

Today

- Shapes of solutions for distinct eigenvalues case.
- General solution for complex eigenvalues case.
- Shapes of solutions for complex eigenvalues case.
- Office hours: Friday 1-2 pm, Monday 1-3 pm (to be confirmed)

Shapes of solution curves in the phase plane

- When matrix A has distinct eigenvalues, the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

- What do solutions look like in the x_1 - x_2 plane (called the **phase plane**)?

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- What do solutions look like in the x_1 - x_2 plane (called the **phase plane**)?
- If the initial condition is an eigenvector, then the solution is a straight line.

Example:

$$x_1' = x_1 + x_2$$

$$x_1(0) = 6$$

$$x_2' = 4x_1 + x_2$$

$$x_2(0) = -12$$

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$$\begin{aligned} x_1' &= x_1 + x_2 & x_1(0) &= 6 \\ x_2' &= 4x_1 + x_2 & x_2(0) &= -12 \end{aligned}$$

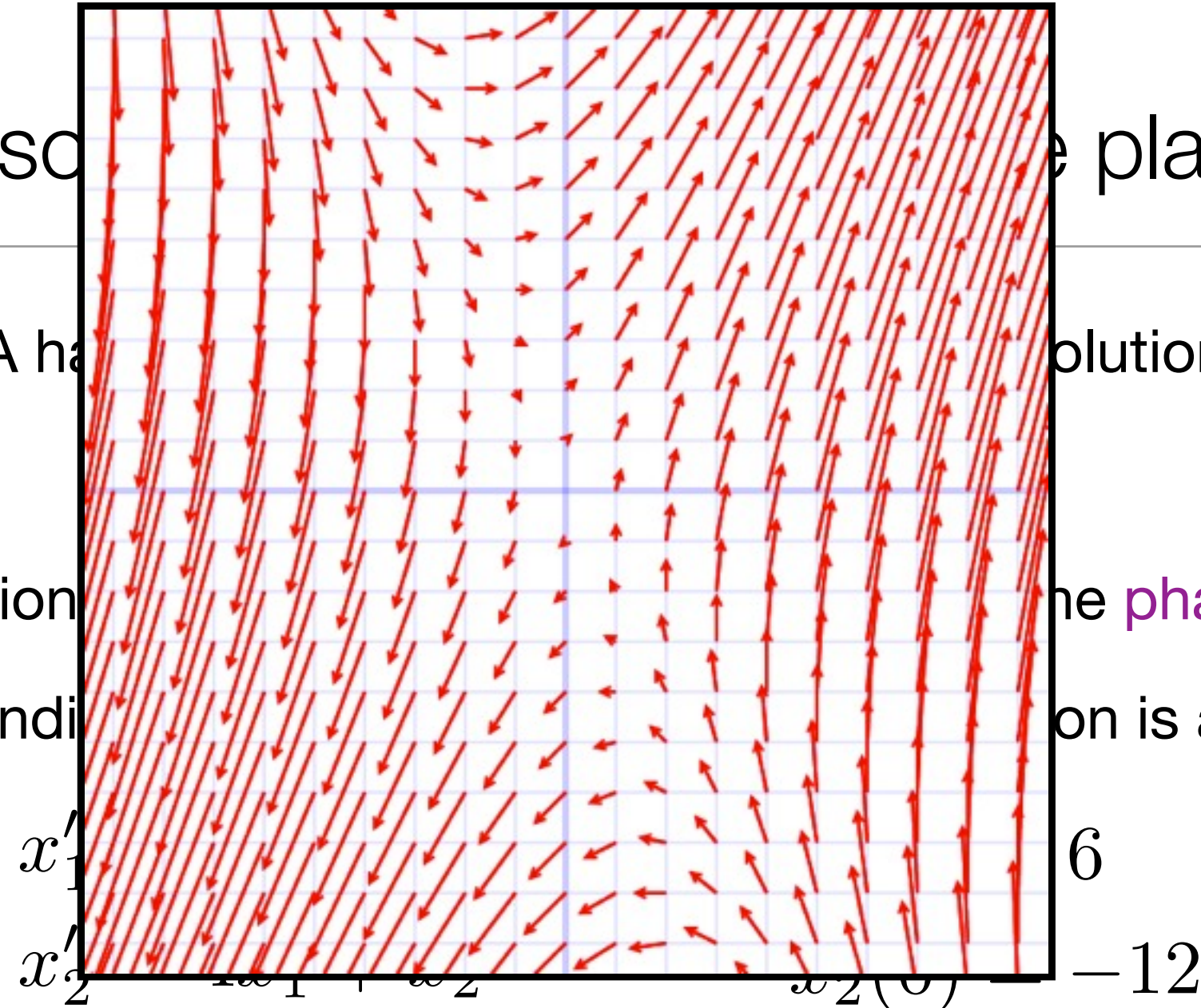
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$C_1 = 6, C_2 = 0$

Shapes of solutions in the phase plane

- When matrix A has eigenvalues λ_1, λ_2 , a solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.
- What do solutions look like in the phase plane?
- If the initial condition $\mathbf{x}(0)$ is a straight line, the solution is a straight line.



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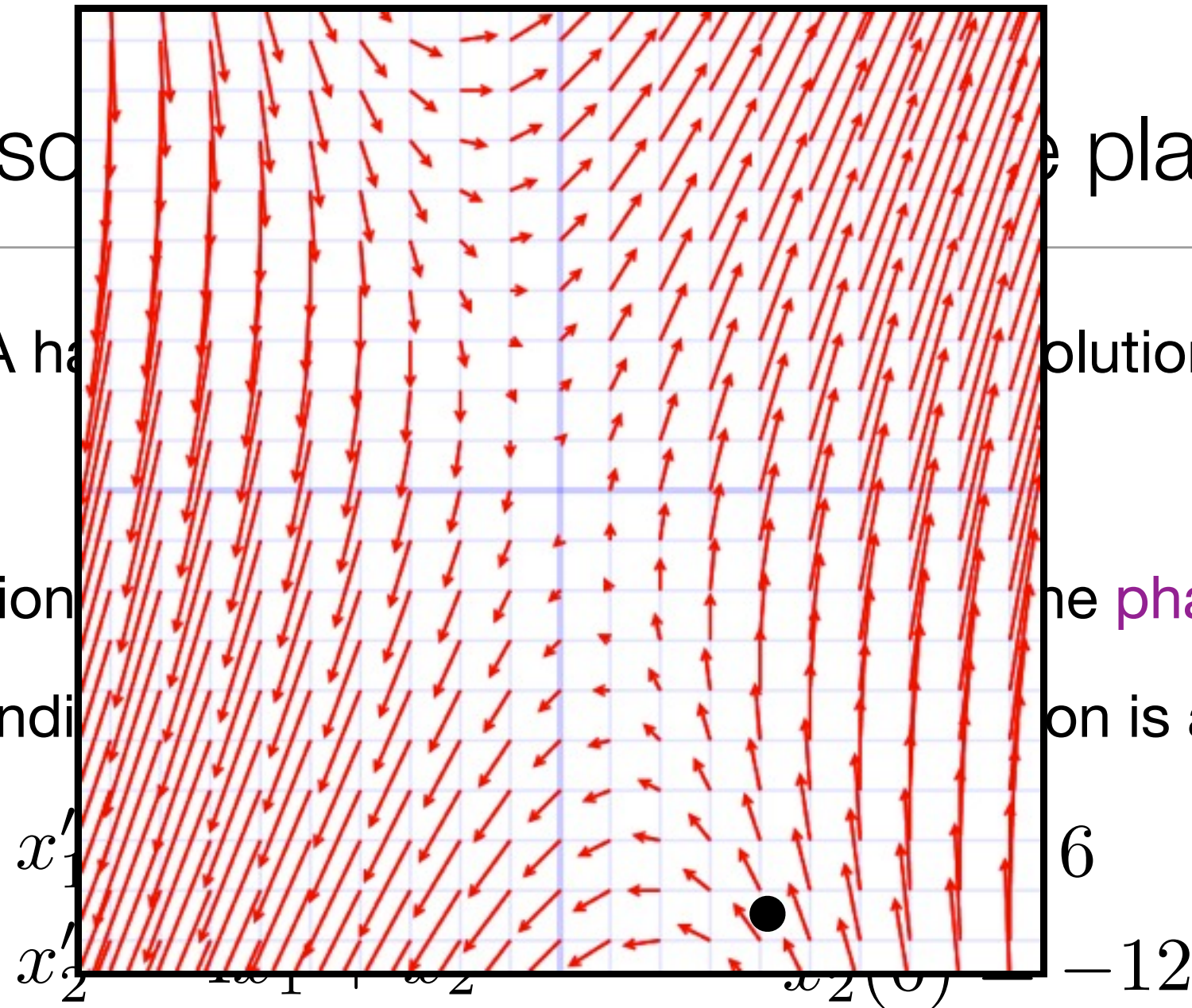
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Shapes of solutions in the phase plane

