Welcome to MATH 256

Differential equations (for Chemical and Biological Engineering students)

Instructor:
Prof. Eric Cytrynbaum

Course goals

- Primary: Learn to solve ordinary and partial differential equations (mostly linear first and second order DEs).

- Secondary: Learn to use DEs to model physical, chemical, biological systems (really just an intro to this skill).

Prerequisites

- First year calculus (MATH 100/101).
- Linear algebra (MATH 152).
- Multivariable calculus (MATH 200 or 253).
- Talk to me if you aren't sure that you're prepared for this course.

Tools we’ll be using this term

- WeBWorK for homework assignments.
- Piazza for online discussion.
- Clickers for in-class responses.
- Cell phones and facebook for getting distracted during lectures and while studying.

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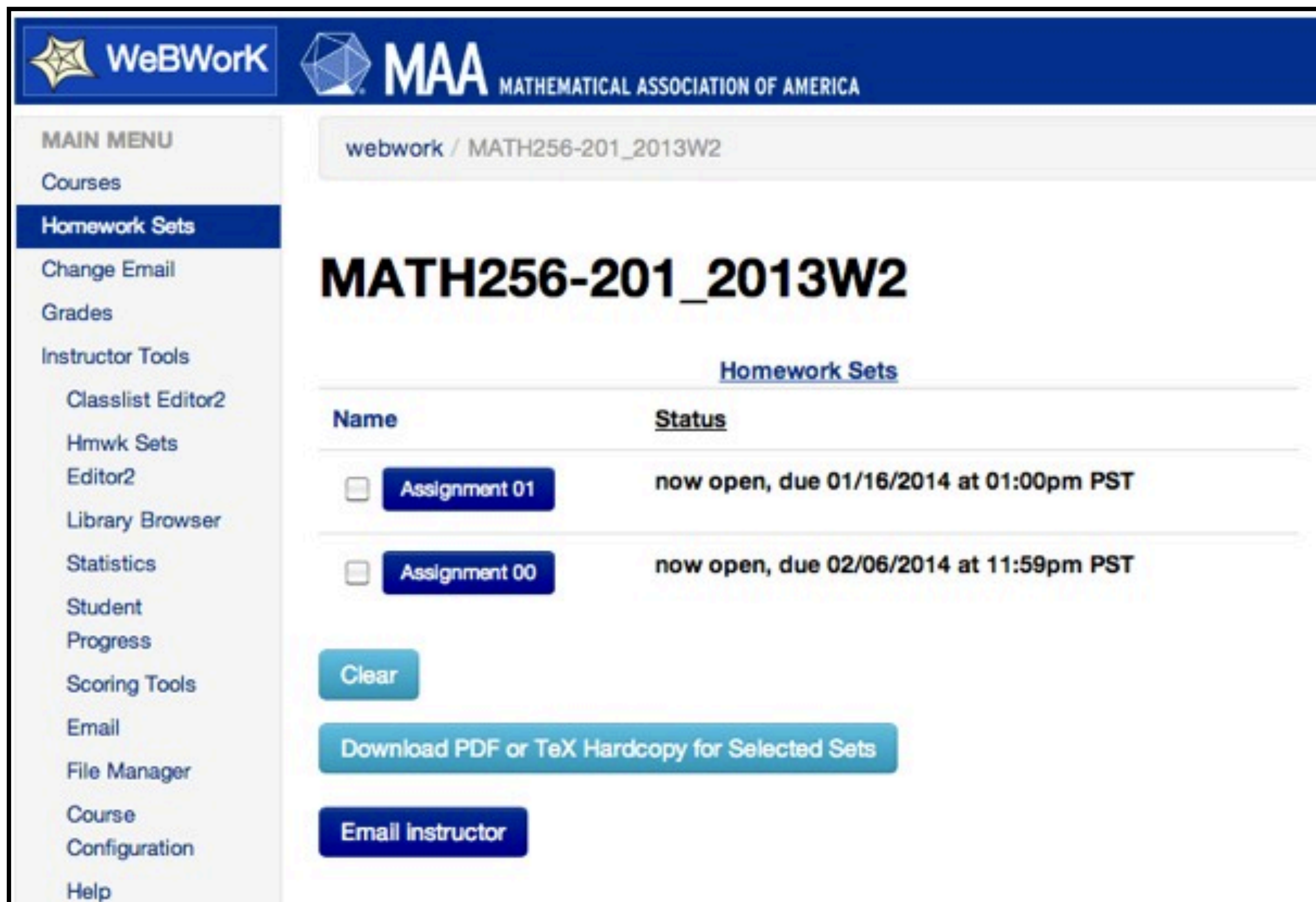
WeBWorK

- Online homework system.
- https://webwork.elearning.ubc.ca/webwork2/MATH256-201_2013W2
- Log in using your CWL.



