Today

- Introduction to the Dirac delta function
- Modelling with delta-function forcing (tanks, springs)

Delta-function forcing

- Suppose a mass is sitting at position x and a force g(t) acts on it:

$$mx'' = g(t)$$

- To find x(t), integrate up:

$$\int_a^b mx''\ dt = \int_a^b g(t)\ dt$$

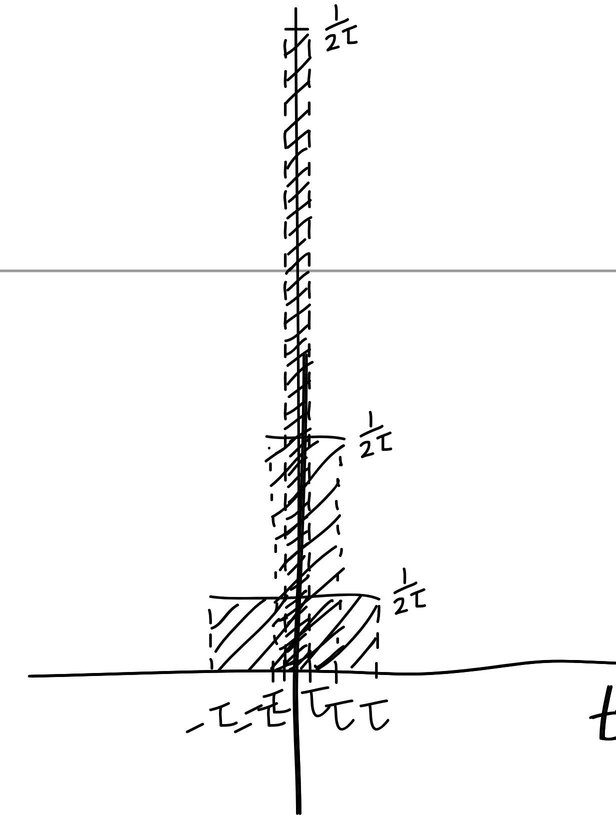
$$mx'\Big|_a^b = \int_a^b g(t)\ dt$$

$$mv(b) - mv(a) = \int_a^b g(t)\ dt$$

- $\int_a^b g(t)\ dt$ is the change in momentum of the mass - called impulse.

- If the force is large and sudden (say a hammer hitting the mass), maybe we just need to get this integral correct and the details don't matter.

Delta-function forcing



- Let's assume $g(t) = \begin{cases} \dfrac{I_0}{2\tau} & -\tau < t < \tau \\ \\ 0 & \text{otherwise} \end{cases}$

$$= (u_{-\tau}(t) - u_\tau(t))\frac{I_0}{2\tau}$$

impulse = $\Delta\text{momentum} = \displaystyle\int_{-\infty}^{\infty} g(t)\ dt \ = \int_{-\tau}^{\tau} \frac{I_0}{2\tau}\ dt \ = I_0$

- For general purposes (any property that might change quickly, not just momentum), we define the Dirac Delta "function" as follows:

$$d_\tau(t) = (u_{-\tau}(t) - u_\tau(t))\frac{1}{2\tau}$$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) = \begin{cases} \text{“}\infty\text{”} & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

$$g(t) = I_0 d_\tau(t)$$

- I_0 can be replaced by any type of quantity
- e.g. m_0 mass added to tank suddenly
- units of δ(t): 1 / time

Some facts about the Delta "function"

$$\int_a^b \delta(t)\ dt = 1 \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$$\int_a^b f(t)\delta(t)\ dt = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(t)\ dt$$

$$= \lim_{\tau \to 0} \frac{F(\tau) - F(-\tau)}{2\tau} \qquad F'(t) = f(t)$$

$$= F'(0) = f(0)$$

$$\int_a^b f(t)\delta(t)\ dt = f(0) \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$\delta(t - c) =$ shift of $\delta(t)$ by c

$$\int_a^b f(t)\delta(t-c)\ dt = \int_{a+c}^{b+c} f(u+c)\delta(u)\ du = f(c)$$ provided a<c<b.

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$$\int_a^b f(t)\delta(t-c)\;dt \;= f(c)$$

Laplace transform of delta function:

$$\mathcal{L}\{\delta(t-c)\} = \int_0^\infty e^{-st}\delta(t-c)\;dt$$
$$= \int_{-c}^\infty e^{-s(u+c)}\delta(u)\;du \;= e^{-sc} \text{ for } c > 0$$

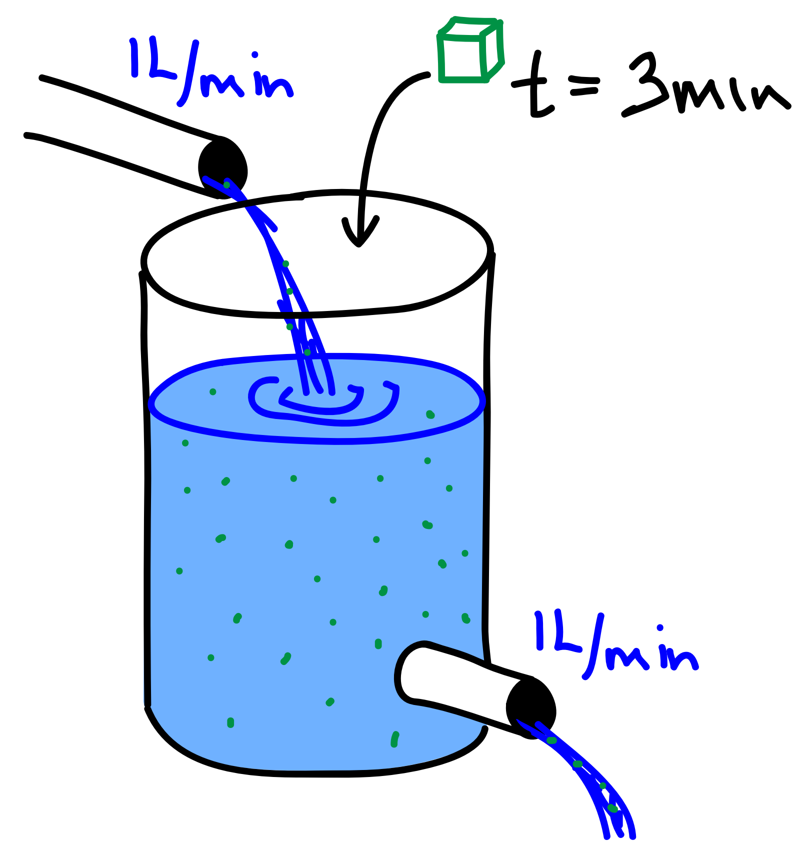
Relationship of delta function to other functions:

$$\frac{d}{dt}|t-c| = \; u_c(t)$$

$$\frac{d}{dt}u_c(t) = \; \delta(t-c)$$

Delta-function forcing

- Water with c_{in} = 2 g/L of sugar enters a tank at a rate of r = 1 L/min. The initially sugar-free tank holds V = 5 L and the contents are well-mixed. Water drains from the tank at a rate r. At t_{cube} = 3 min, a sugar cube of mass m_{cube} = 3 g is dropped into the tank.



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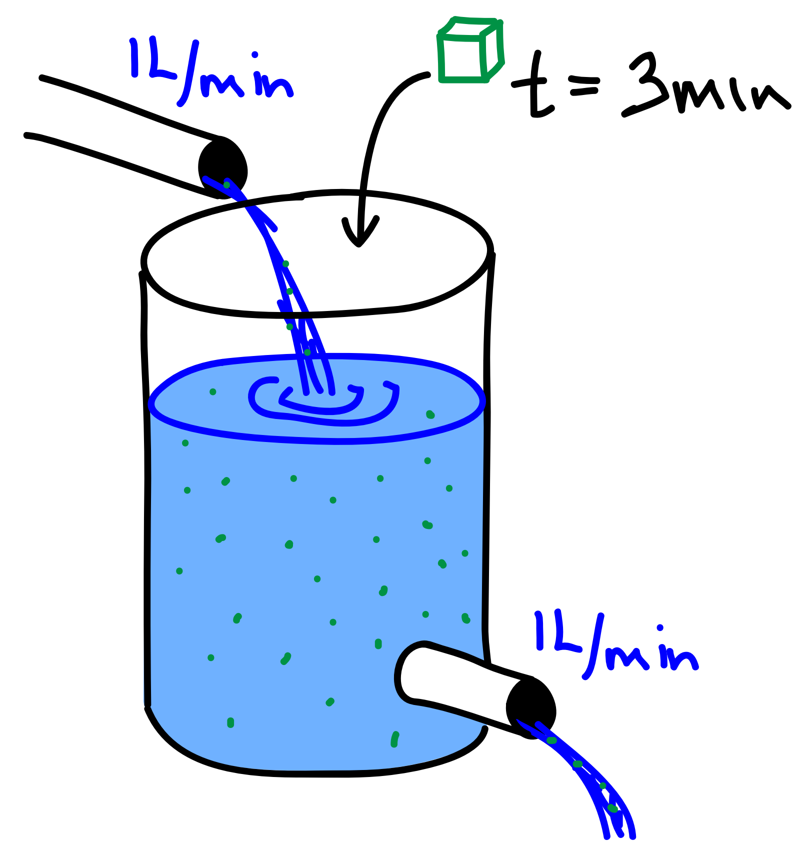

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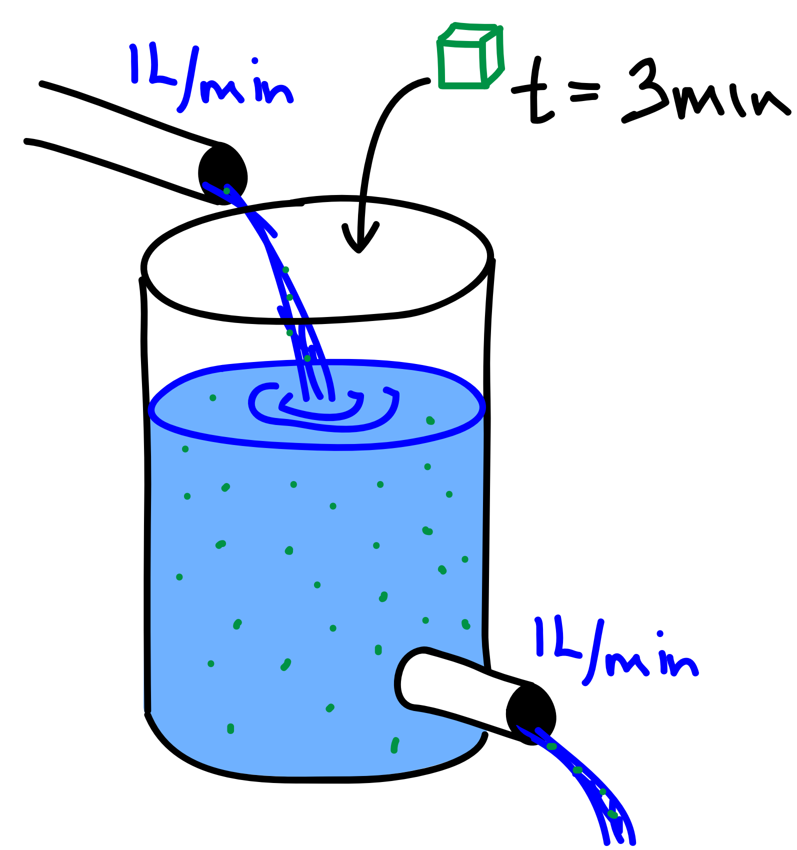