- When matrix A has eigenvalues $\lambda_1 < 0 < \lambda_2$, the solution to $\mathbf{x}' = A\mathbf{x}$ is a curve in the phase plane.
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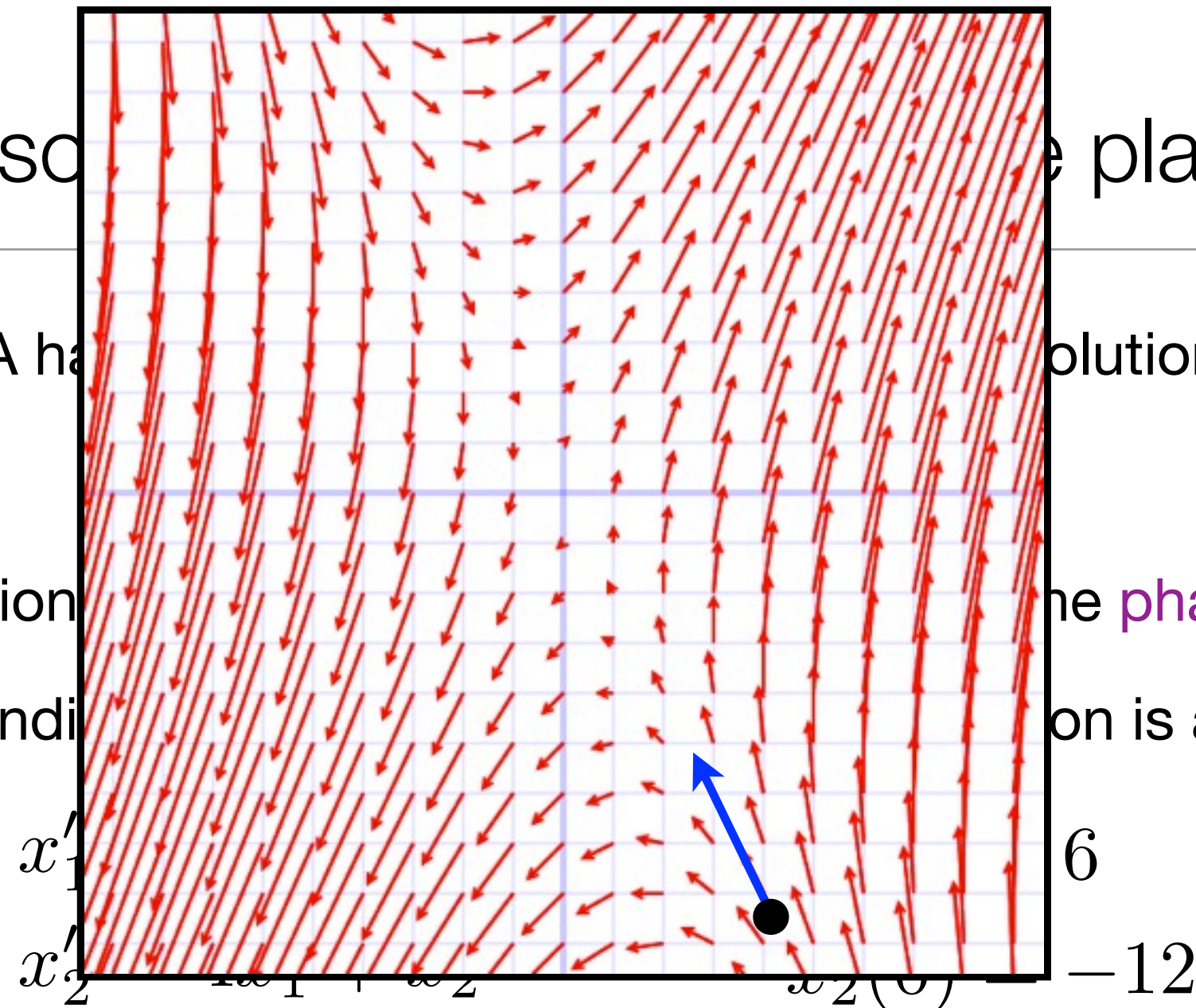
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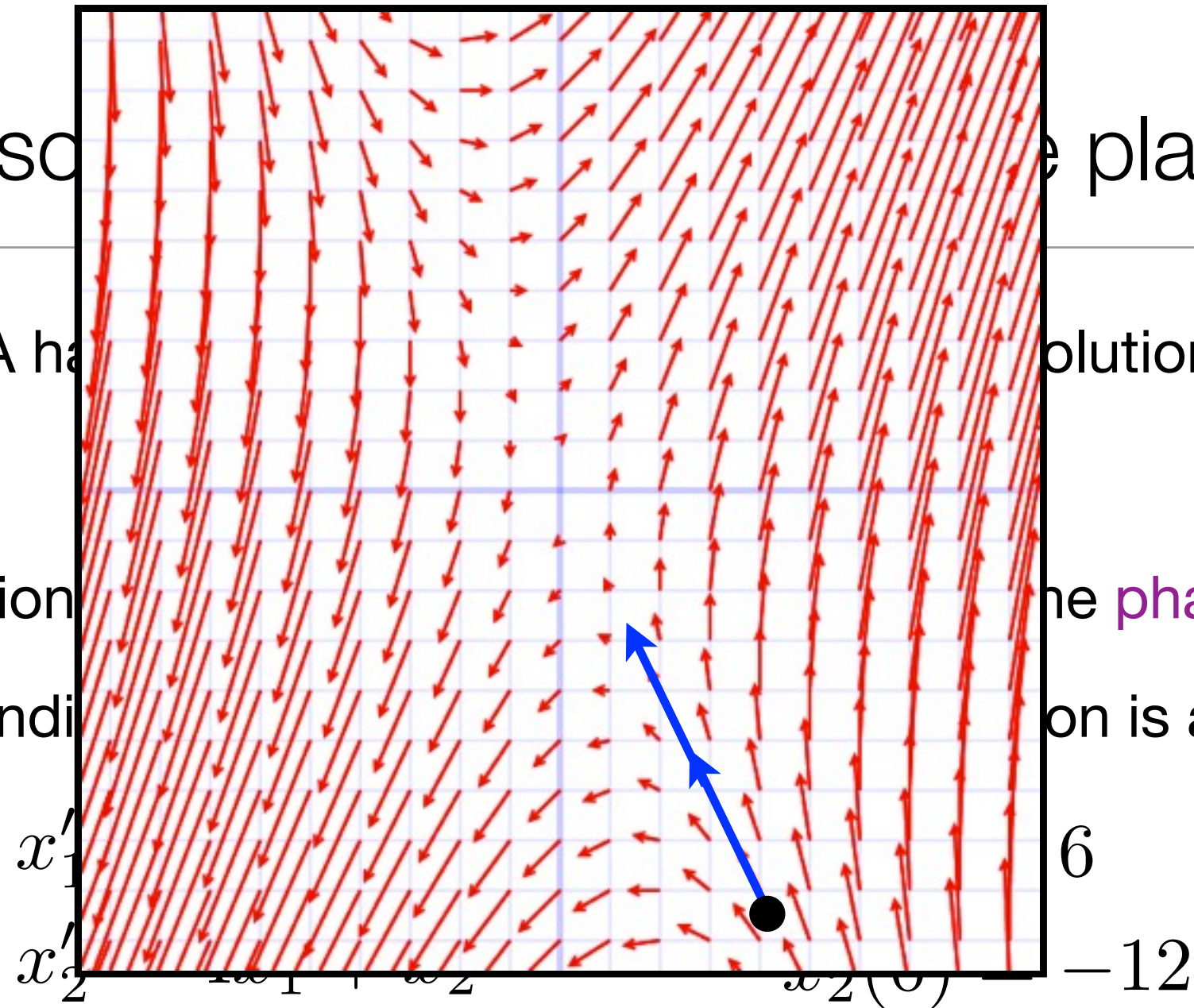
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Shapes of solutions in the phase plane

- When matrix A has real eigenvalues, the solution to $\mathbf{x}' = A\mathbf{x}$ is a straight line in the phase plane?
- What do solutions look like in the phase plane?
- If the initial condition is a straight line. Example:



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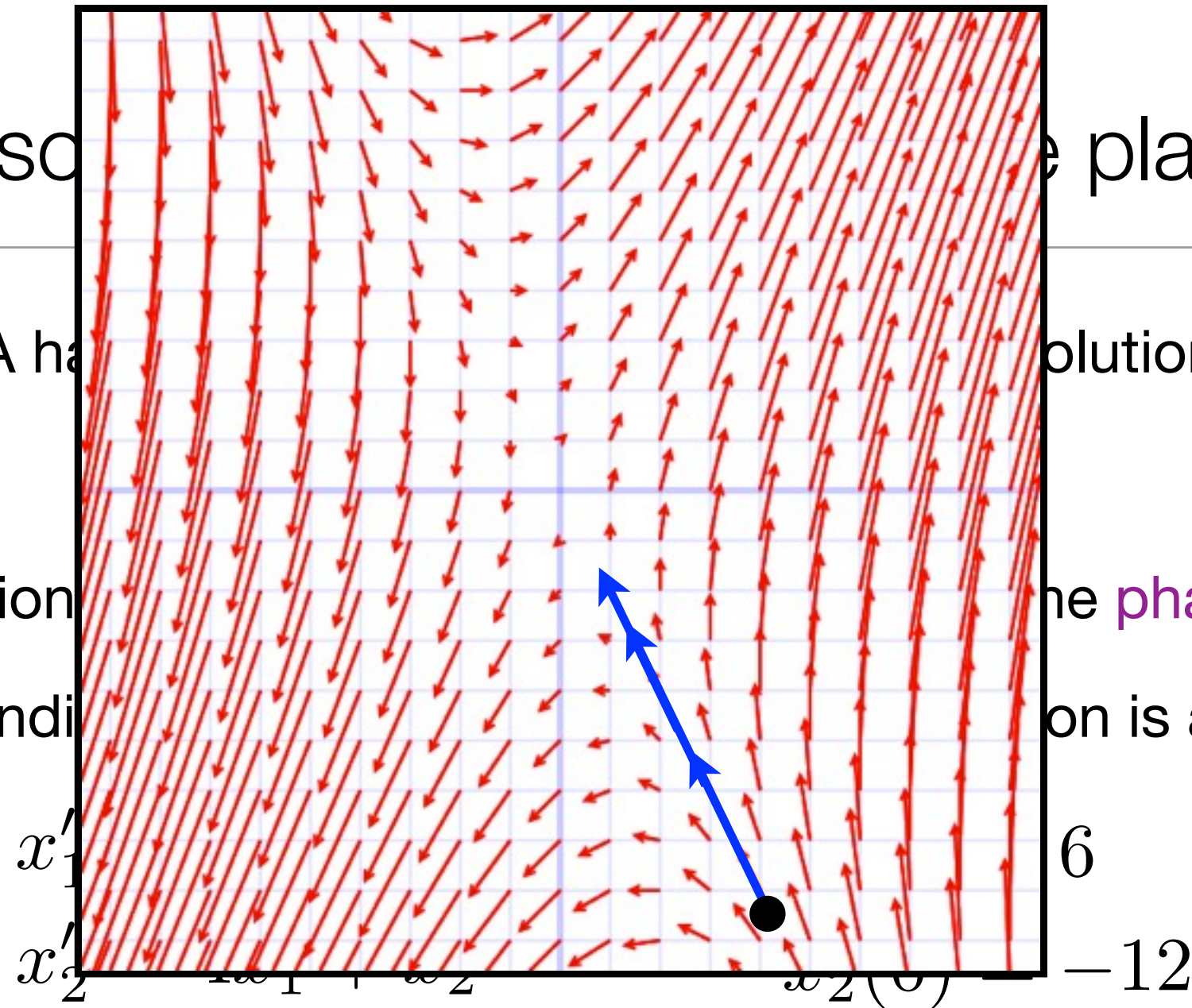
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- When matrix A has two real eigenvalues, the solution to $\mathbf{x}' = A\mathbf{x}$ is a straight line in the phase plane.
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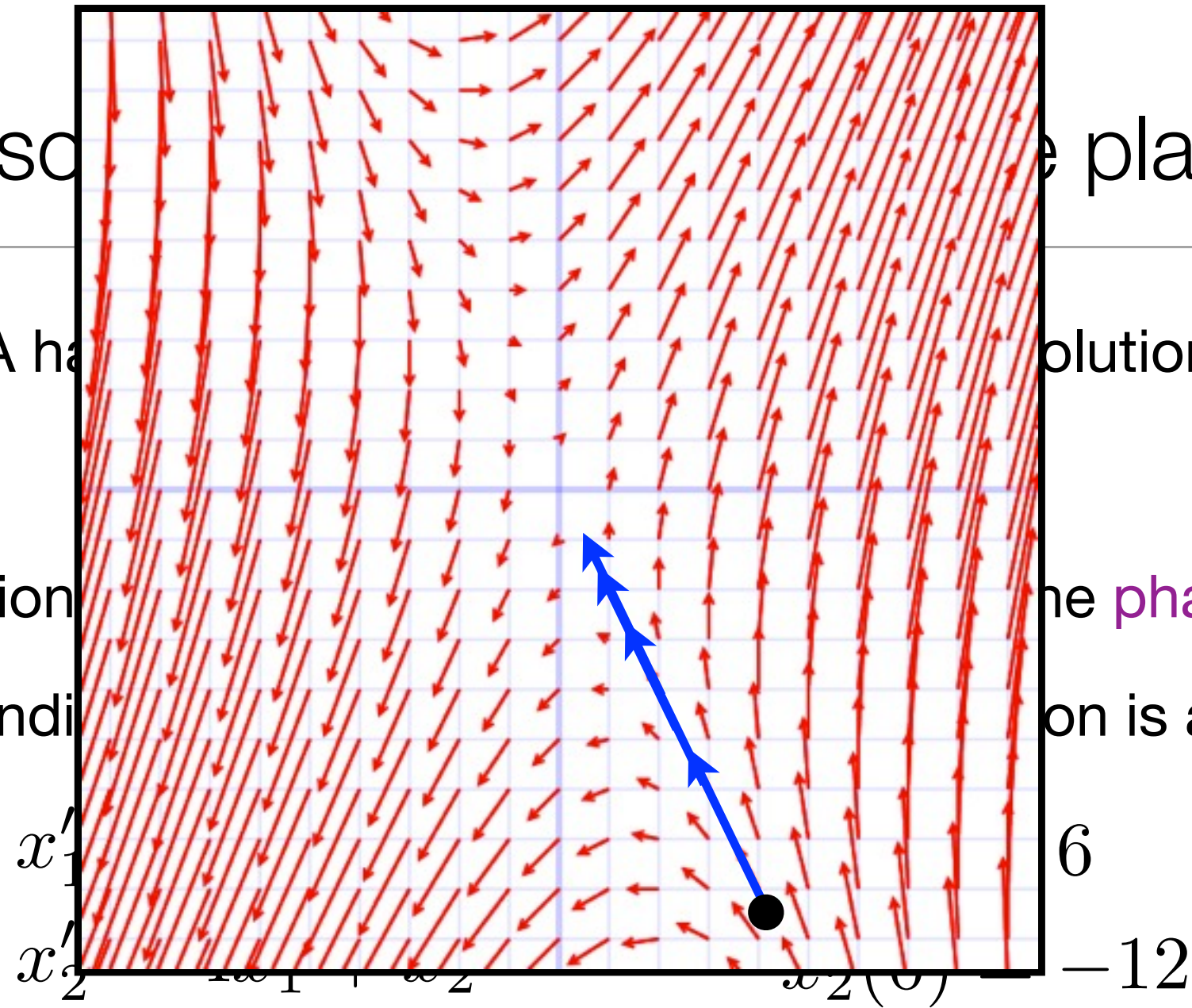
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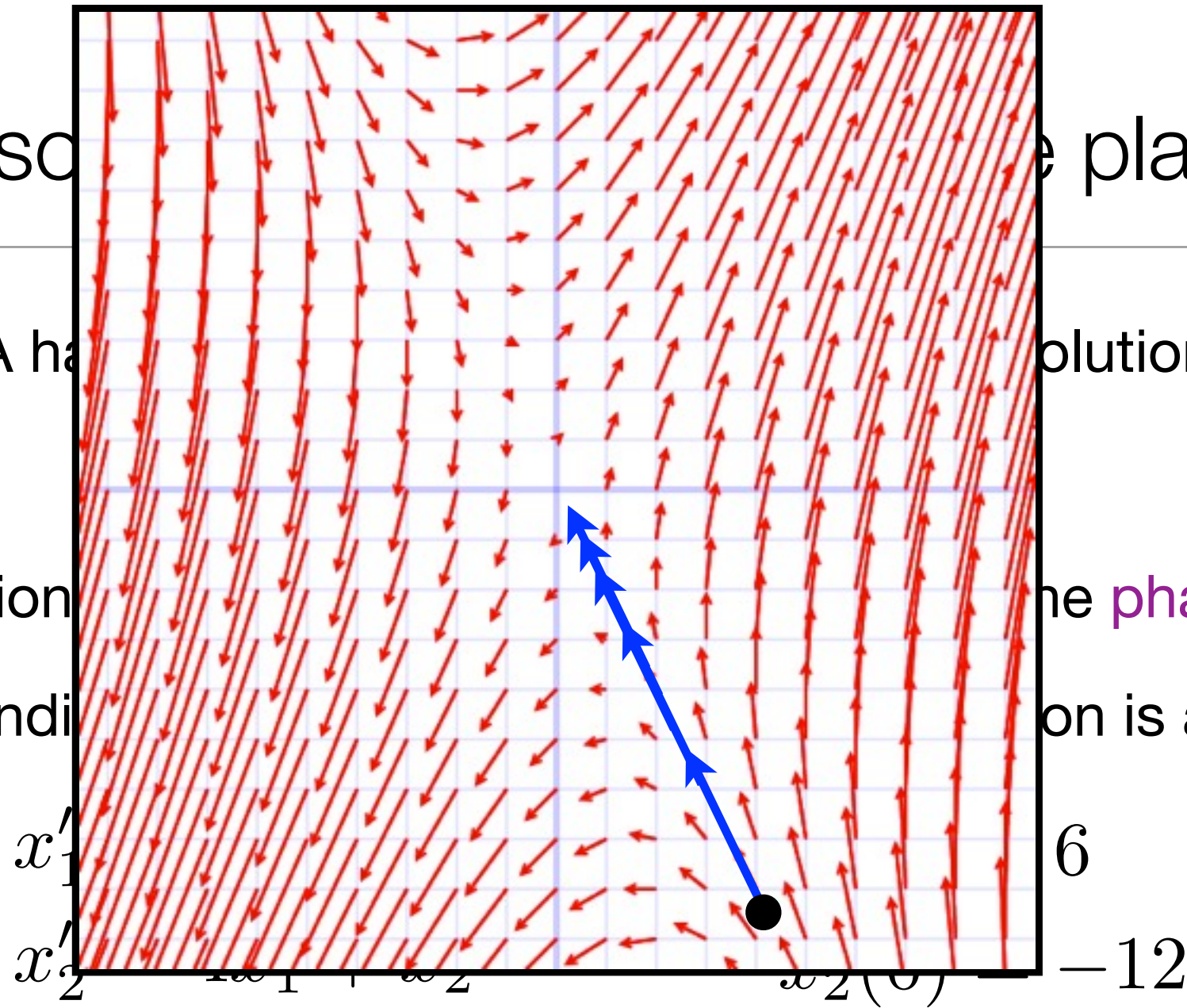
