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- General solution:

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)}$$

Complex roots

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i$, $r_2 = 1 - 2i$

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(C) $r_1 = 1 - 2i$, $r_2 = -1 + 2i$

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- Reduction of order - a method for guessing another solution.

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- Heuristic explanation for exponential solutions and Reduction of order.

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$$v'' = 0 \quad \Rightarrow \quad v' = C_1$$

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$$4y_2(t) = 4v(t)e^{-2t} \qquad 4y_2'(t) = 4v'(t)e^{-2t} - 8v(t)e^{-2t}$$

$$y_2''(t) = v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$$

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$$0 = y_2'' + 4y_2' + 4y_2 = v''e^{-2t}$$

$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

Reduction of order

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So yes!

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$$y = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

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(C) $y(t) = e^{2t}(C_1 \cos(4t) + C_2 \sin(4t))$

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