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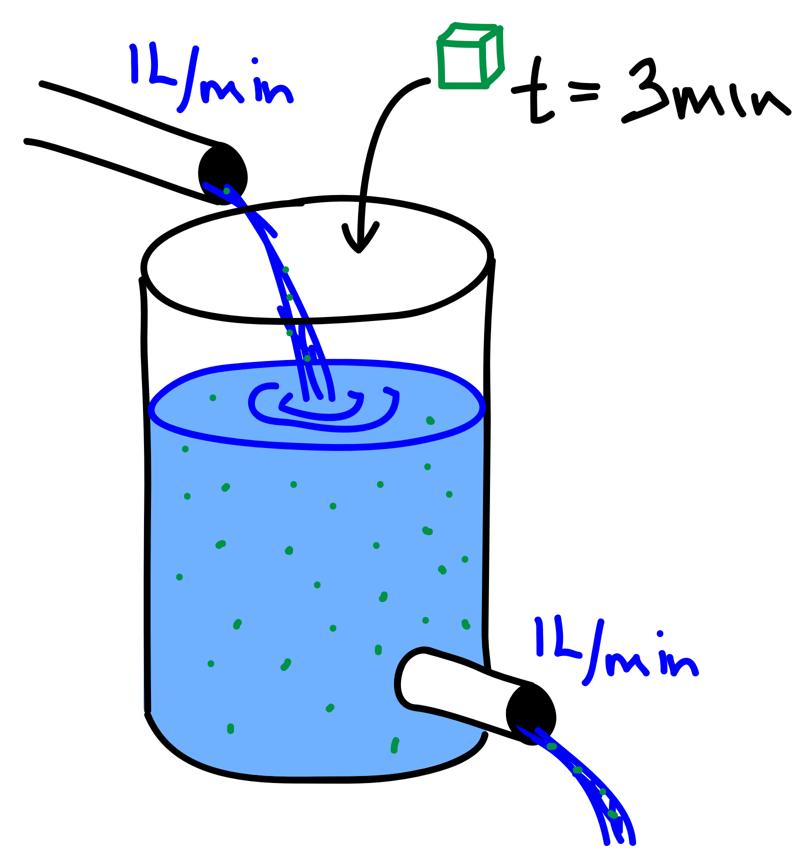
$$m' = rc_{in} - \frac{r}{V}m + m_{cube}\delta(t - t_{cube})$$



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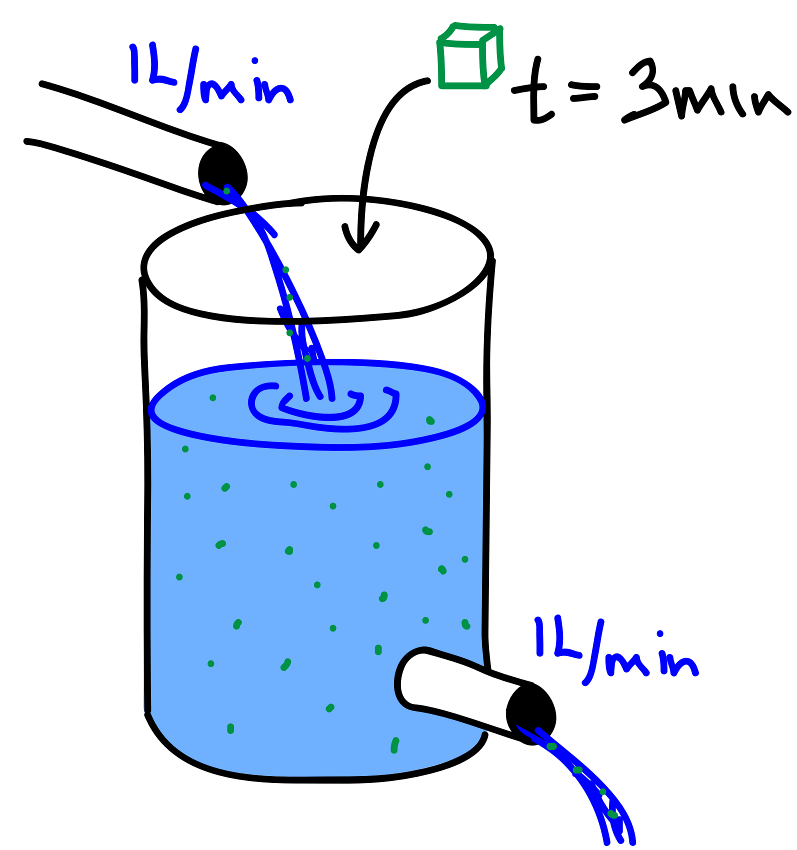

- Note: δ(t) has units of 1/time.

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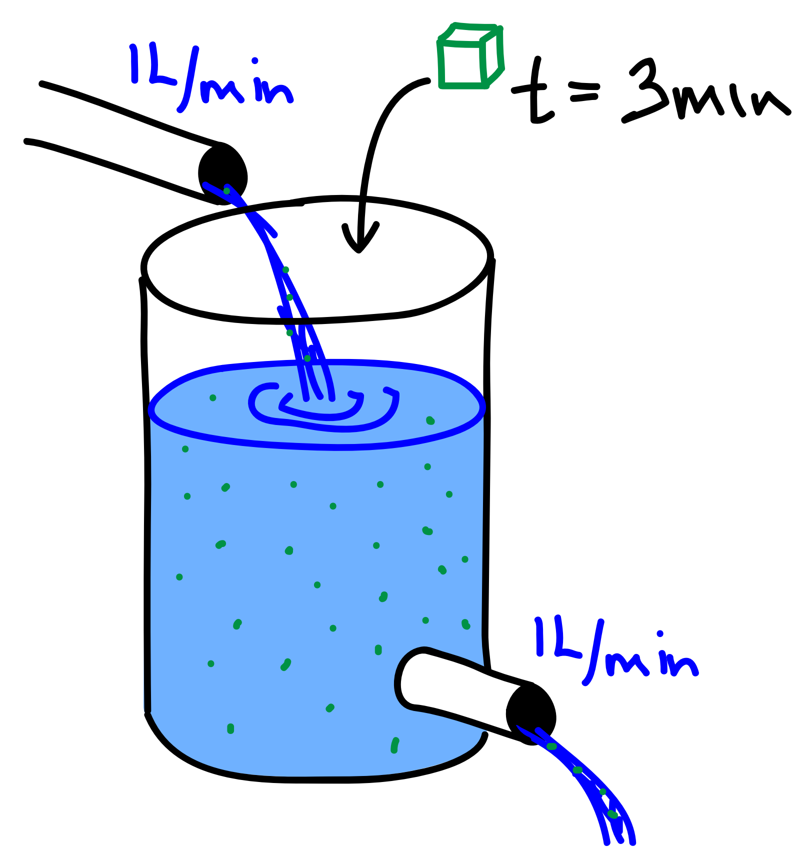

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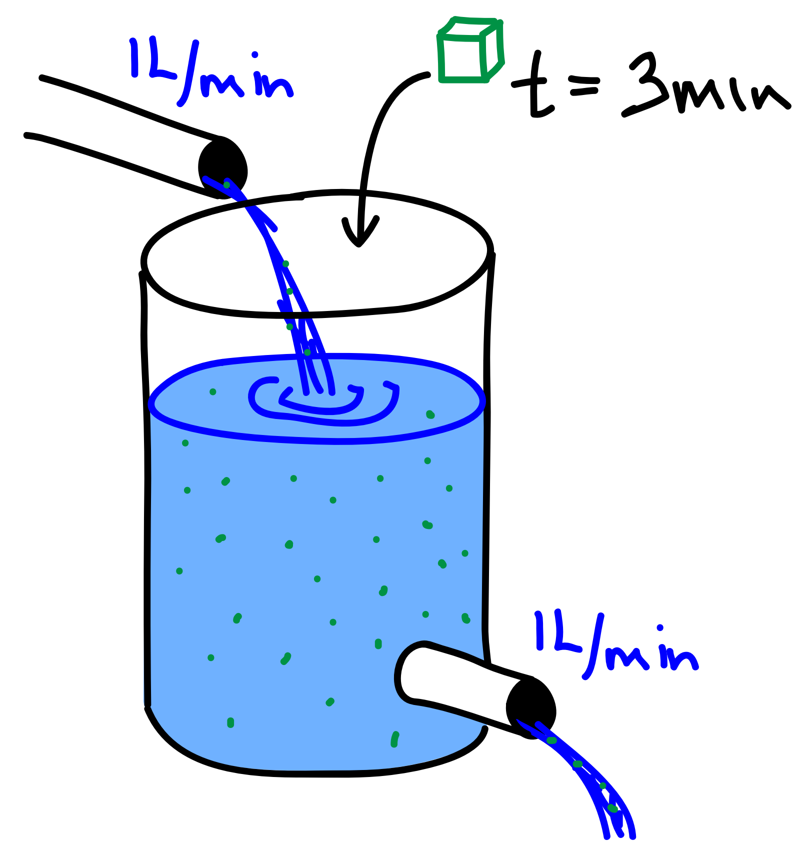

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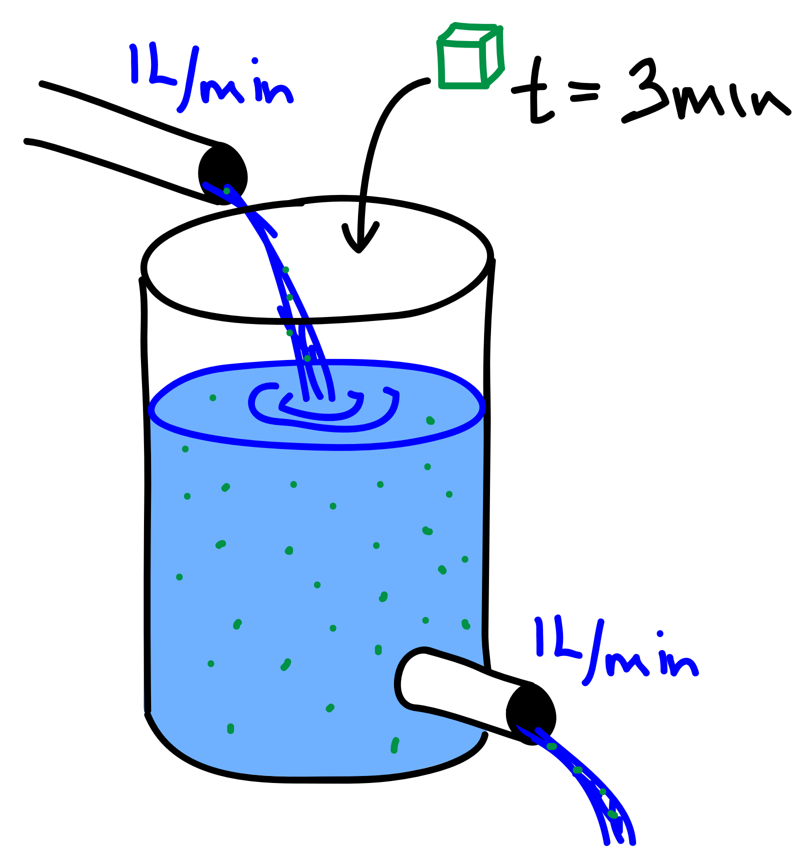