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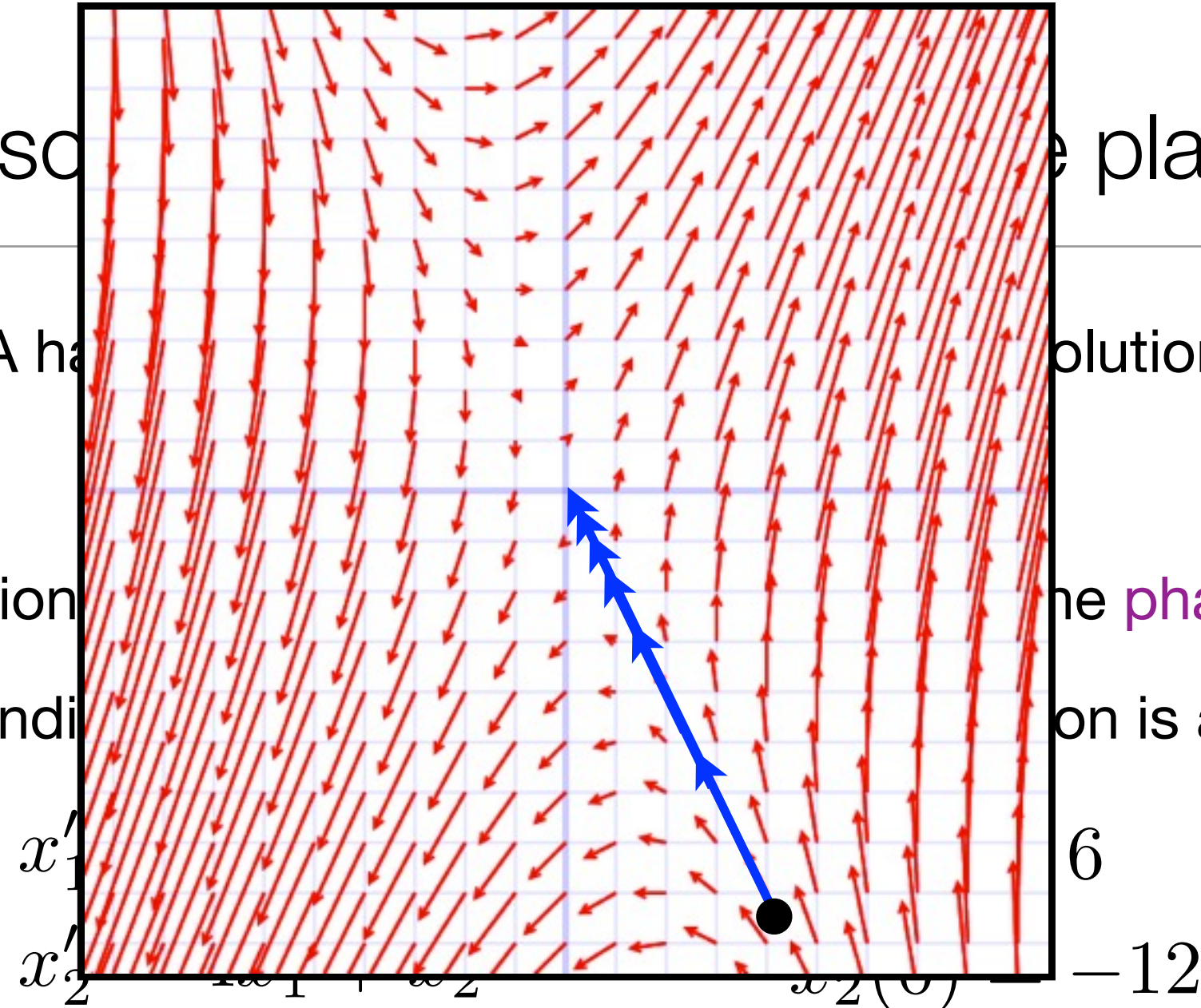
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- When matrix A has eigenvalues $\lambda_1 < 0 < \lambda_2$, the solution to $\mathbf{x}' = A\mathbf{x}$ is a curve in the phase plane that approaches the origin as $t \rightarrow \infty$.
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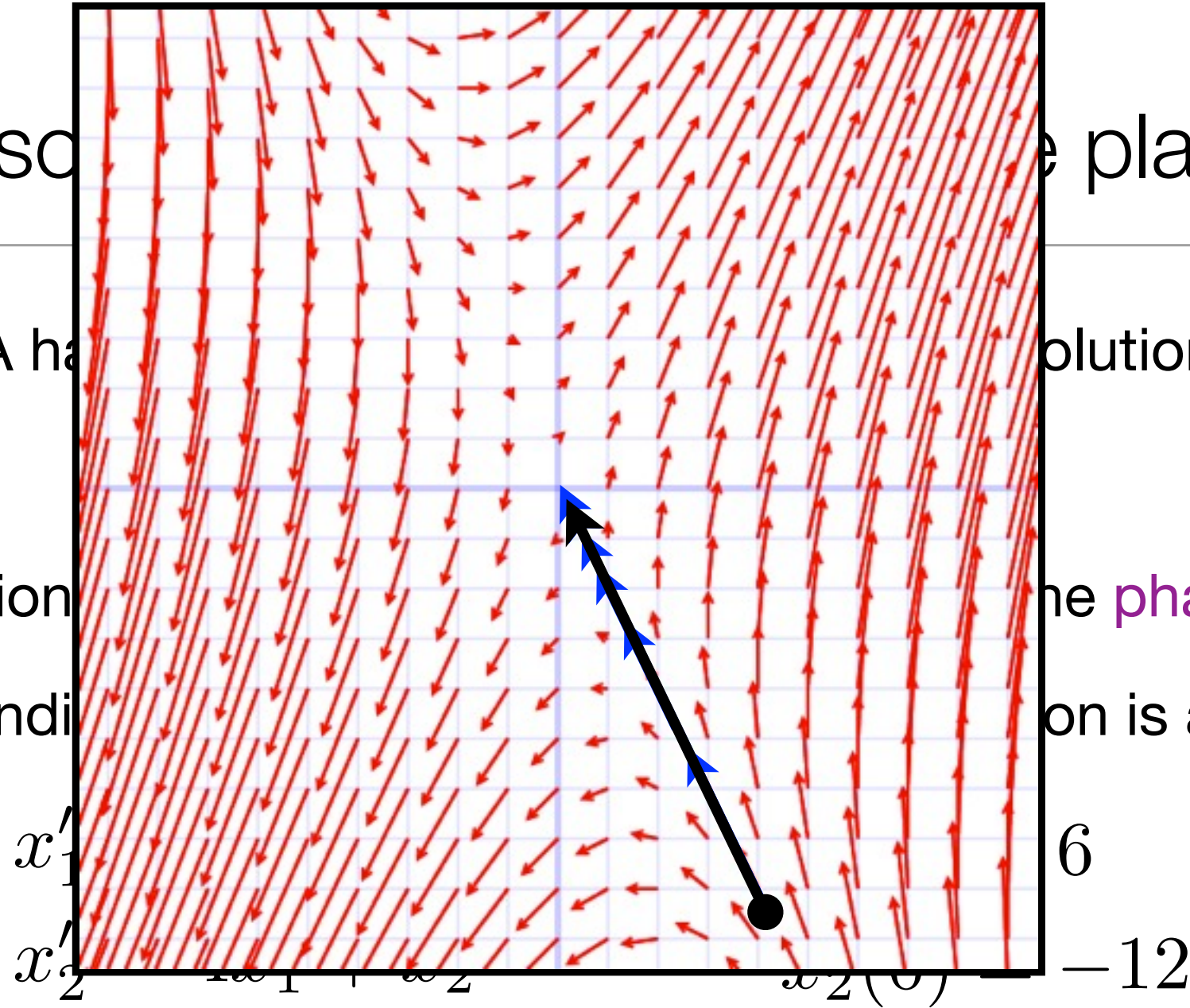
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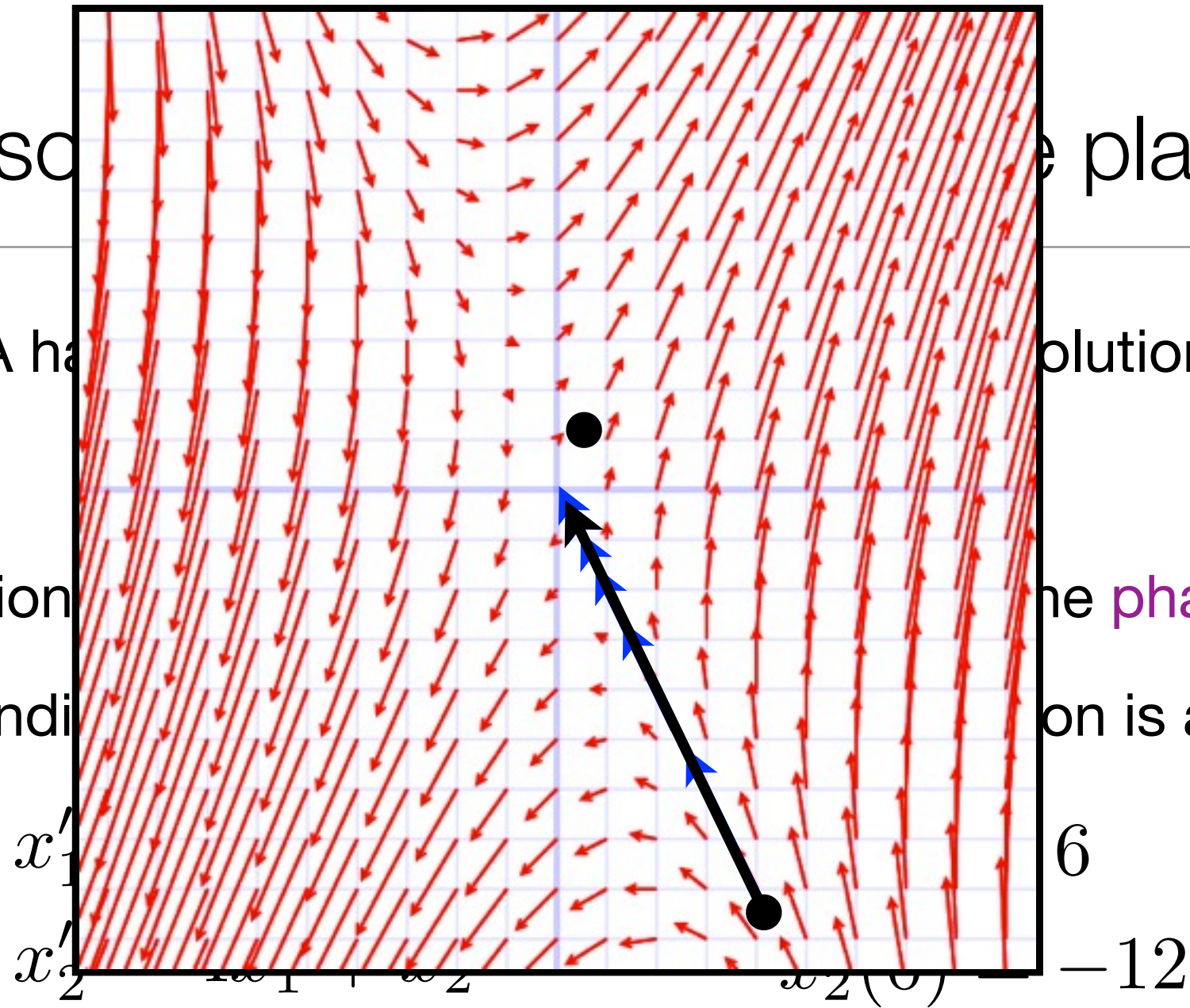