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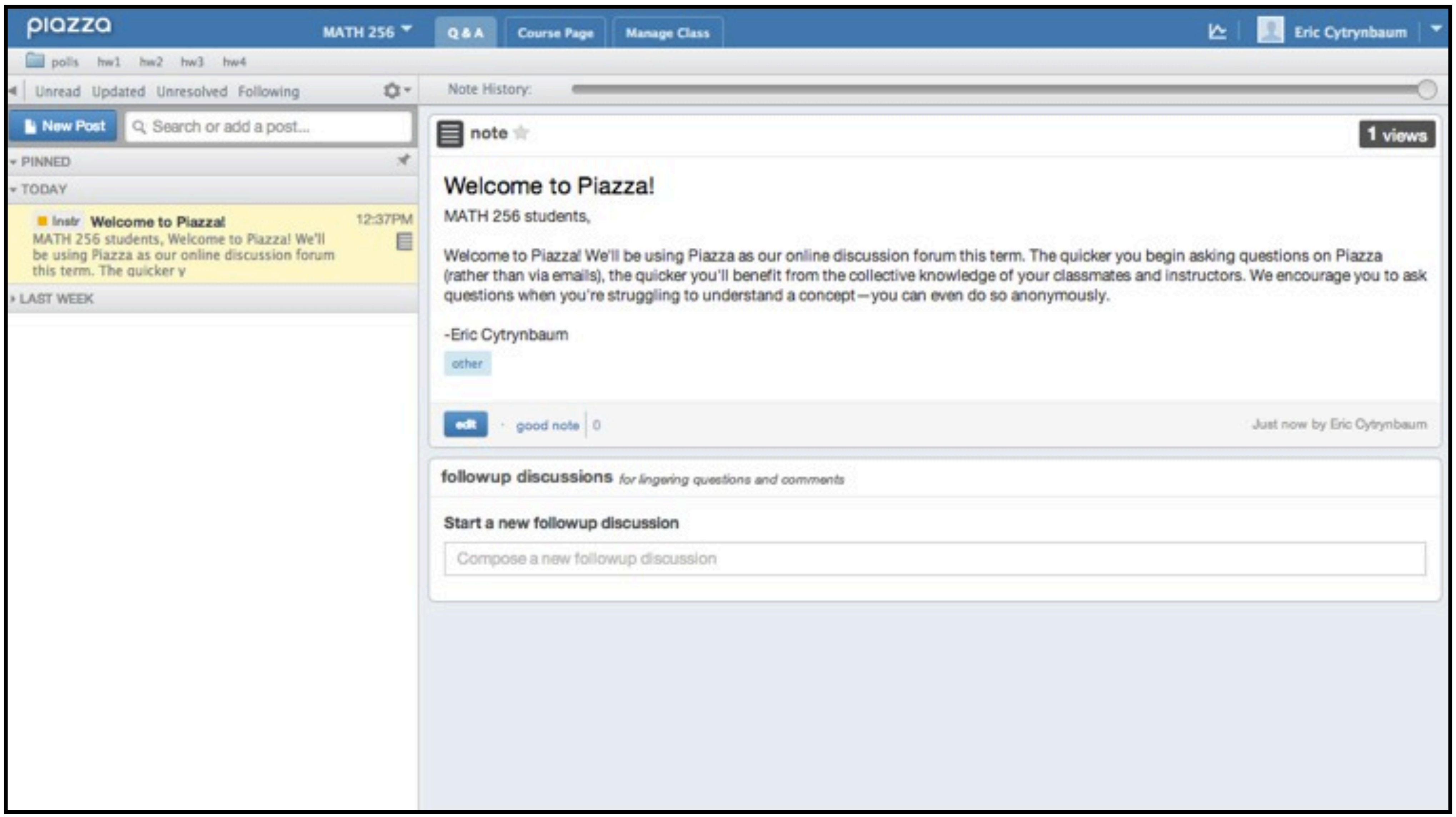
Piazza

- Online discussion forum.
- Sign up at https://piazza.com



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Clickers

- Personal response system.
- Register your clicker at https://connect.ubc.ca

Why clickers?

- Active learning - you should be thinking during class.
- My goal is to make clicker Qs that many of you get wrong - they help us to target what you don’t understand yet.
- Points are for (thinking and then) clicking, not for getting answers correct.
- I don’t look at the results on an individual basis so they are effectively anonymous.