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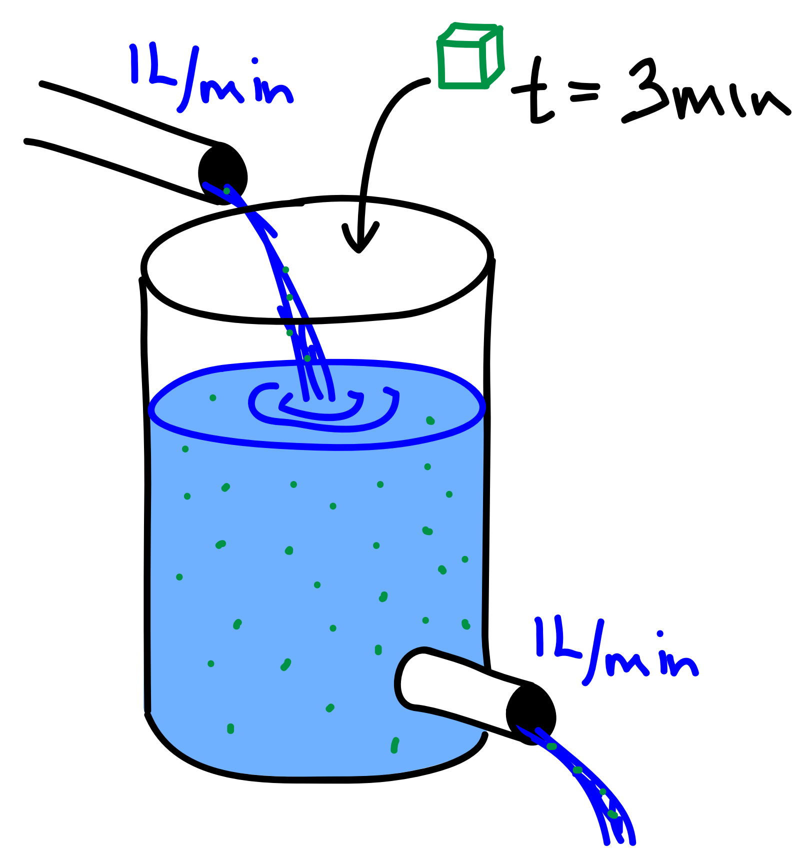
  - Sketch the solution to the ODE. How would it differ if t_{cube}=10 min?



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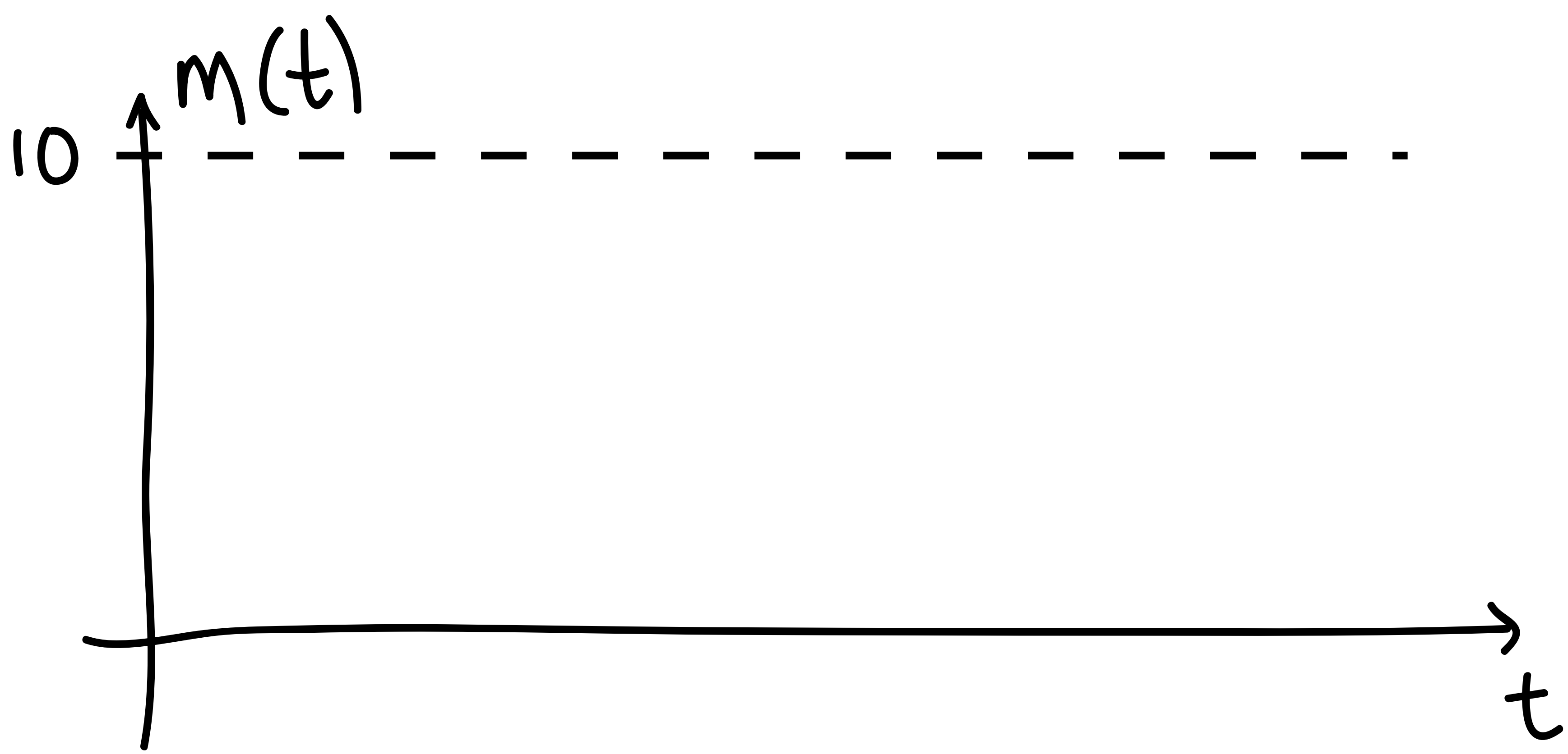

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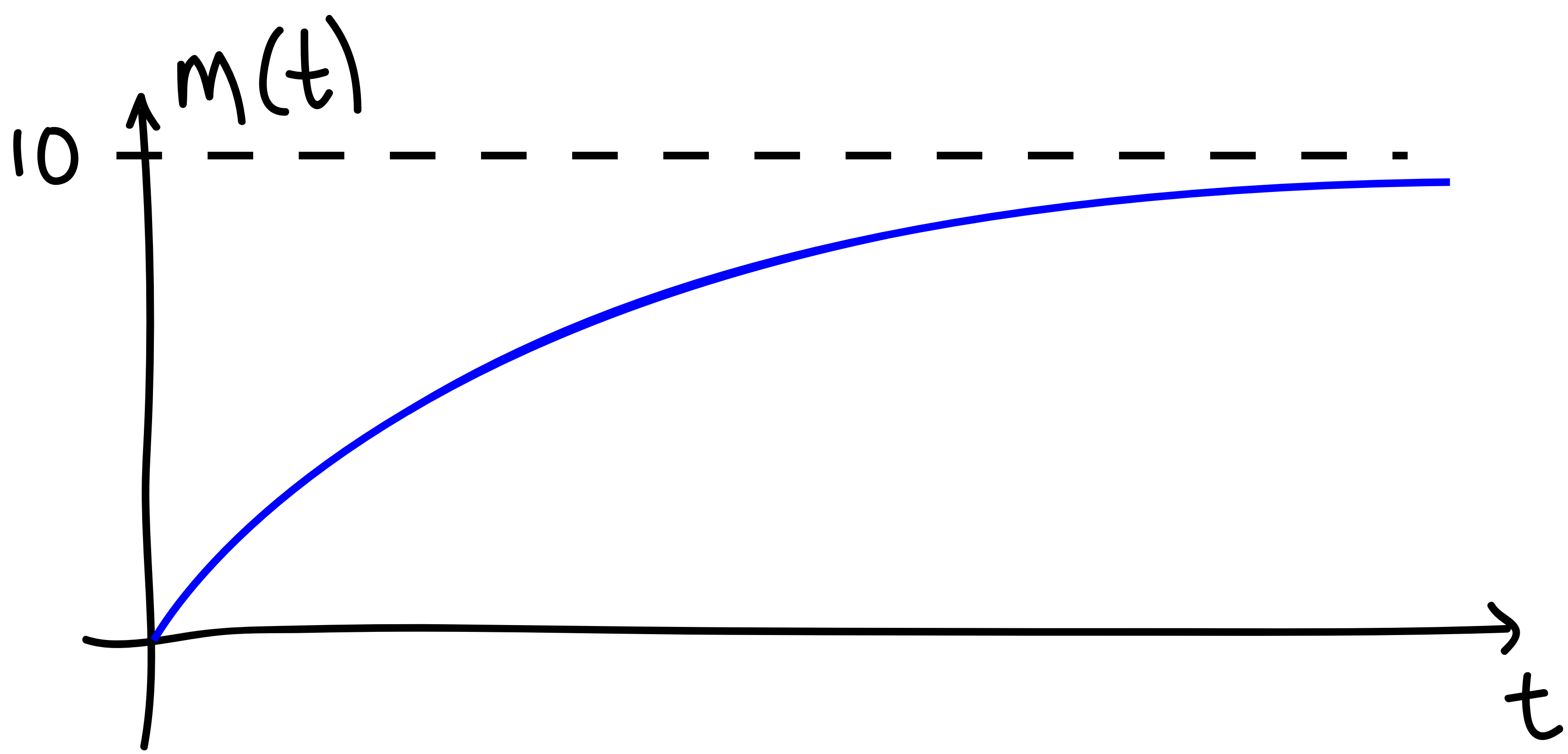