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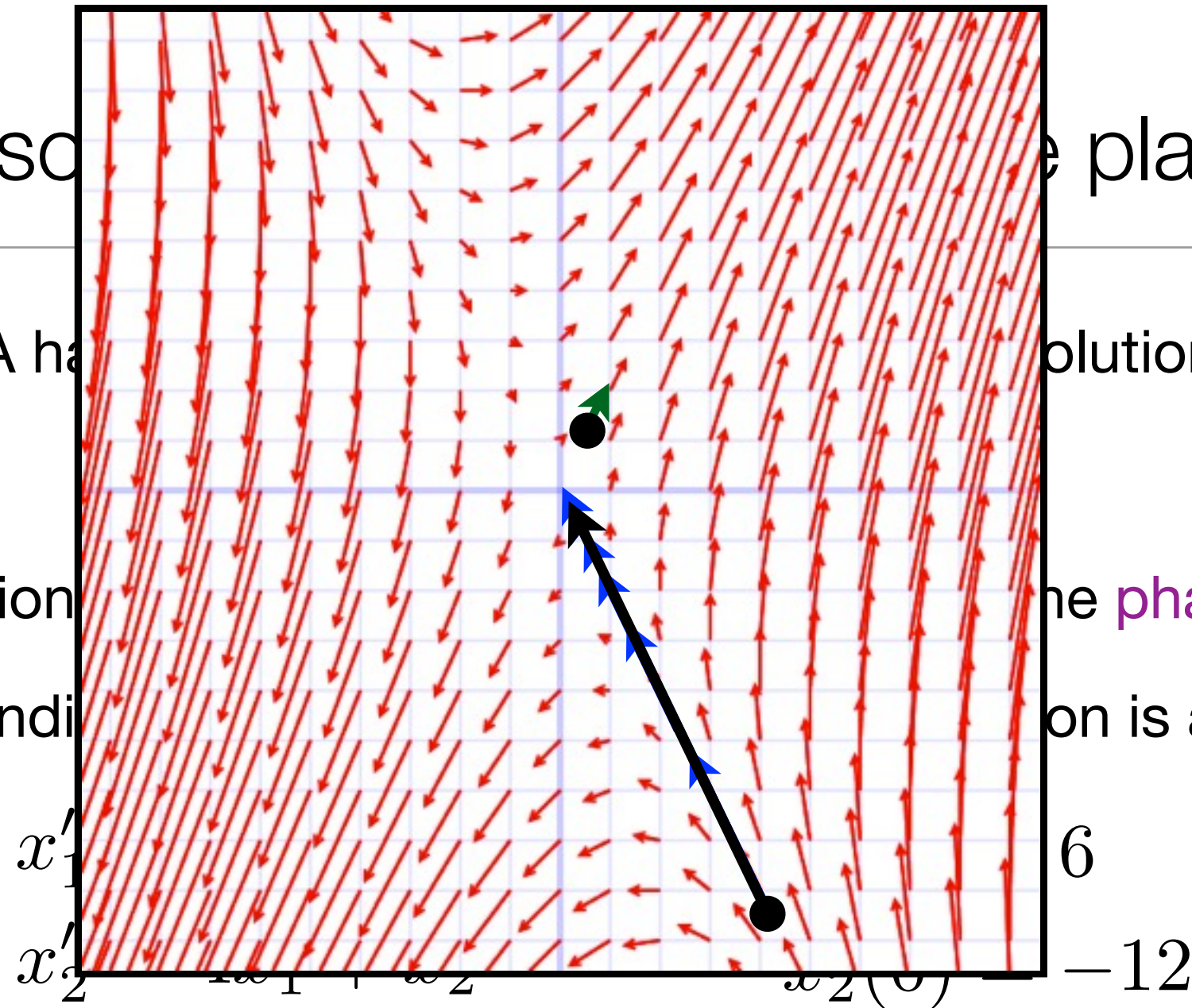
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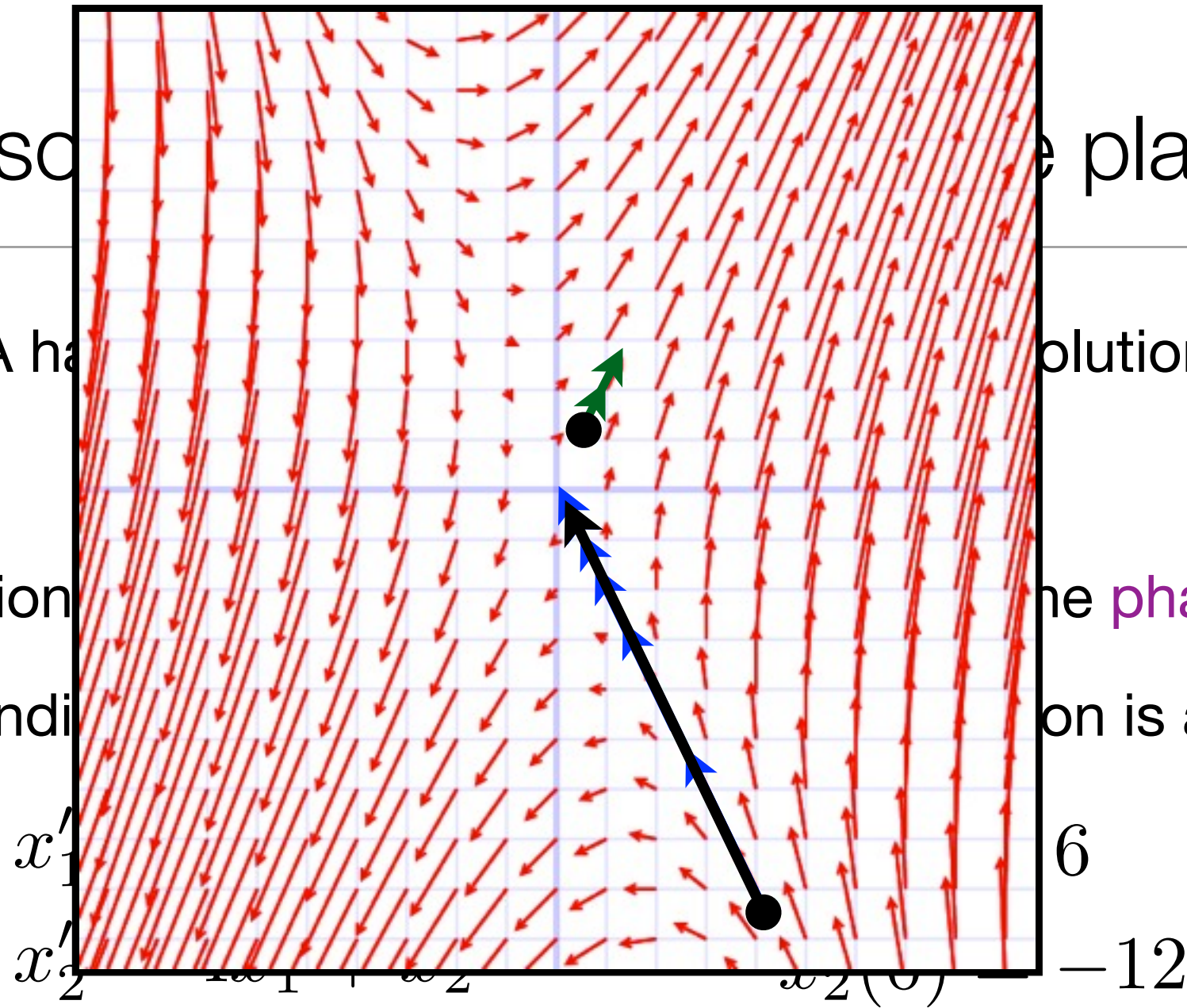
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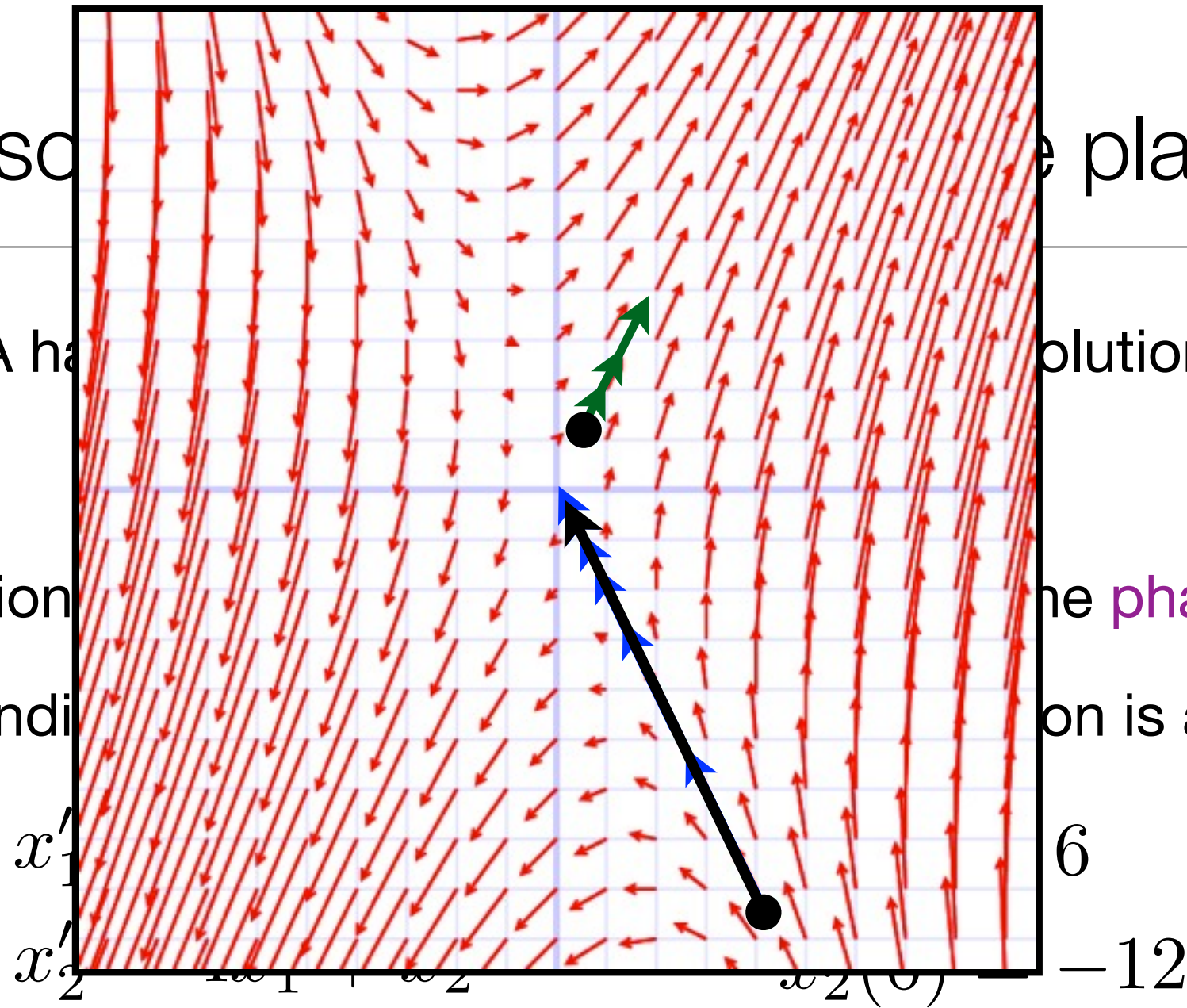
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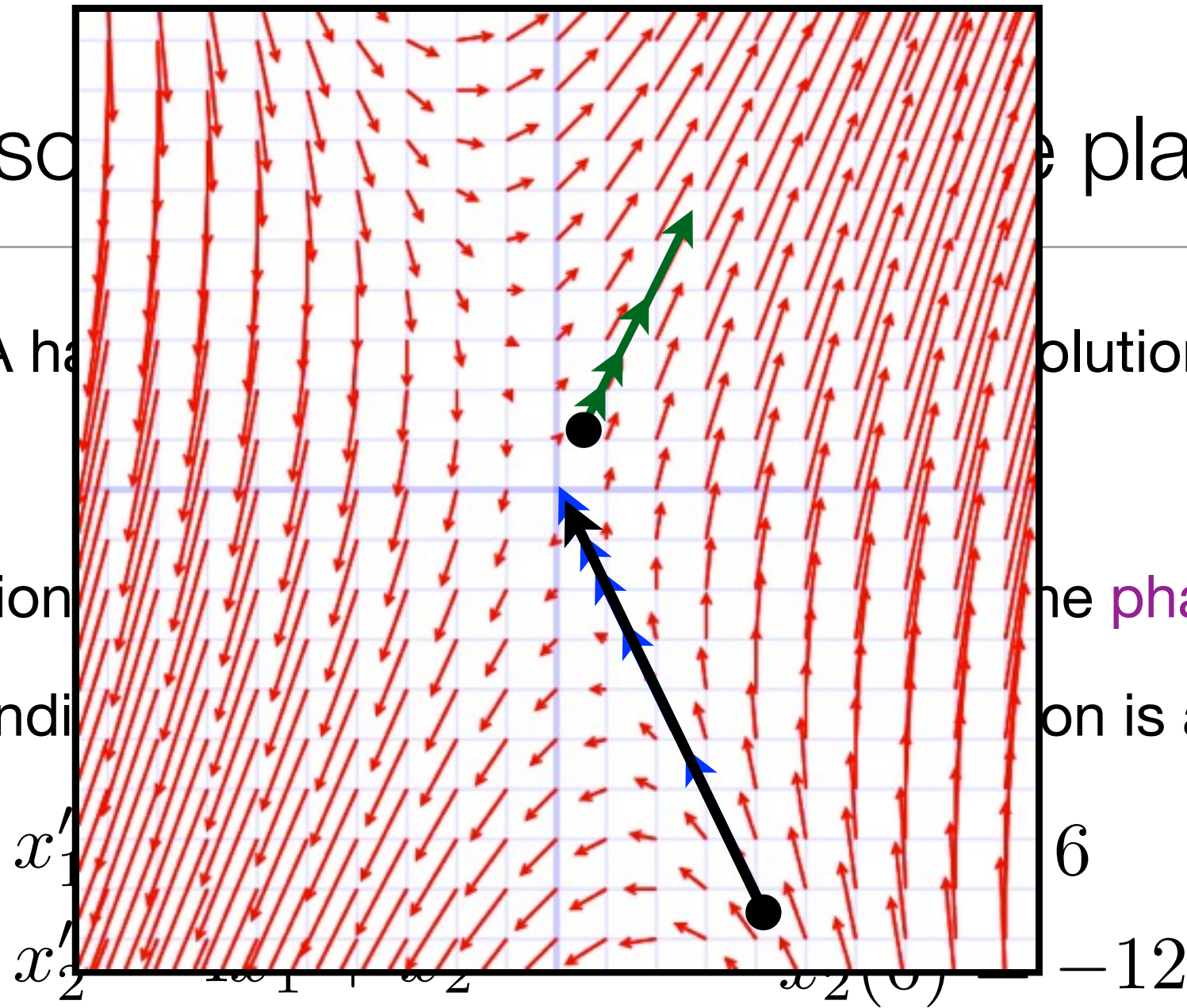
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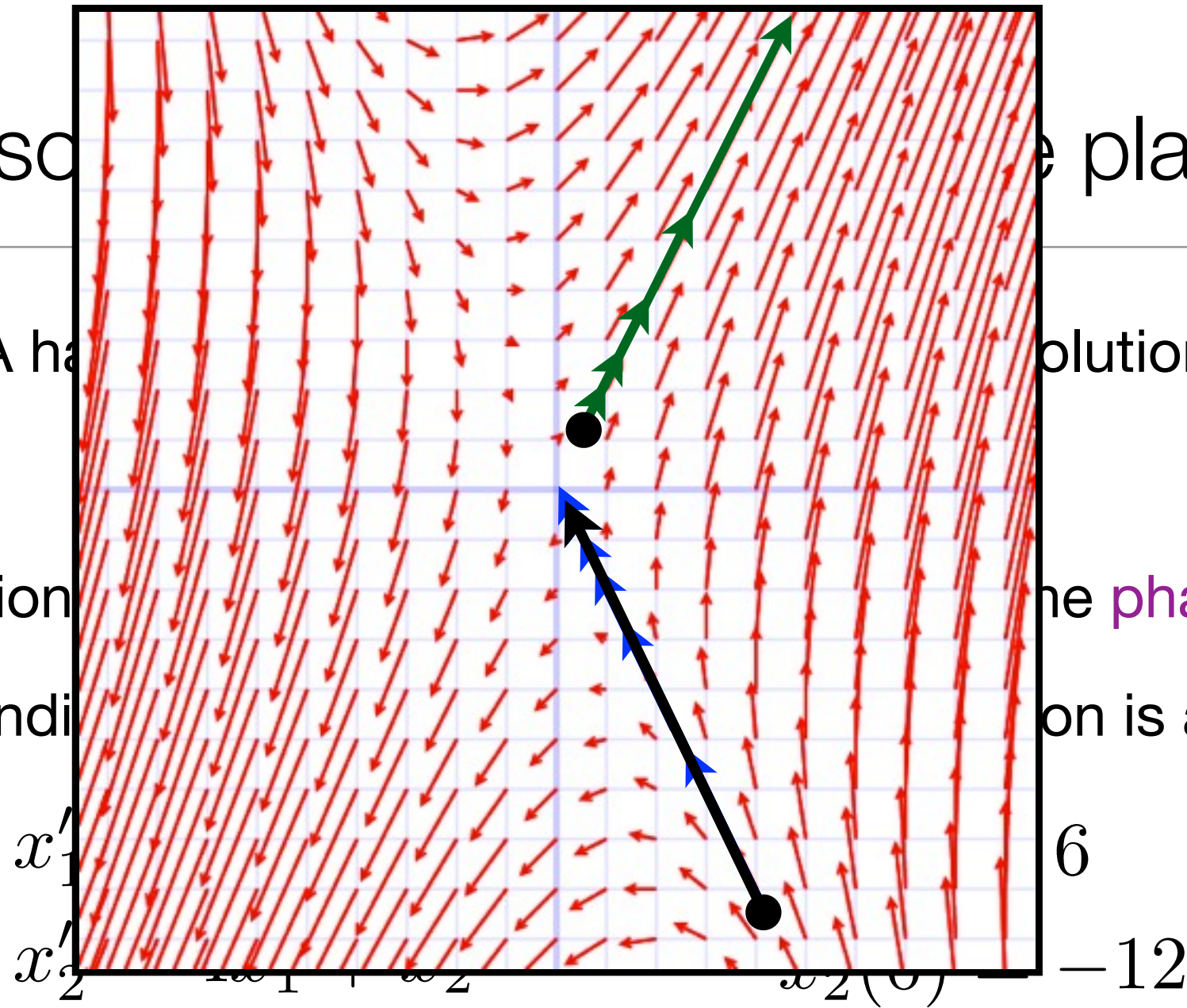