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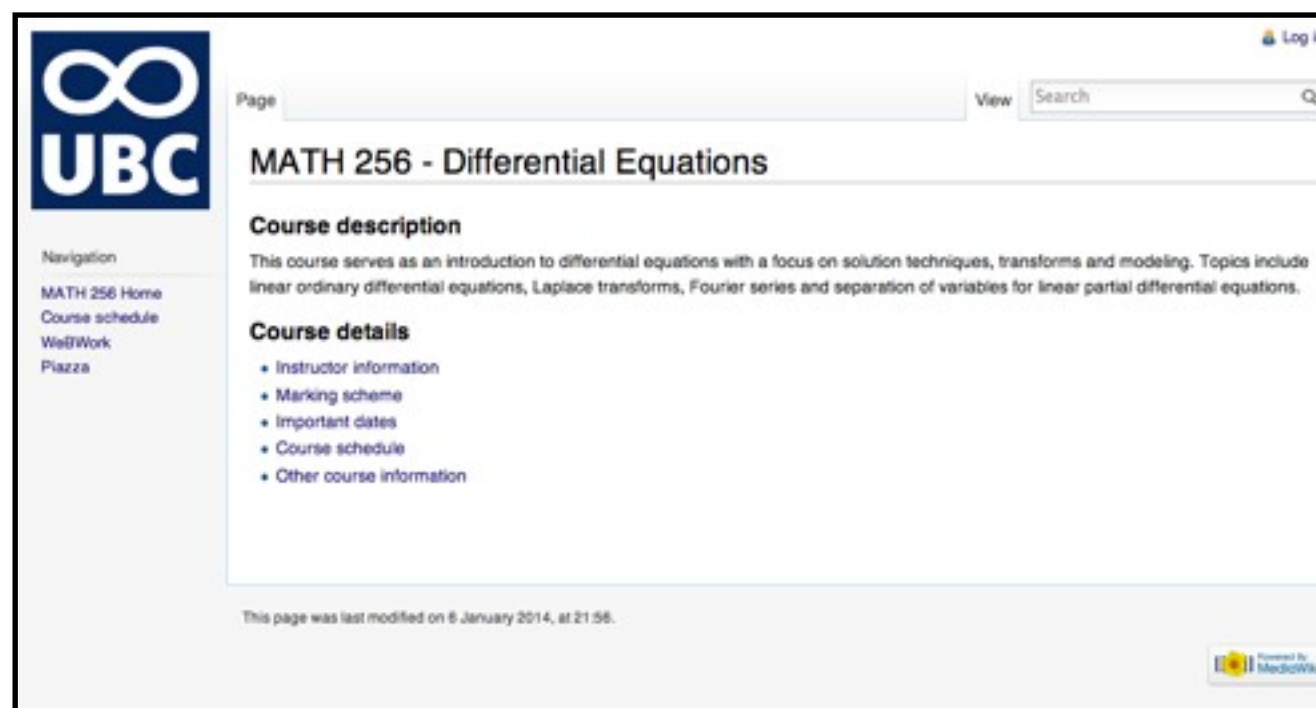
More info online...

- Check the course website for
    - office hour info,
    - info on additional help,
    - textbook,
    - course policies (e.g. marking scheme)
    - week-by-week schedule.

- https://wiki.math.ubc.ca/mathbook/M256/MATH_256_-_Differential_Equations



Felix Baumgartner’s freefall from 40 km up

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- Simple model to predict how fast he'll go, how long it will take etc.

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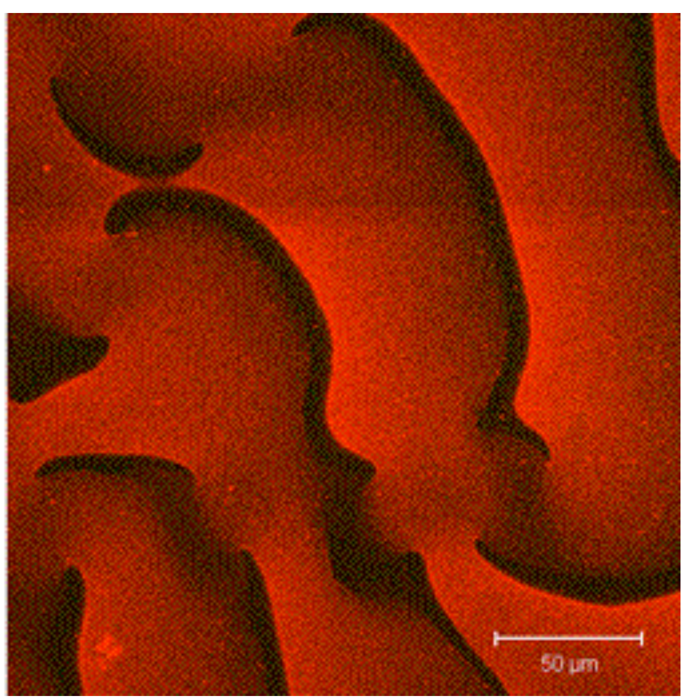
- Flaws with this model?
- g is not constant...
- ...but 6371 km ≈ 6411 km so not bad.
- k is not constant either (depends on air density) - this is significant!