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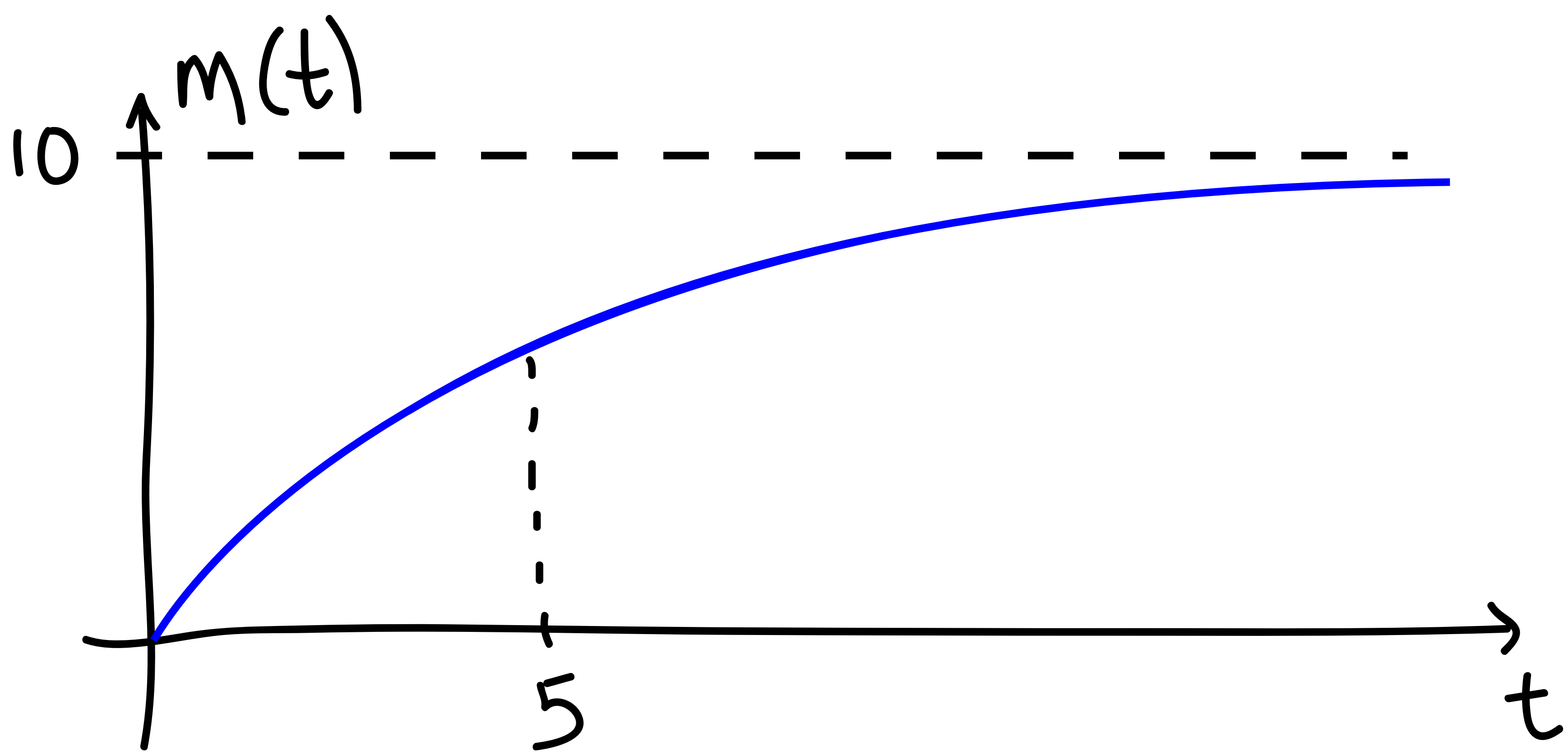

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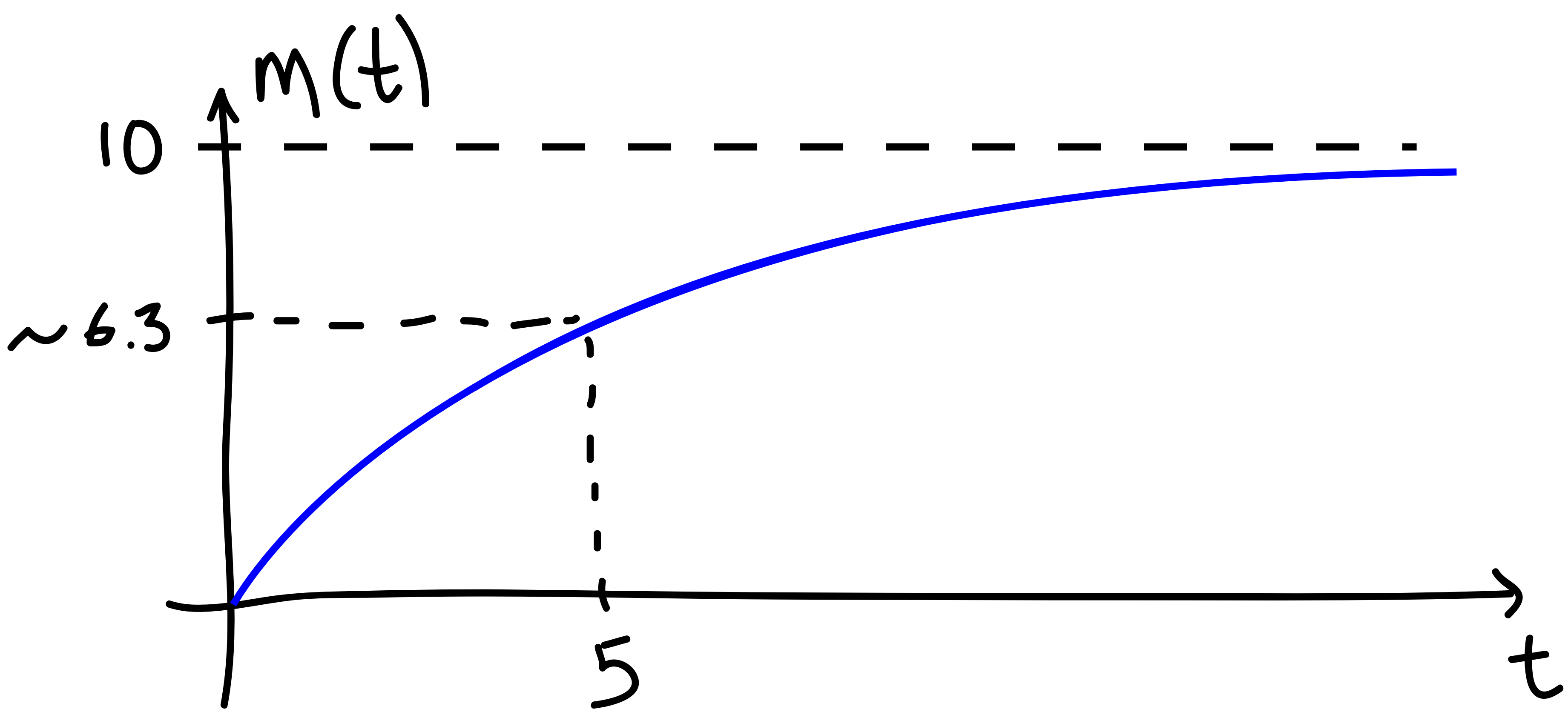

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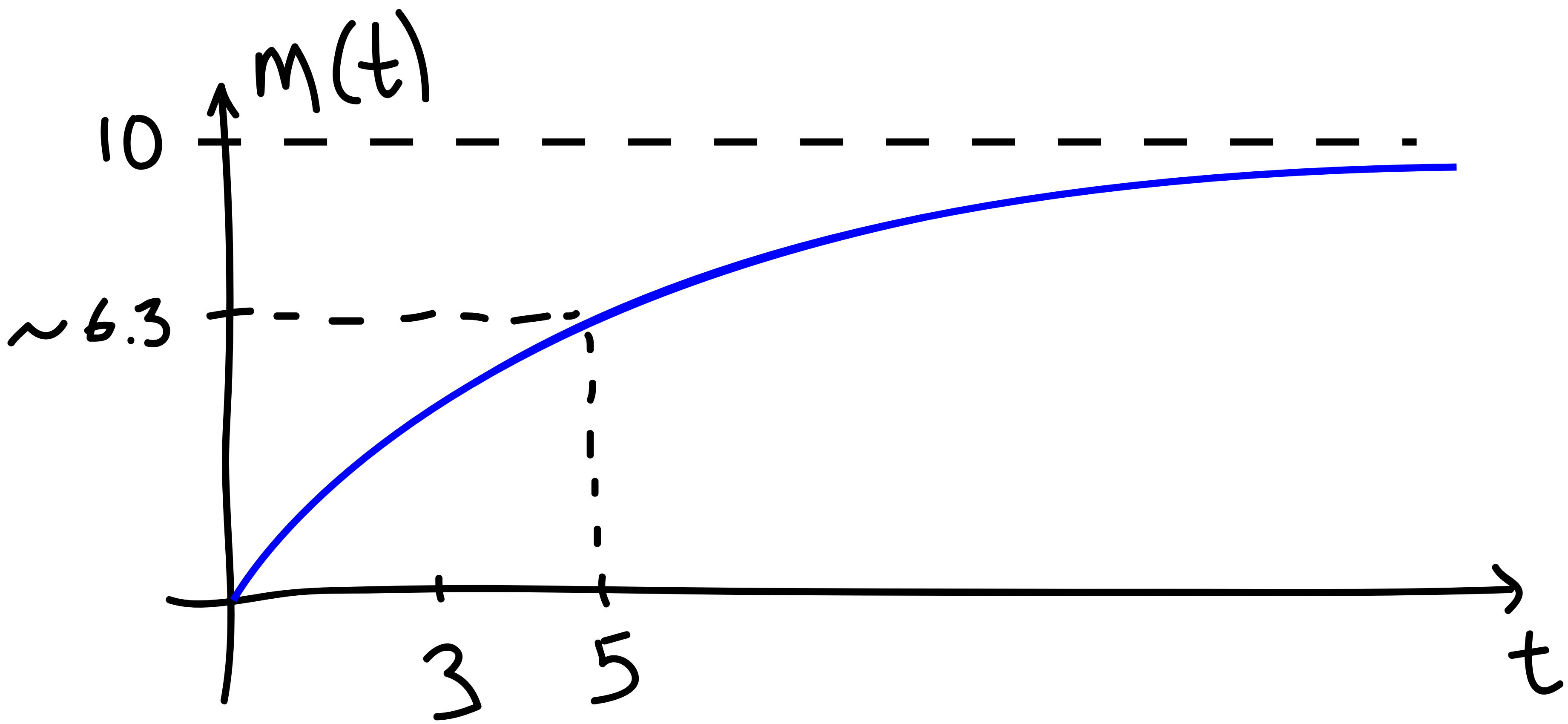