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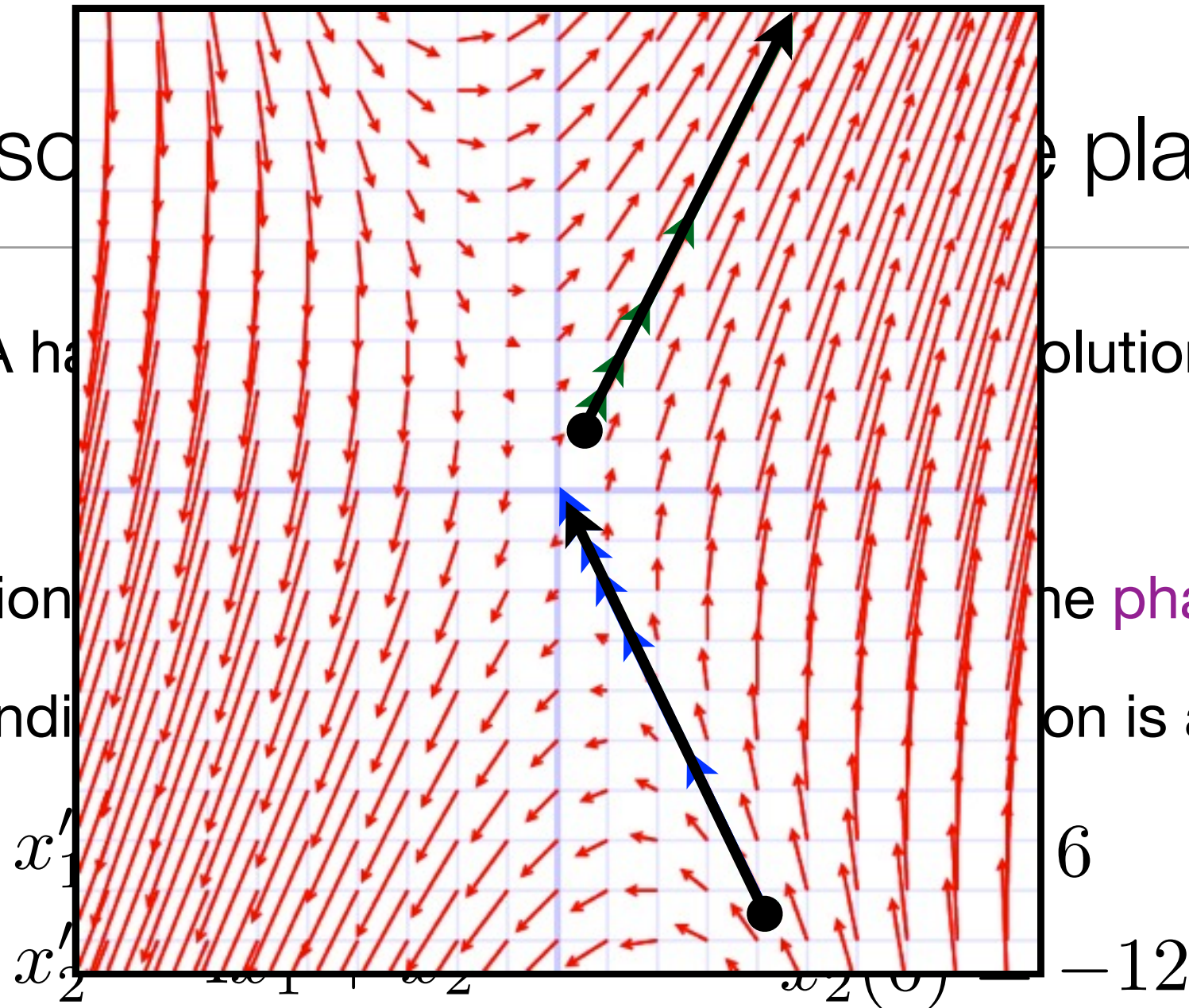
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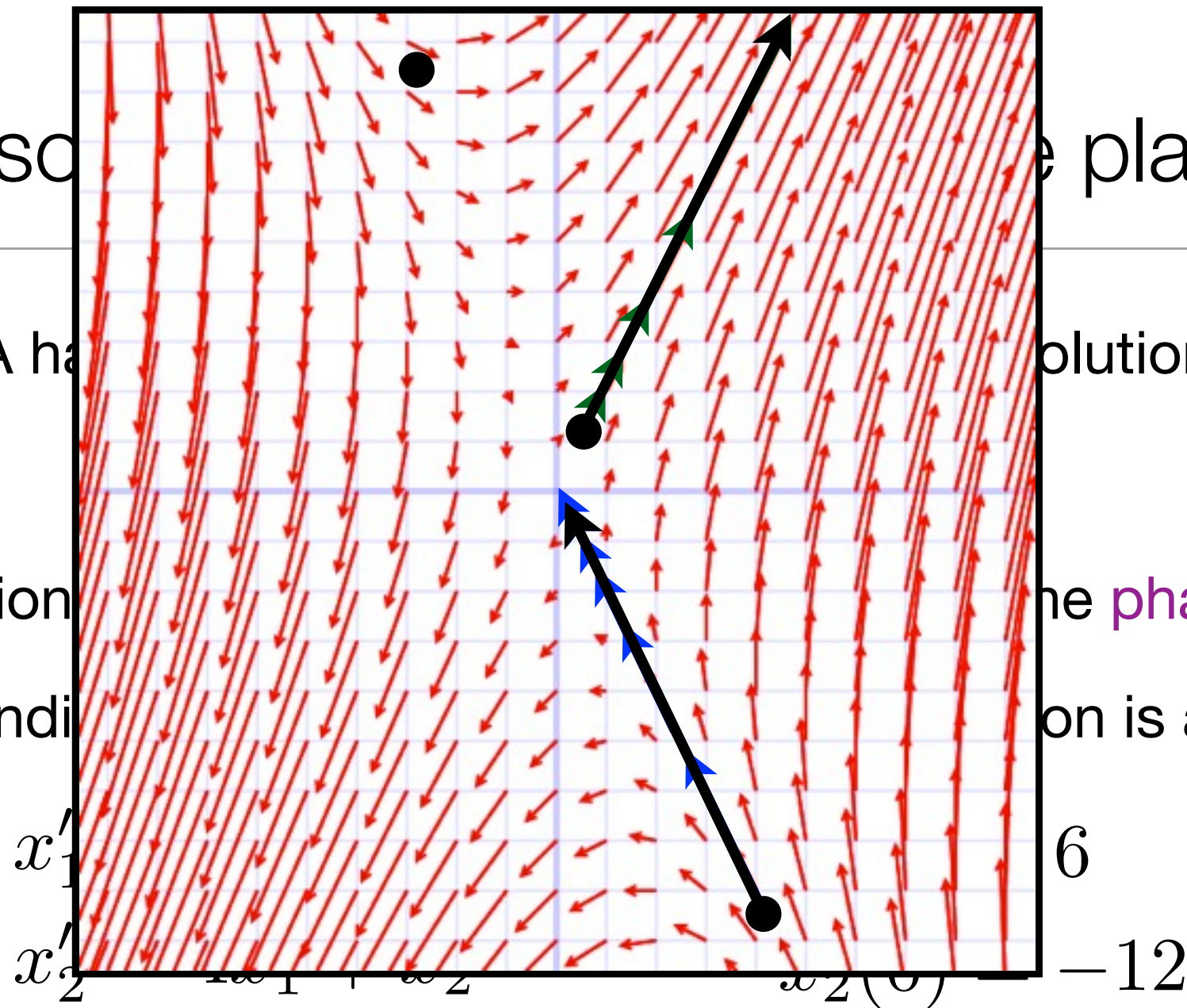
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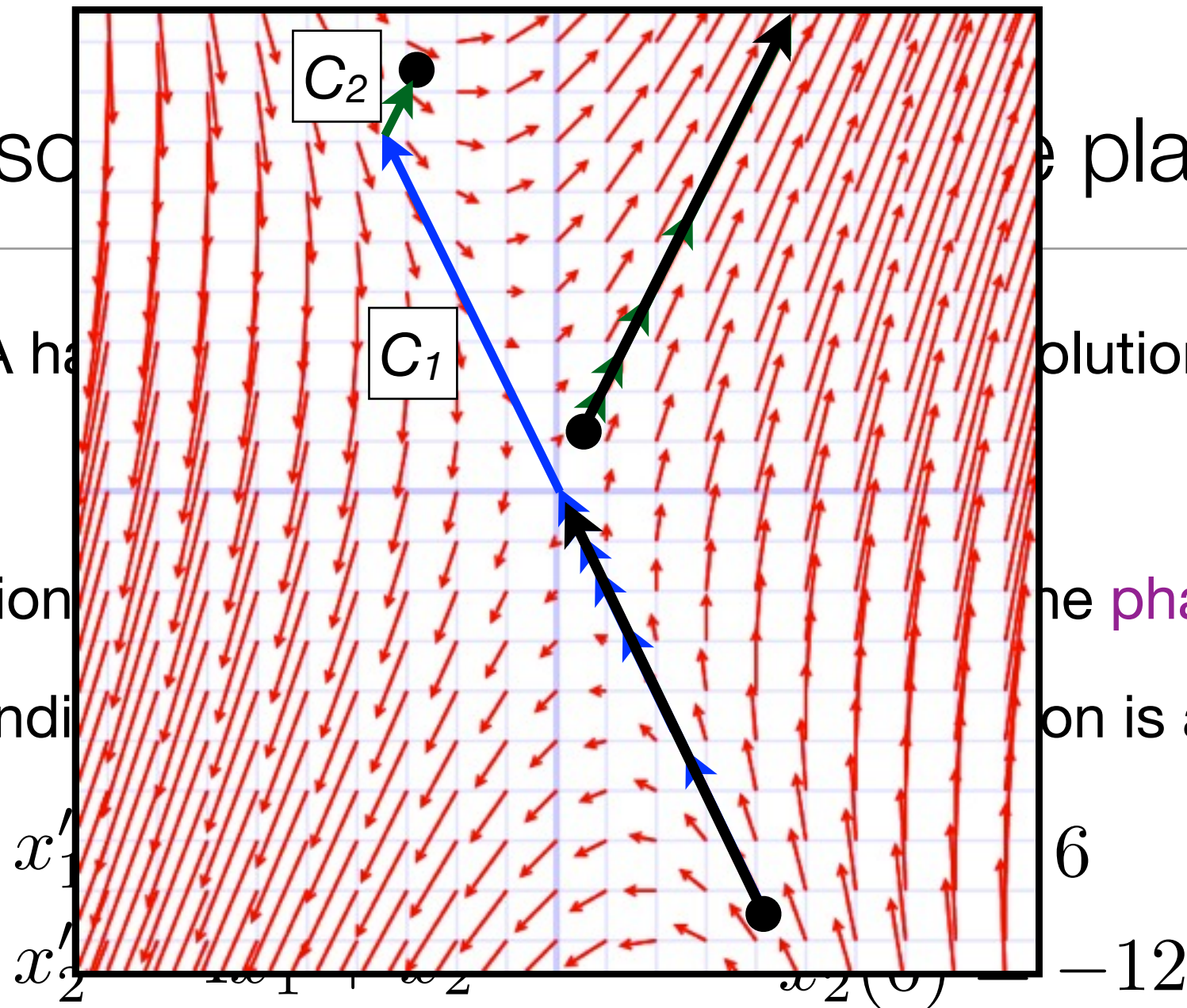
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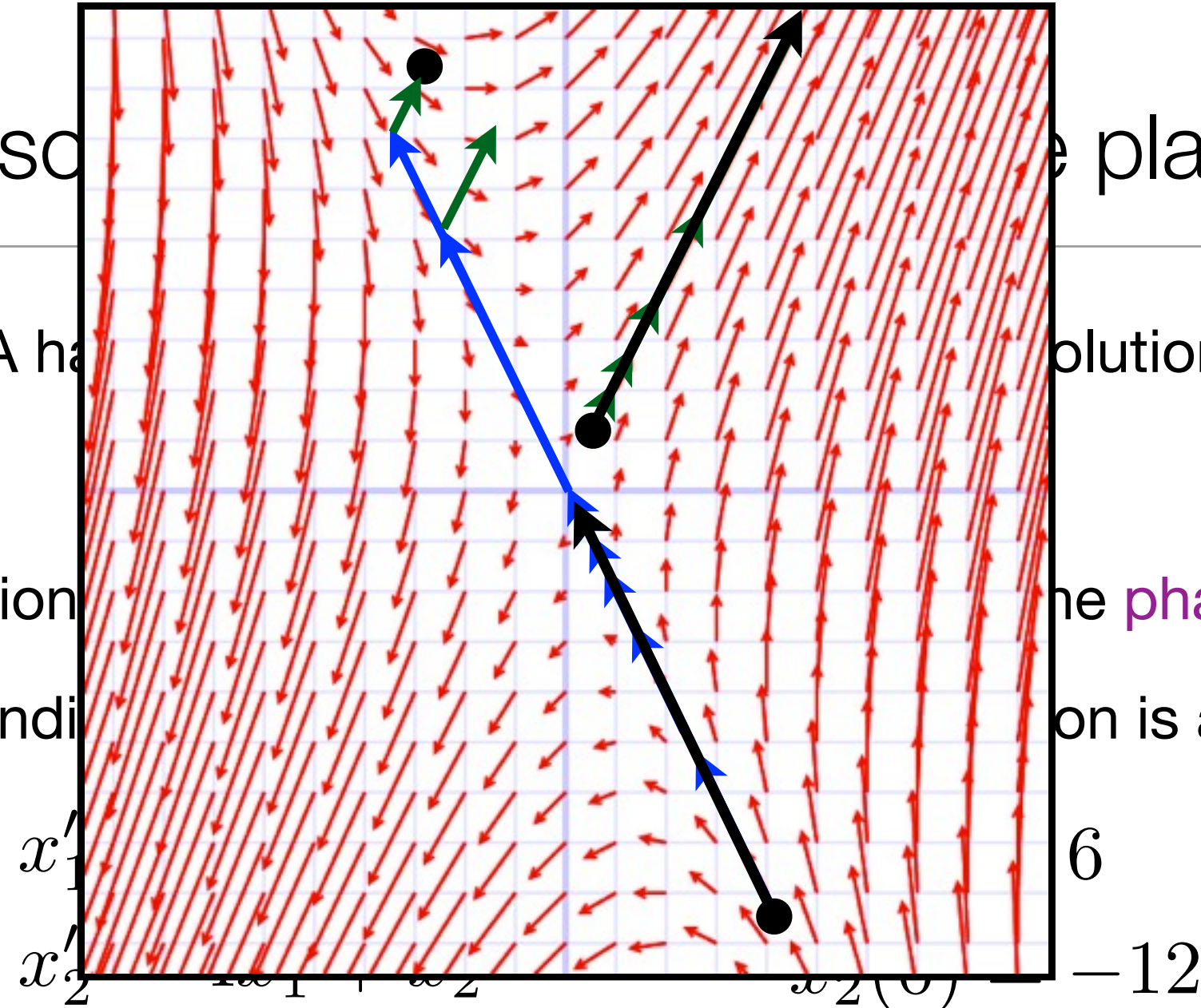
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$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