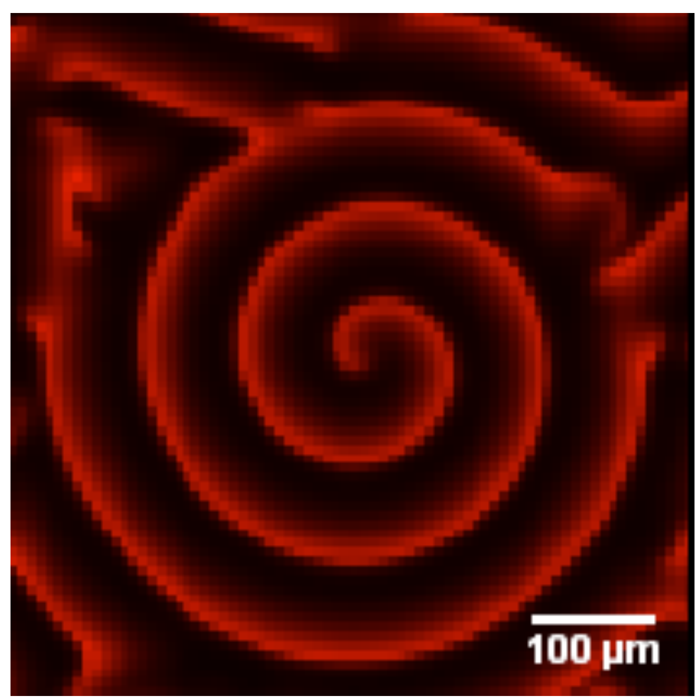
A bacterial cell division regulator

- Two interacting bacterial proteins that undergo complicated dynamics.
- Differential equation model help understand how they work.

Experiment



Model



$$\frac{\partial u}{\partial t} = u - uv + D\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial v}{\partial t} = uv - v + D\frac{\partial^2 v}{\partial x^2}$$

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- The general solution - a solution with one or more arbitrary constants that encompass ALL possible solutions to the DE.

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A cylindrical bucket has a hole in the bottom. If $h(t)$ is the height of the water at any time t in hours, then the differential equation describing this leaky bucket is given by the equation:

$$\frac{dh(t)}{dt} = -6\sqrt{h(t)}.$$

If initially, there are 4 inches of water in the bucket $(h(0) = 4)$, what is the solution to this differential equation?

A. $h(t) = (2 - 3t)^2$
B. $h(t) = \sqrt{16 - 2t}$
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Method of integrating factors (Section 2.1)

$$\frac{d}{dt}\left(t^2 y(t)\right) =$$

(A) $2t\dfrac{dy}{dt}$

(B) $t^2\dfrac{dy}{dt}$

(C) $2ty$

(D) $t^2\dfrac{dy}{dt} + 2ty$

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- Given that $\dfrac{d}{dt}\left(t^2 y(t)\right) = \ t^2\dfrac{dy}{dt} + 2ty$

- if you're given the equation $t^2\dfrac{dy}{dt} + 2ty = 0$

- you can rewrite is as $\dfrac{d}{dt}\left(t^2 y(t)\right) = 0$

- so the solution is $t^2 y(t) = C$ or equivalently $y(t) = \dfrac{C}{t^2}$.

arbitrary constant that appeared at an integration step

Method of integrating factors (Section 2.1)

- Solve the equation $t^2 \frac{dy}{dt} + 2ty(t) = \sin(t)$ (not brute force checking).

(A) $y(t) = -\frac{1}{t^2}\sin(t)$

(B) $y(t) = -\cos(t) + C$

(C) $y(t) = \frac{C - \cos(t)}{t^2}$

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(C) $y(t) = \frac{C - \cos(t)}{t^2}$ ← general solution (although that's not obvious)

(D) $y(t) = \sin(t) + C$

(E) $y(t) = -\frac{1}{t^2} \cos(t)$ ← a particular solution

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- An Initial Value Problem (IVP) is a ODE together with an IC.

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$$t\frac{dy}{dt} + 2y(t) = 1 \qquad \to f(t) = t$$

$$t^2\frac{dy}{dt} + 4ty(t) = \frac{1}{t} \qquad \to f(t) = t^2$$

$$\frac{dy}{dt} + y(t) = 0 \qquad \to f(t) = e^t$$

$$\frac{dy}{dt} + \cos(t)y(t) = 0 \qquad \to f(t) = e^{\sin(t)}$$

$$\frac{dy}{dt} + g'(t)y(t) = 0 \qquad \to f(t) = e^{g(t)}$$

Method of integrating factors (Section 2.1)

- General case - all first order linear ODEs can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

- The appropriate integrating factor is $e^{\int p(t)dt}$.

- The equation can be rewritten $\frac{d}{dt}\left(e^{\int p(t)dt}y\right) = e^{\int p(t)dt}q(t)$ which is solvable provided you can find the antiderivative of the right hand side.