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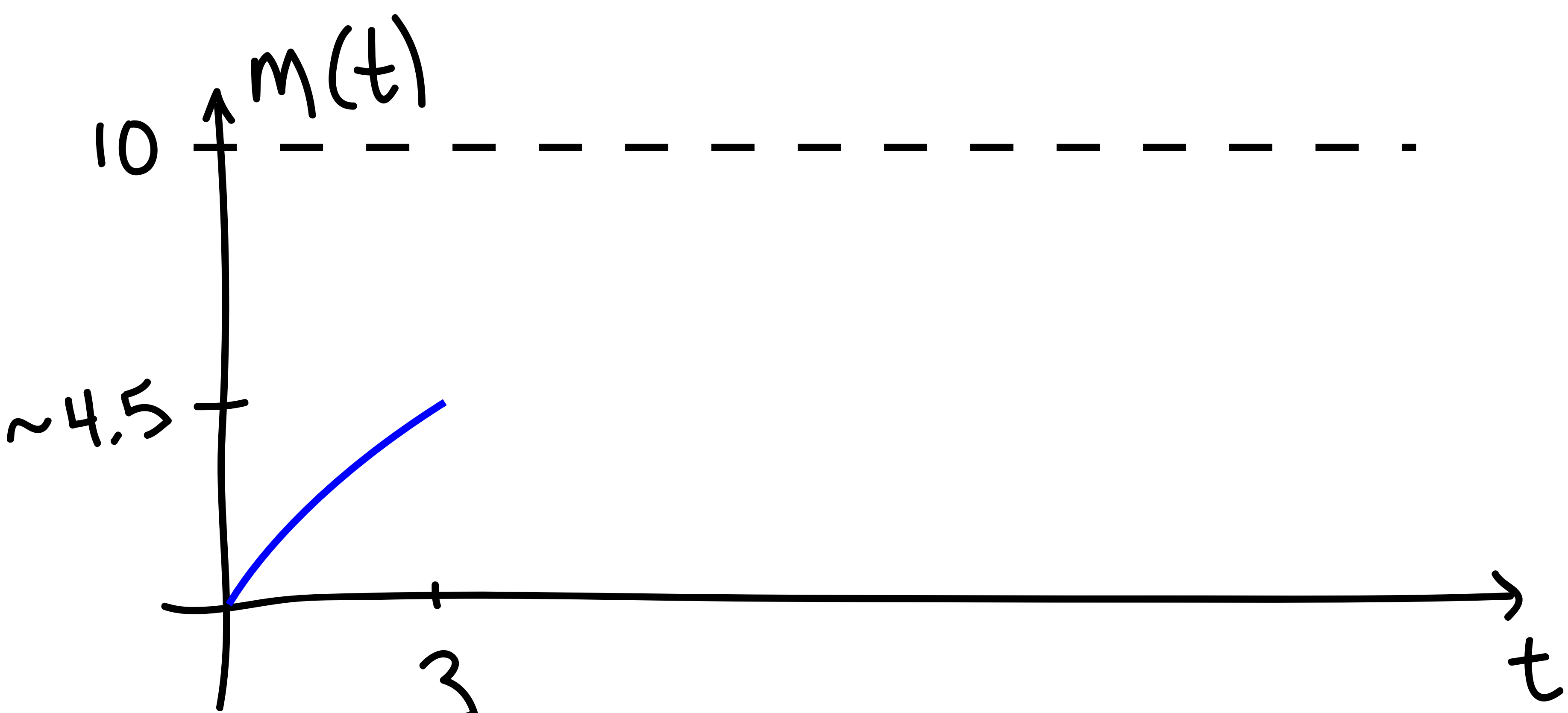

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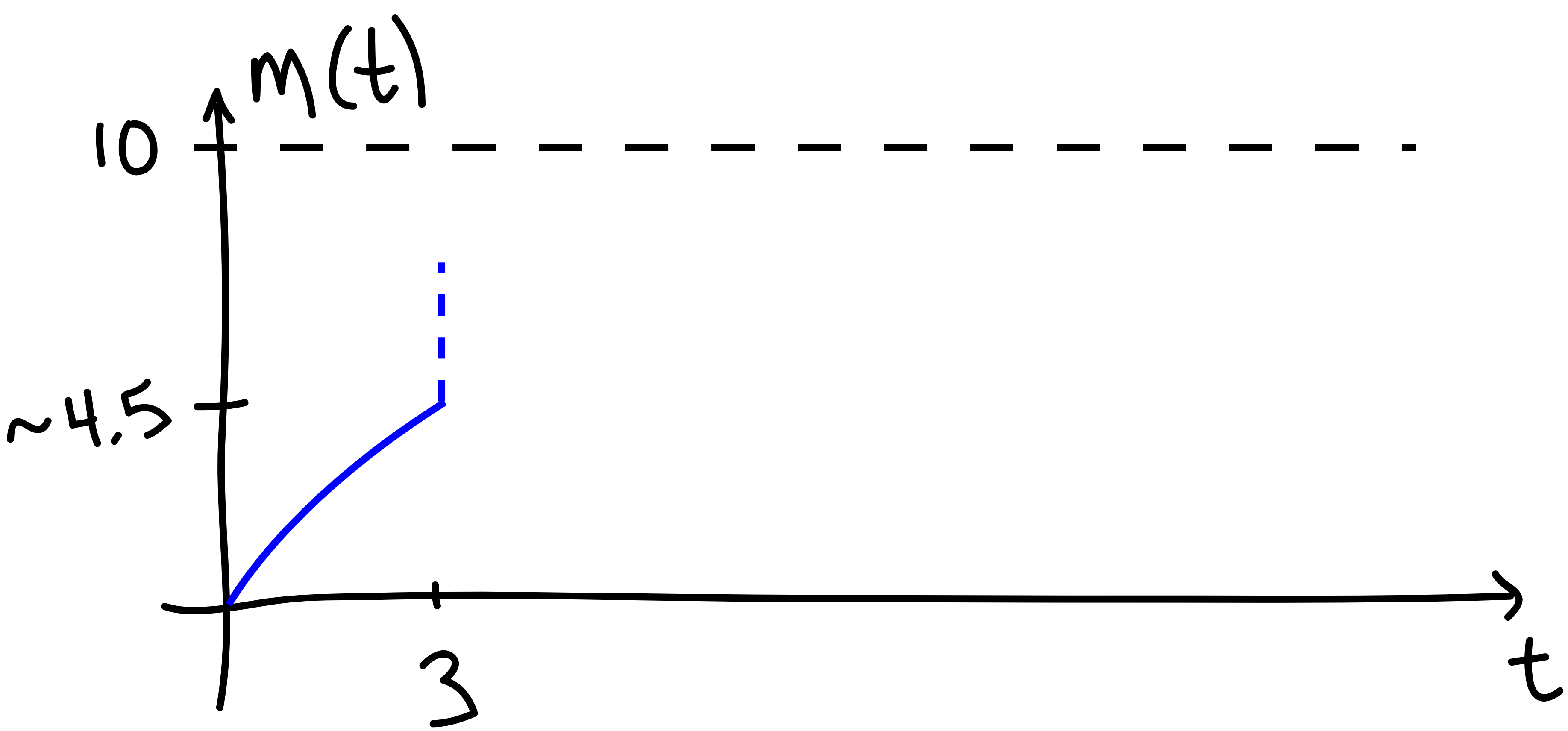

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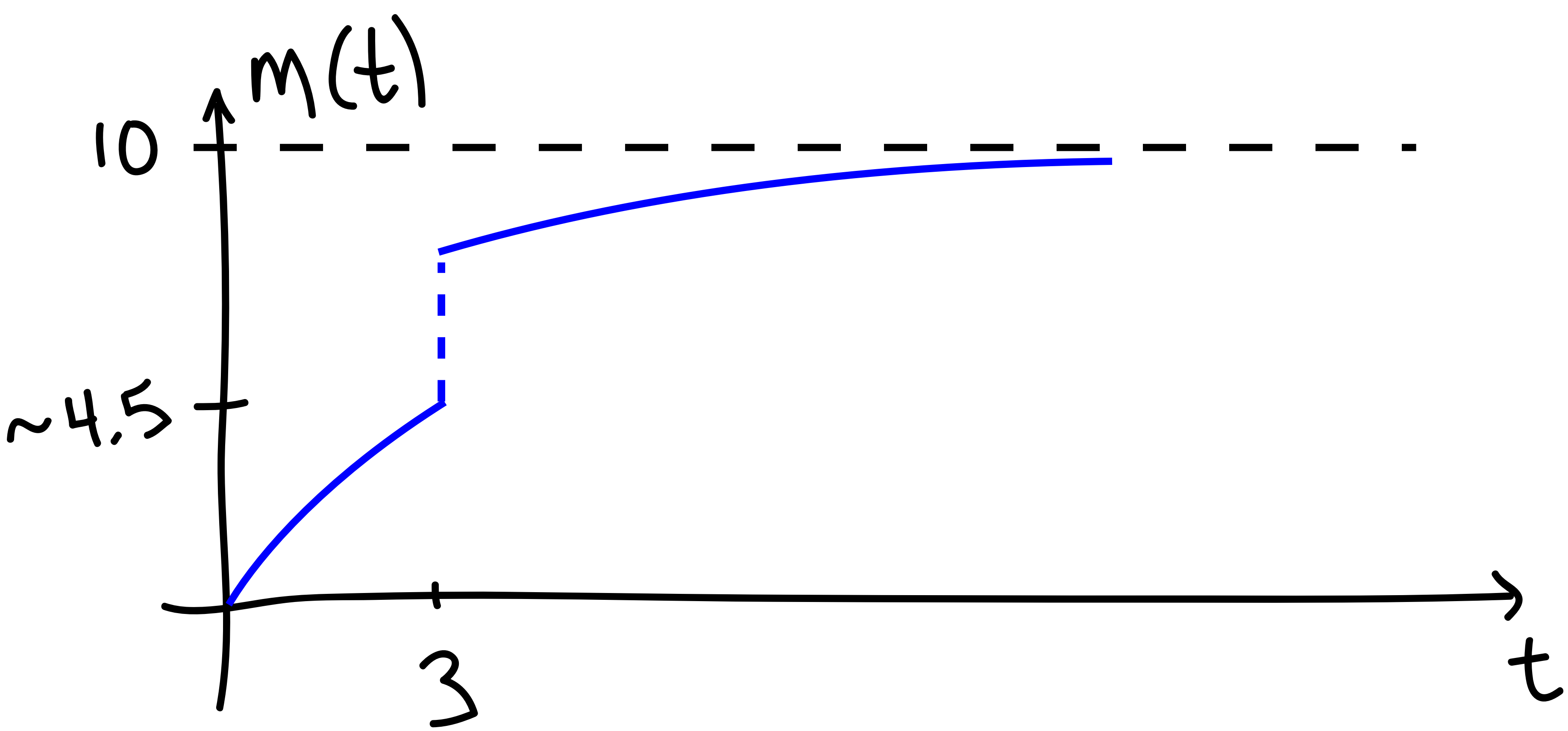

