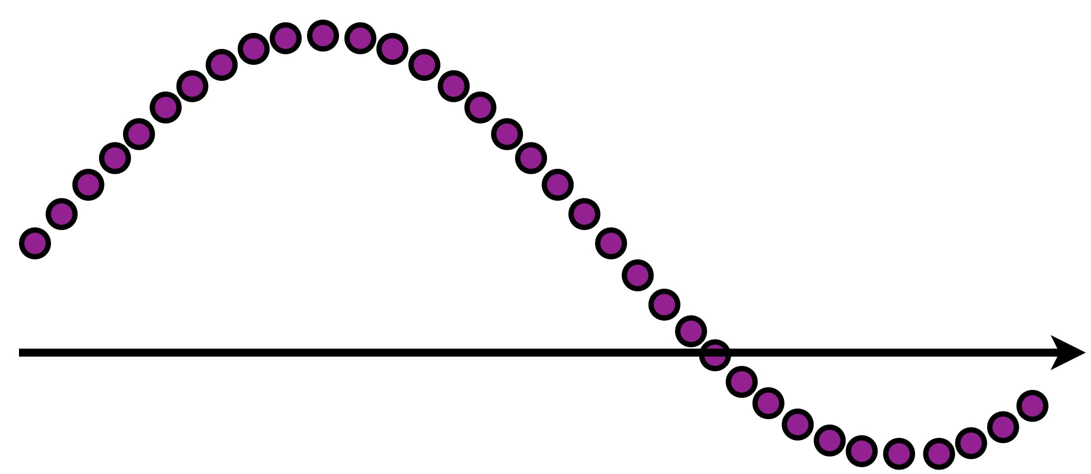

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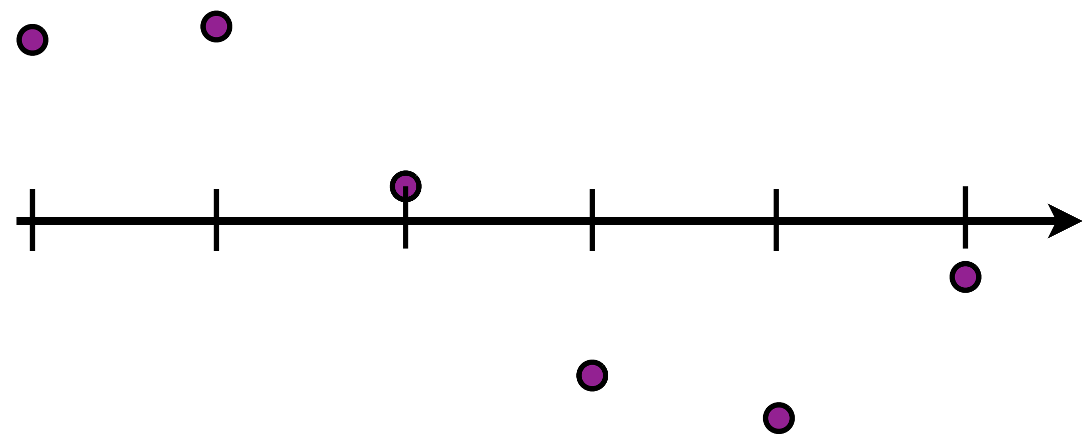

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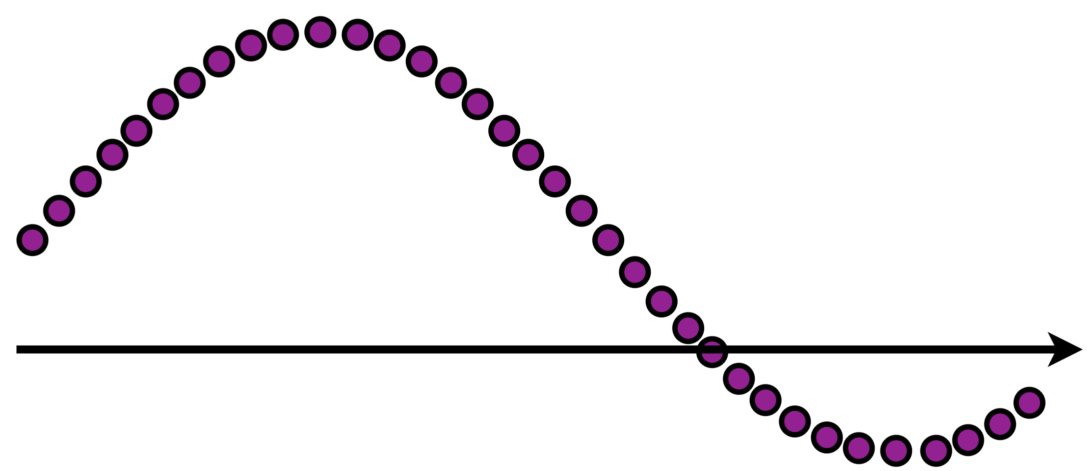

  - A differential operator is just a really big matrix.