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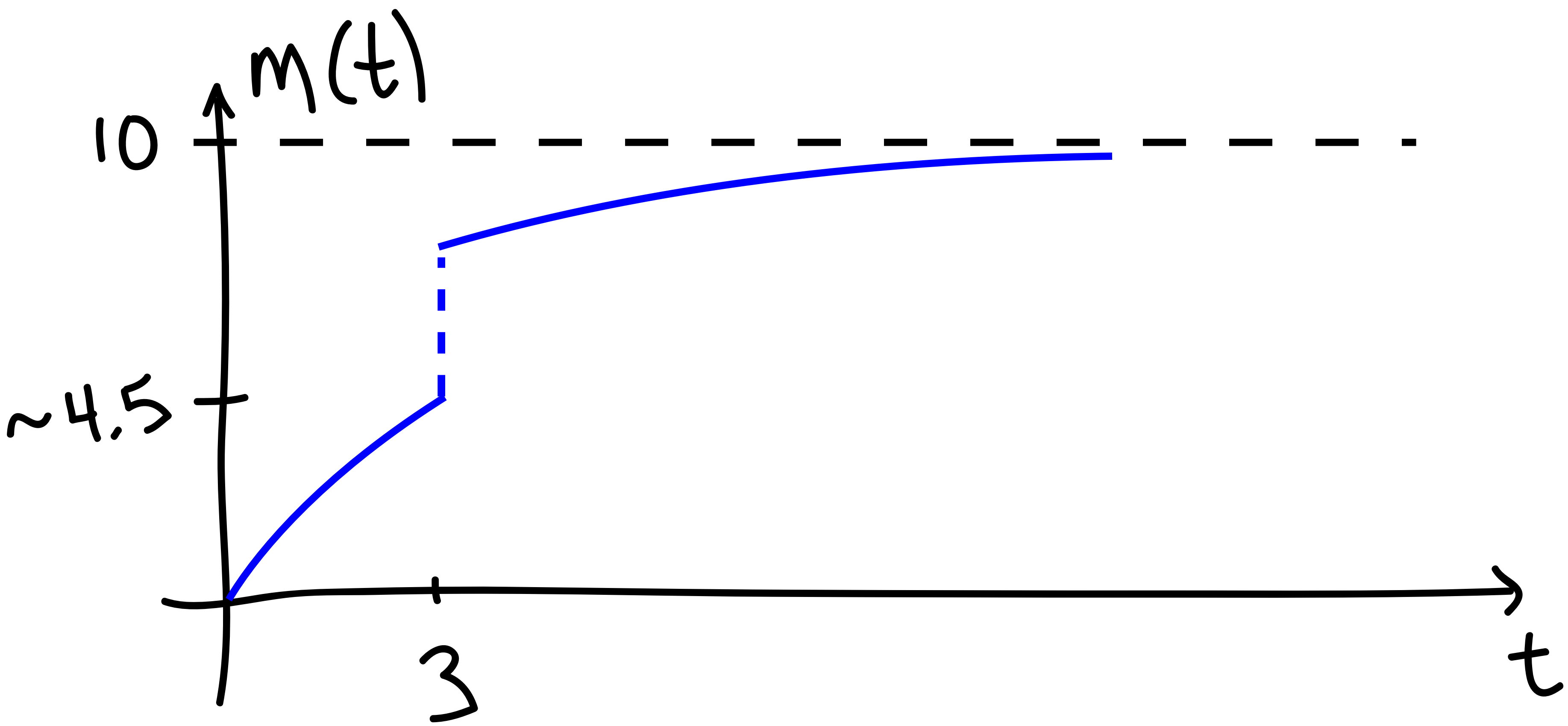

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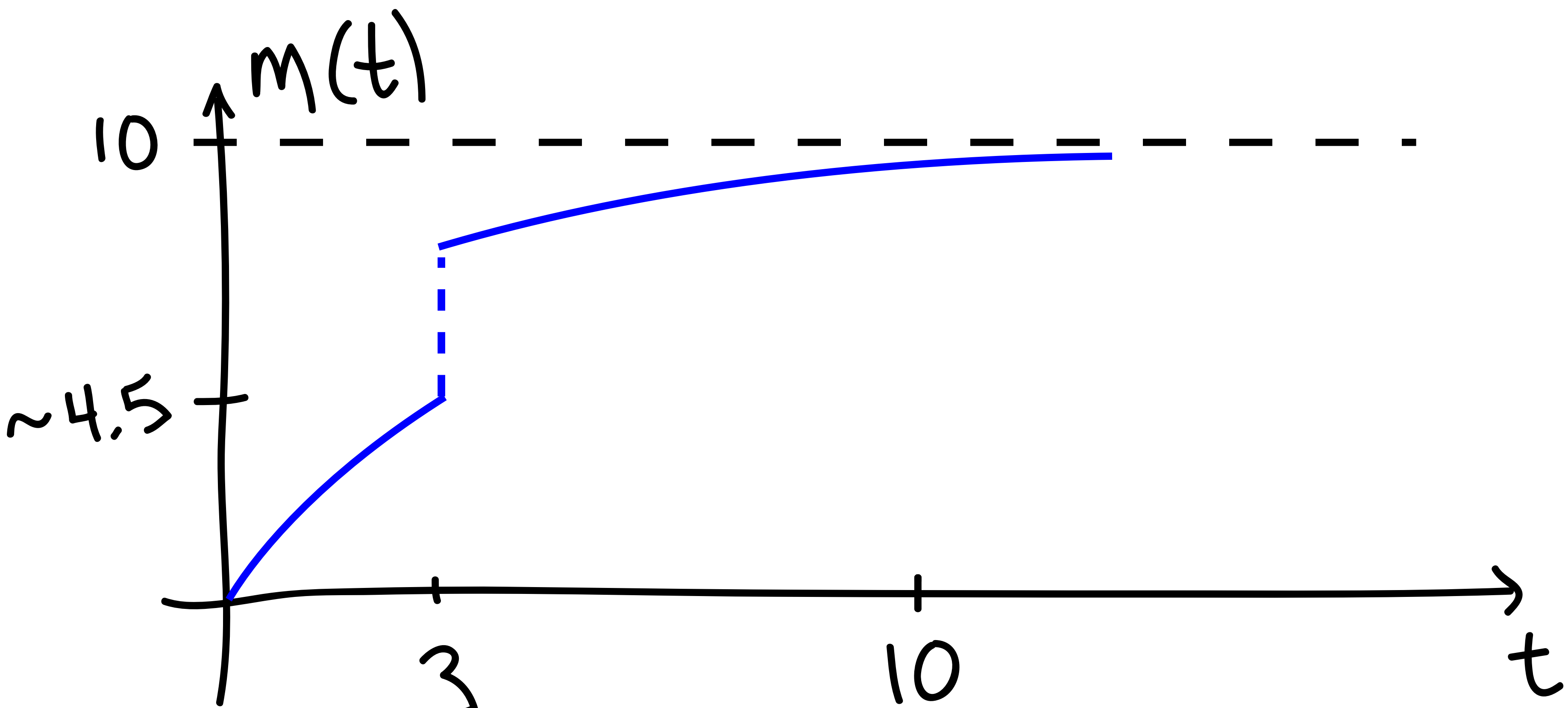

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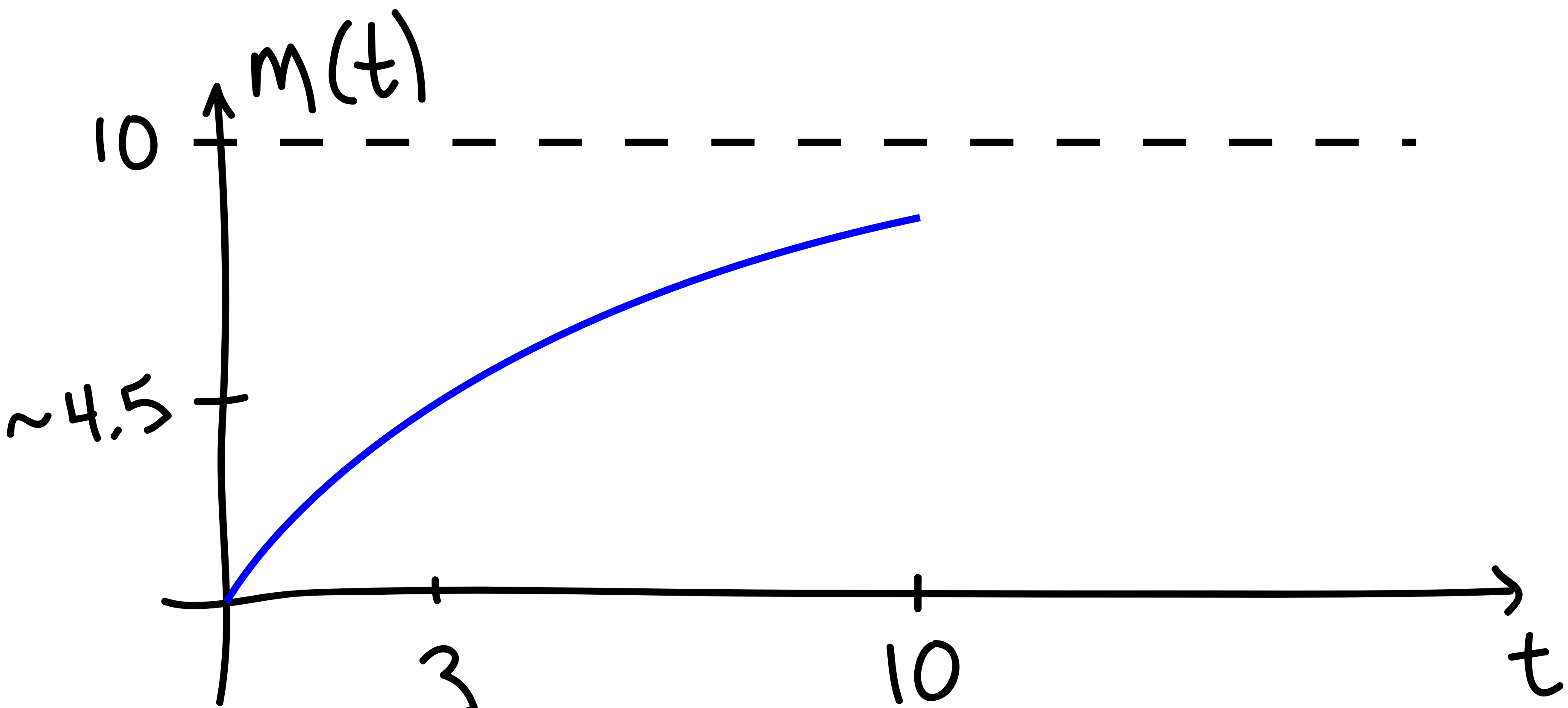

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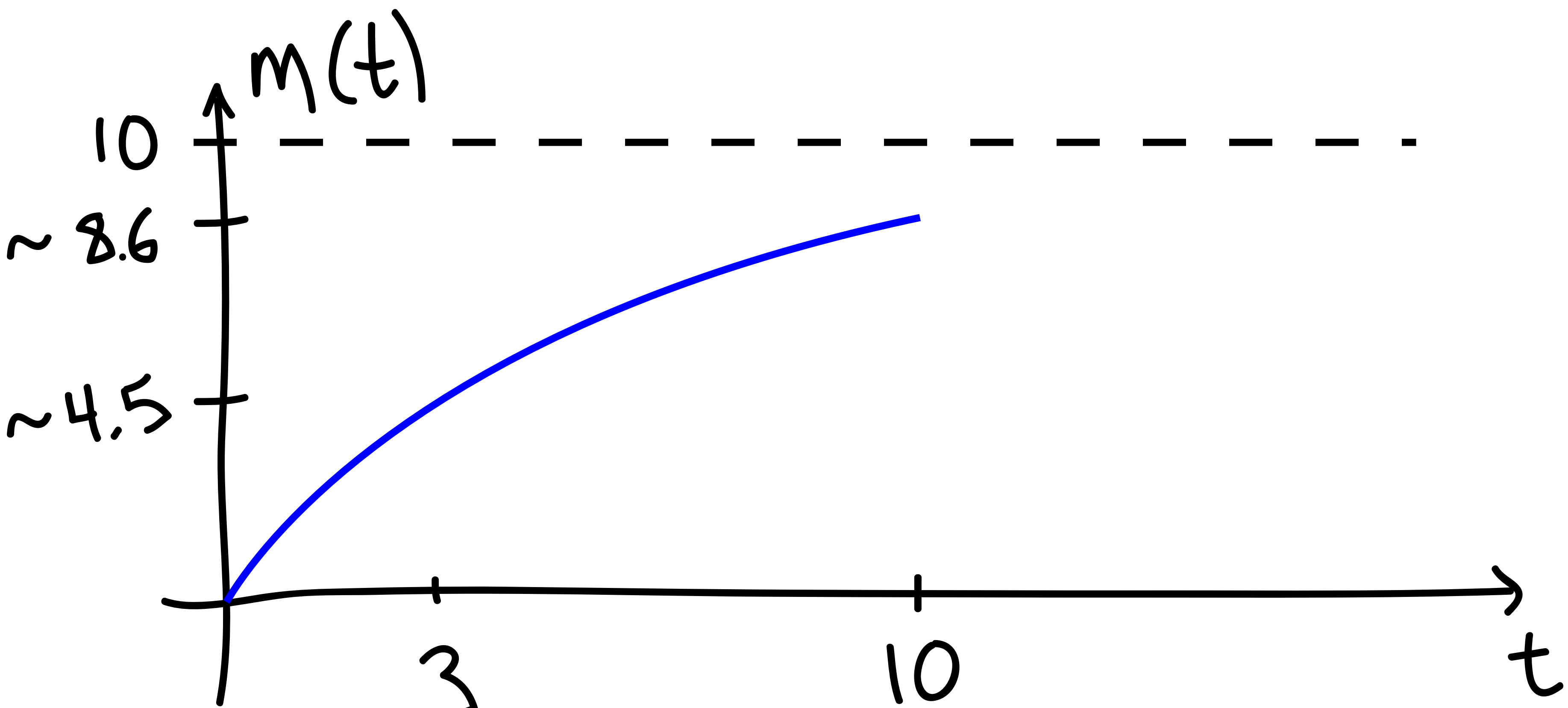

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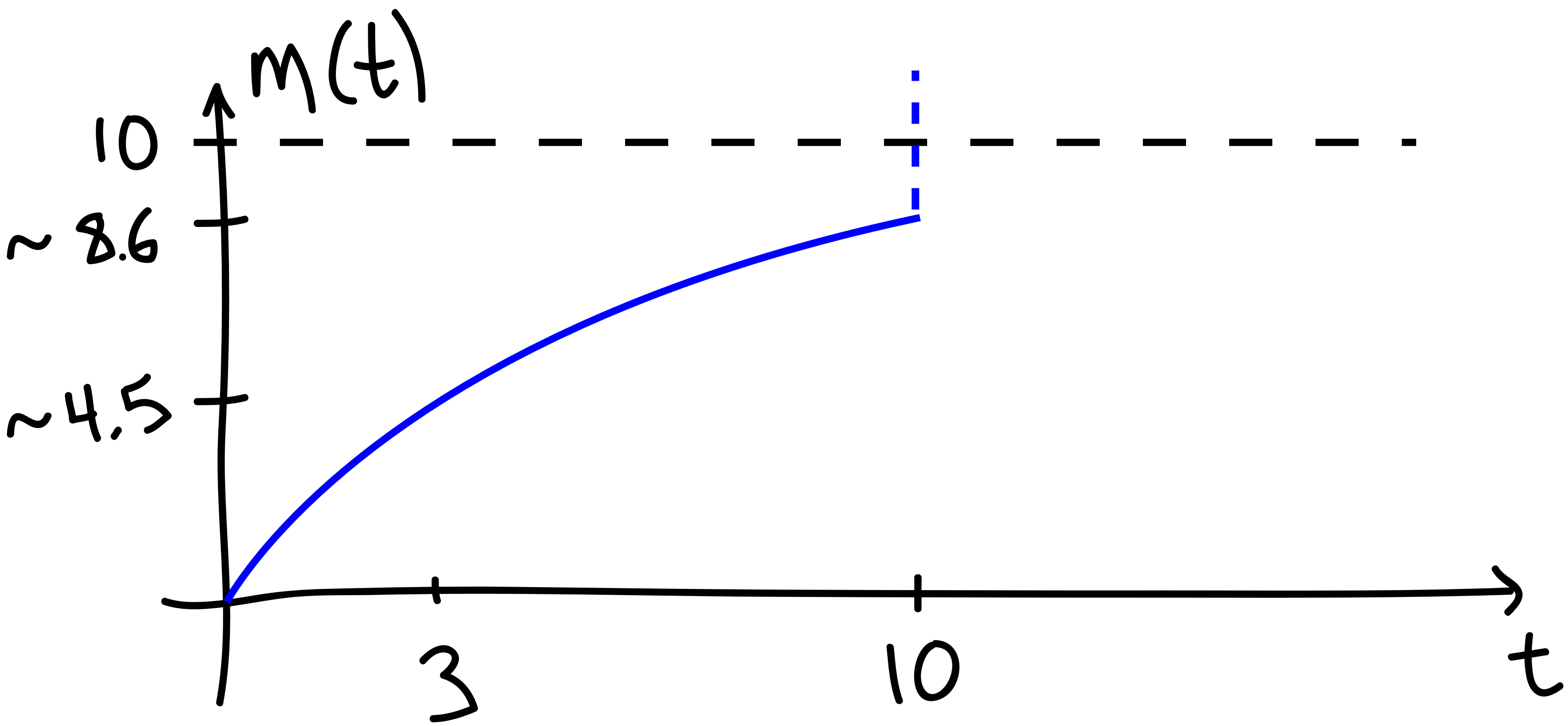

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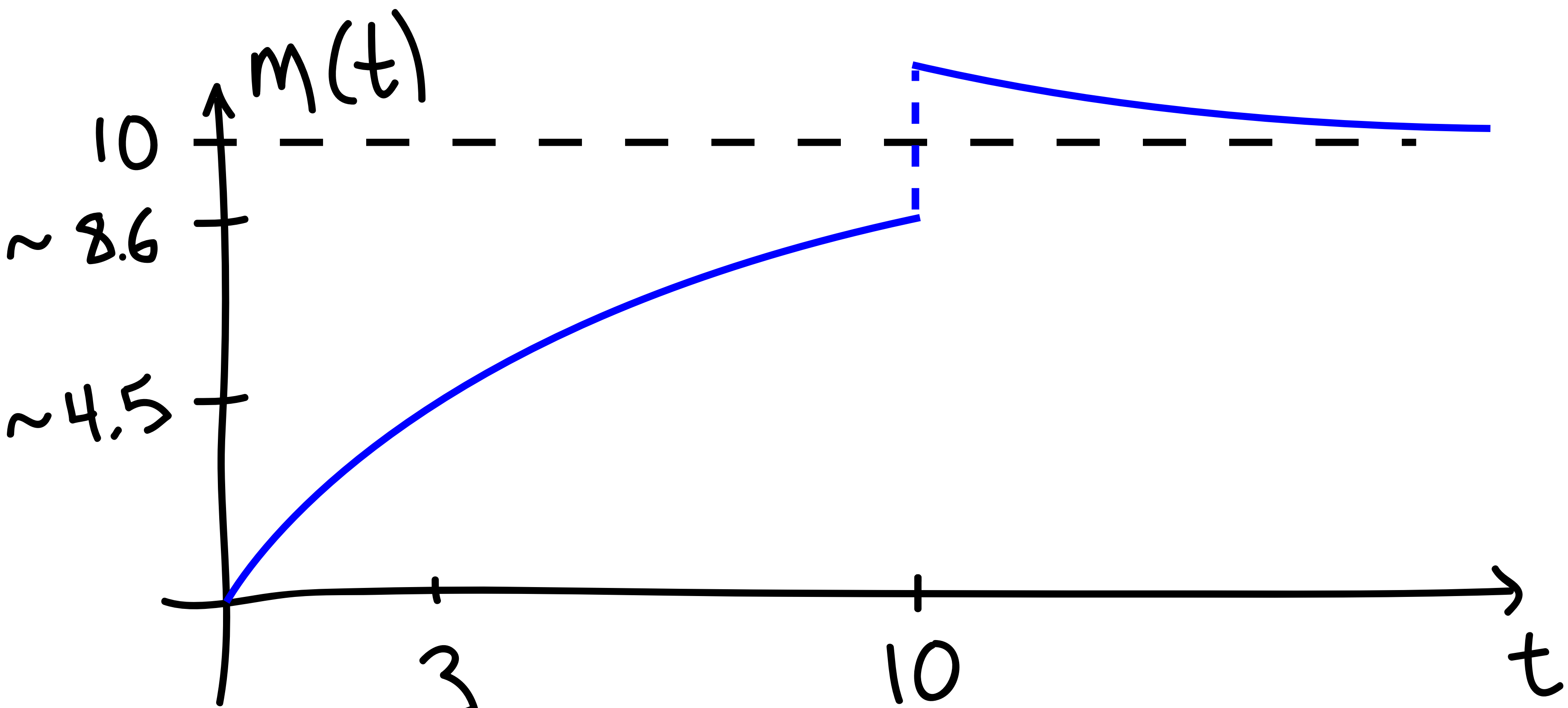

https://www.desmos.com/calculator/eizjtgf3jc

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- A hammer hits a mass-spring system imparting an impulse of $I_0 = 2 \text{ N s}$ at $t = 5 \text{ s}$. The mass of the block is $m = 1 \text{ kg}$. The drag coefficient is $\gamma = 2 \text{ kg/s}$ and the spring constant is $k = 10 \text{ kg/s}^2$. The mass is initially at $y(0) = 2 \text{ m}$ with velocity $y'(0) = 0 \text{ m/s}$.

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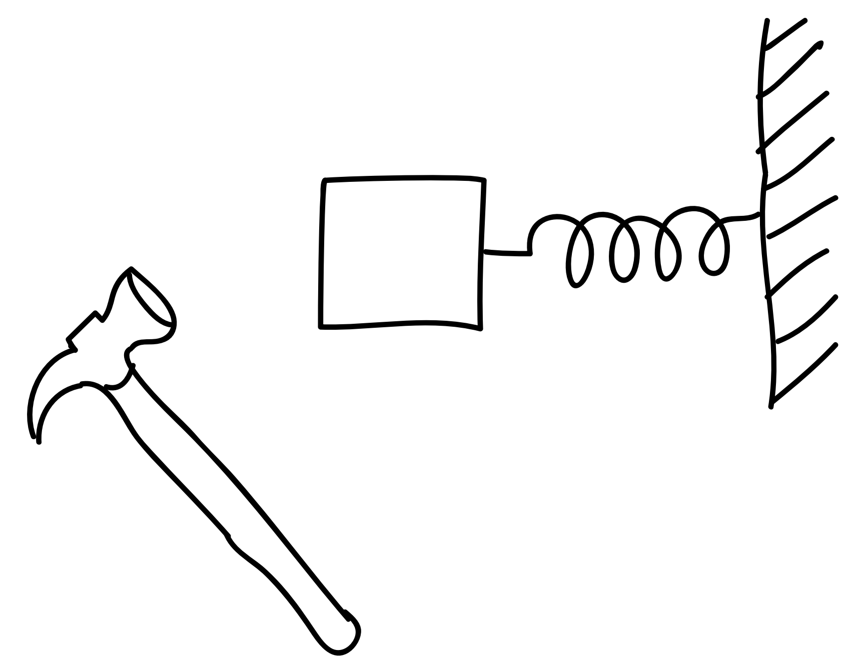

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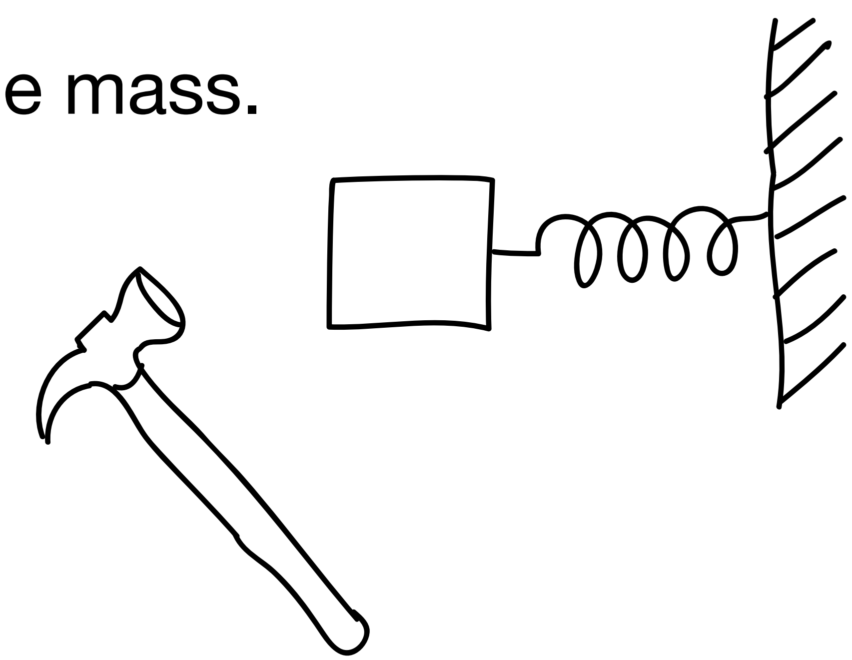

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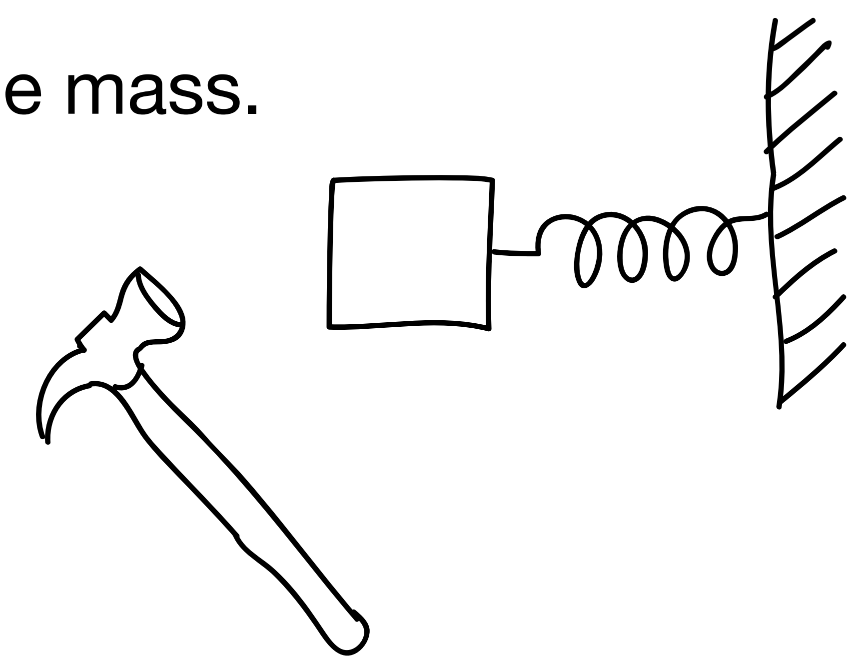
(A) $y'' + 2y' + 10y = 2 \ u_0(t)$

(B) $y'' + 2y' + 10y = 2 \ u_5(t)$

(C) $y'' + 2y' + 10y = 2 \ \delta(t)$

(D) $y'' + 2y' + 10y = 2 \ \delta(t-5)$



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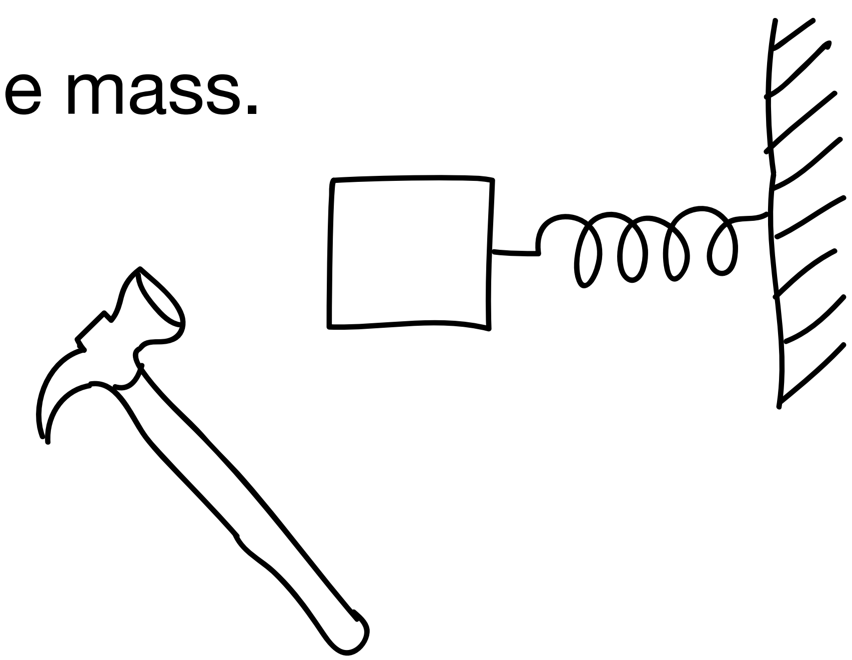
  - Write down an equation for the position of the mass.



(A) $y'' + 2y' + 10y = 2\ u_0(t)$

(B) $y'' + 2y' + 10y = 2\ u_5(t)$

(C) $y'' + 2y' + 10y = 2\ \delta(t)$

☆ (D) $y'' + 2y' + 10y = 2\ \delta(t-5)$

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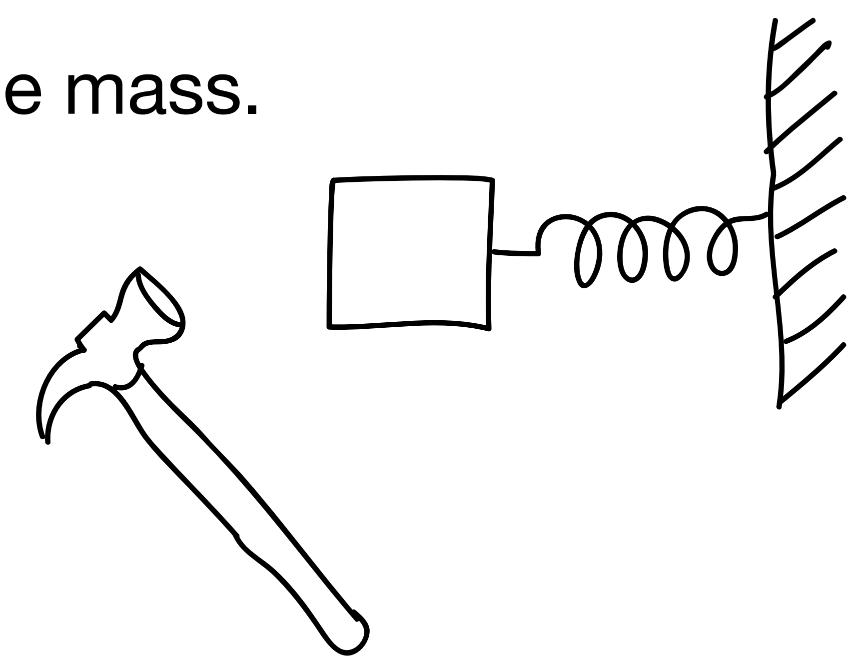

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$$s^2Y - 2s + 2sY - 4 + 10Y = 2e^{-5c}$$

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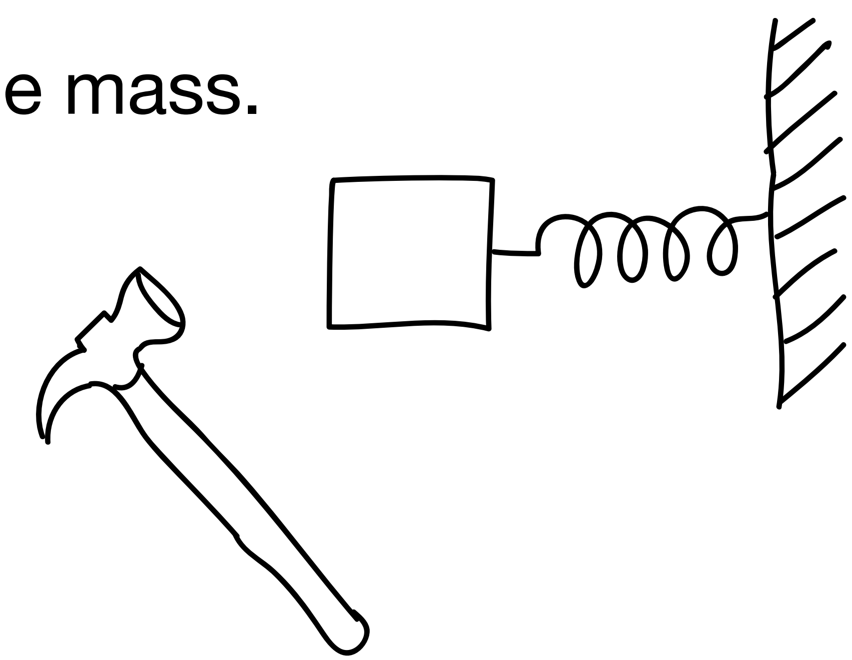

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$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10}$$

Delta-function forcing

- Inverting Y(s)... (go through this on your own)

$$Y(s) = \frac{2(e^{-5s}+s+2)}{s^2+2s+10} = \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+2}{s^2+2s+10}$$

$$= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+2}{(s+1)^2+9}$$

$$= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{(s+1)^2+9}$$

$$= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{3}\frac{3}{(s+1)^2+9}$$

$$= \frac{2}{3}\frac{3e^{-5s}}{(s+1)^2+9} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{3}\frac{3}{(s+1)^2+9}$$

$$y(t) = \underbrace{\frac{2}{3}u_5(t)e^{-(t-5)}\sin(3(t-5))}_{\text{particular solution from } \delta \text{ forcing}} + \underbrace{2e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)}_{\text{homogeneous part}}$$