Shapes of solutions

in the phase plane

- When matrix A has eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, the general solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$.
- What do solutions look like in the phase plane?
- If the initial condition is a straight line, the solution is a straight line.



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$C_1 = 6, \quad C_2 = 0$$

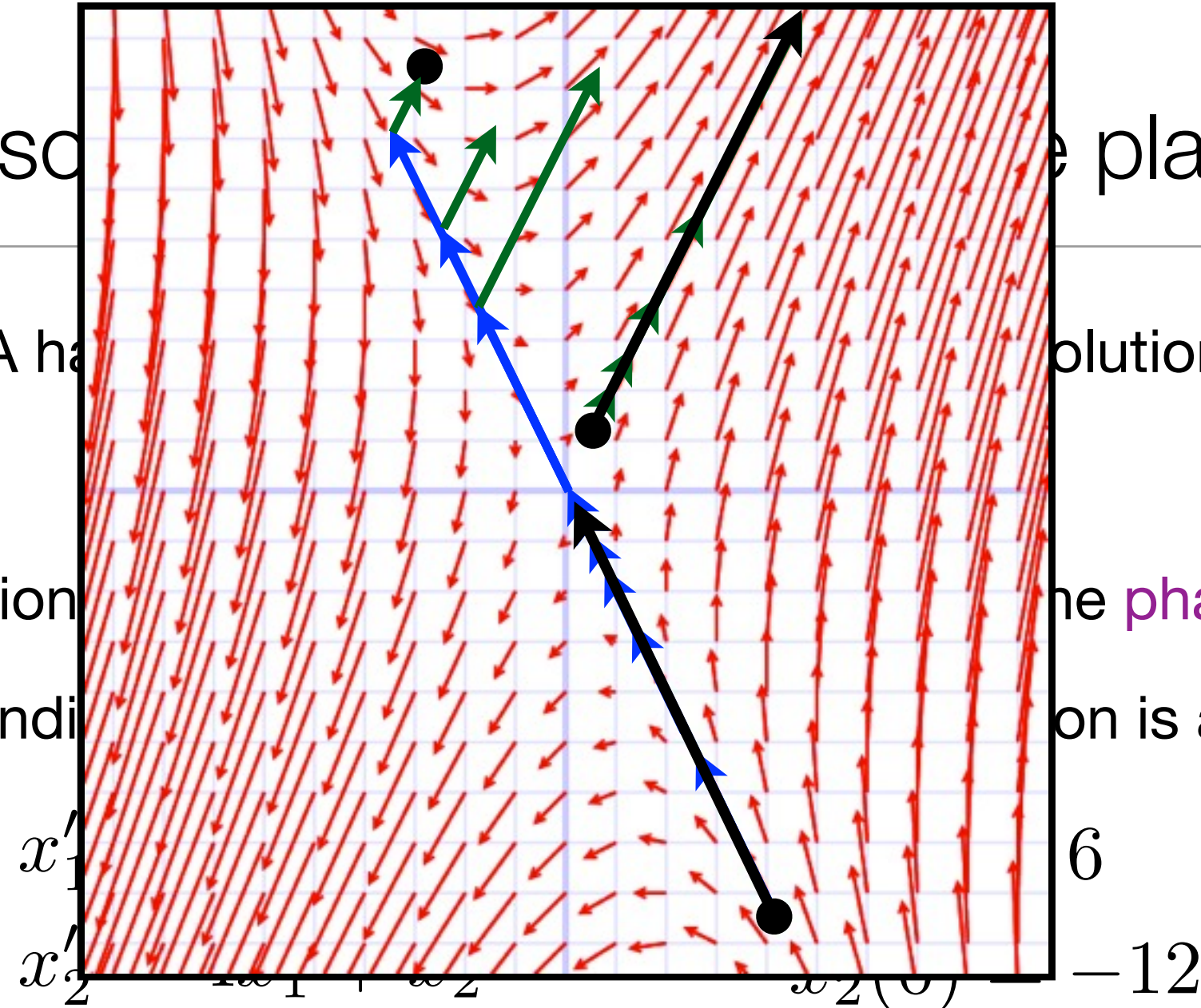
• IVP solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

Shapes of so

the plane

- When matrix A has...
 - What do solution...
 - If the initial condi...
- Example:



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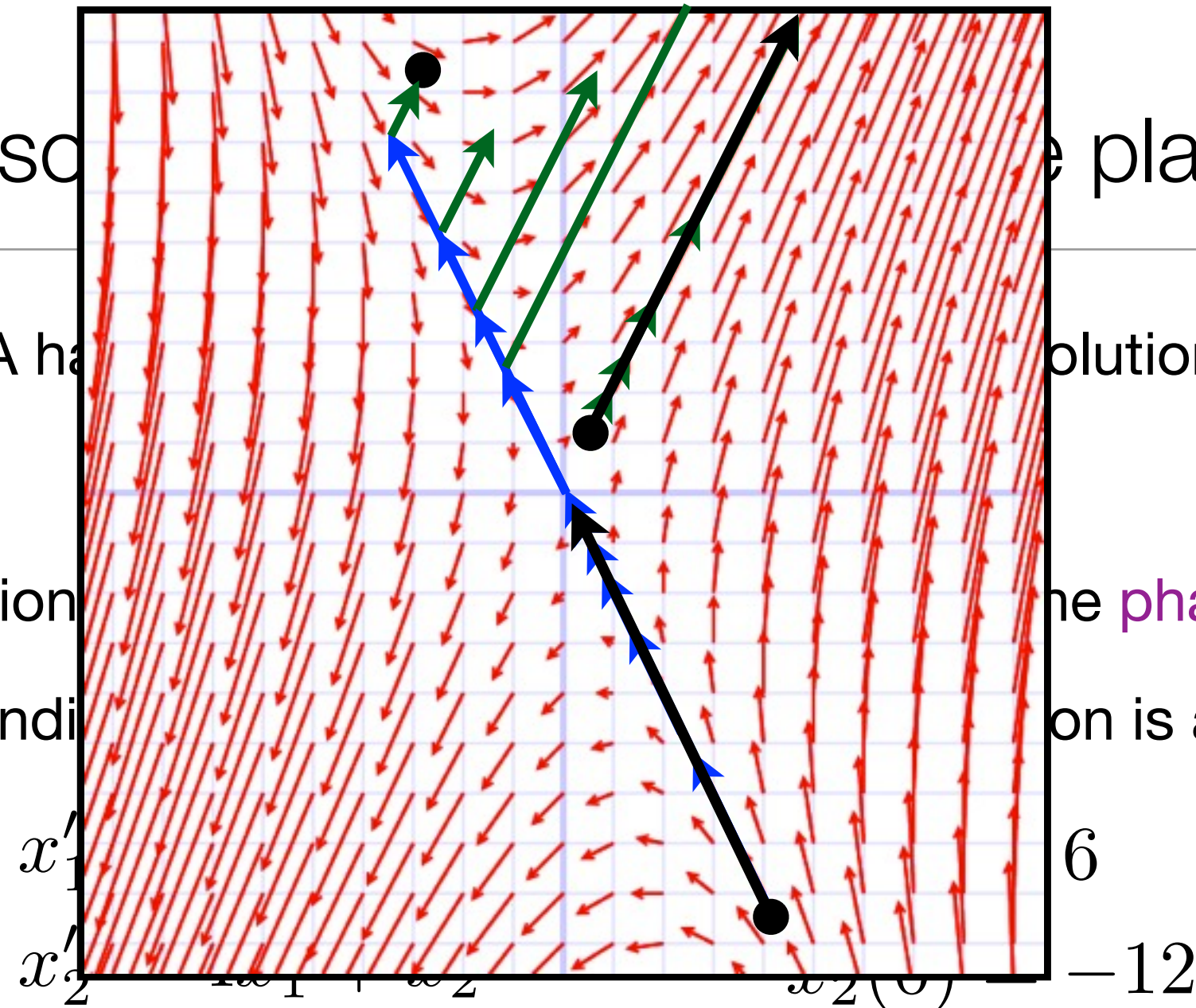
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Shapes of solutions

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- When matrix A has...
 - What do solutions look like in the phase plane?
 - If the initial condition is a straight line.
- Example:



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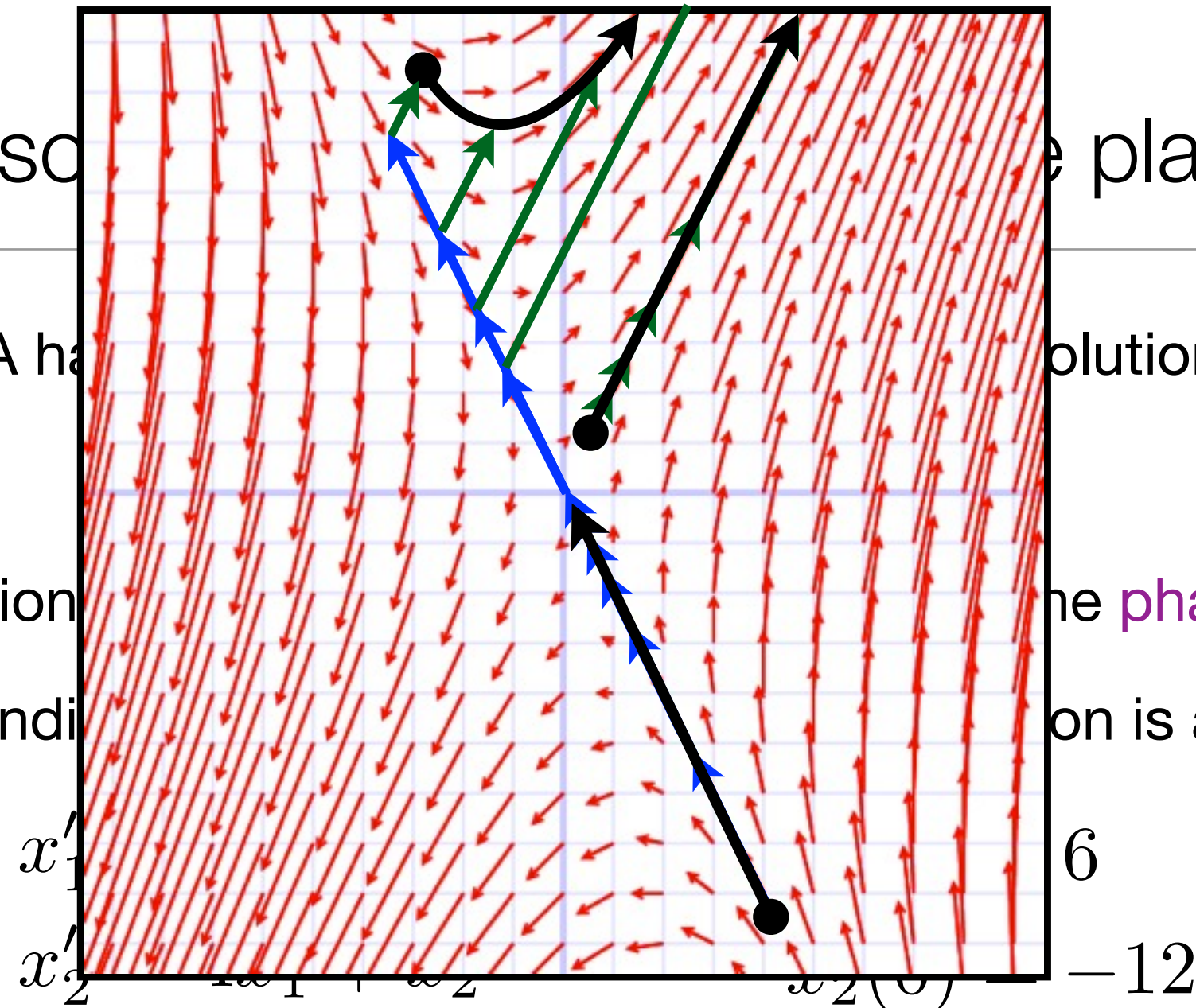
• IVP solution:

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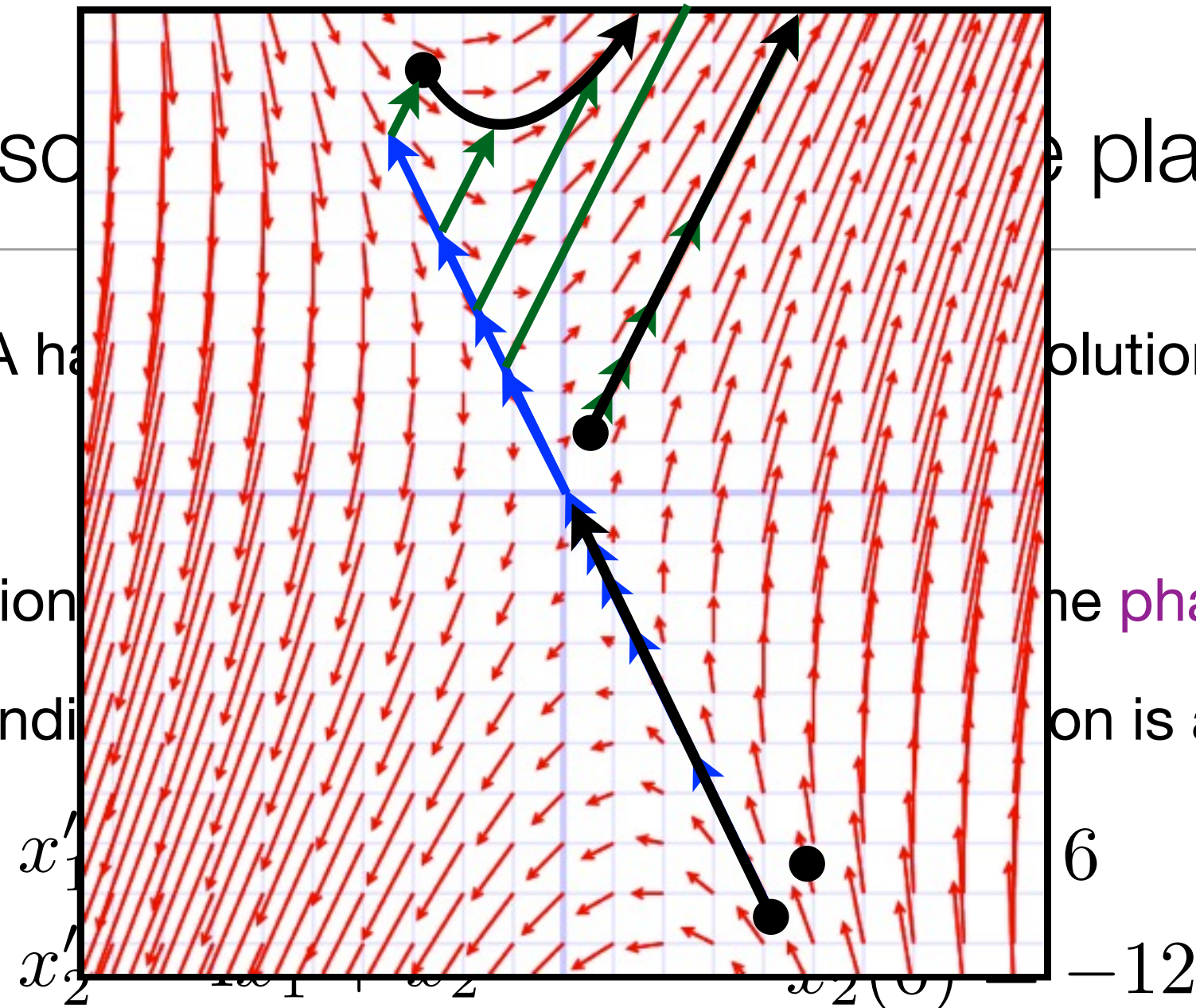
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Shapes of so

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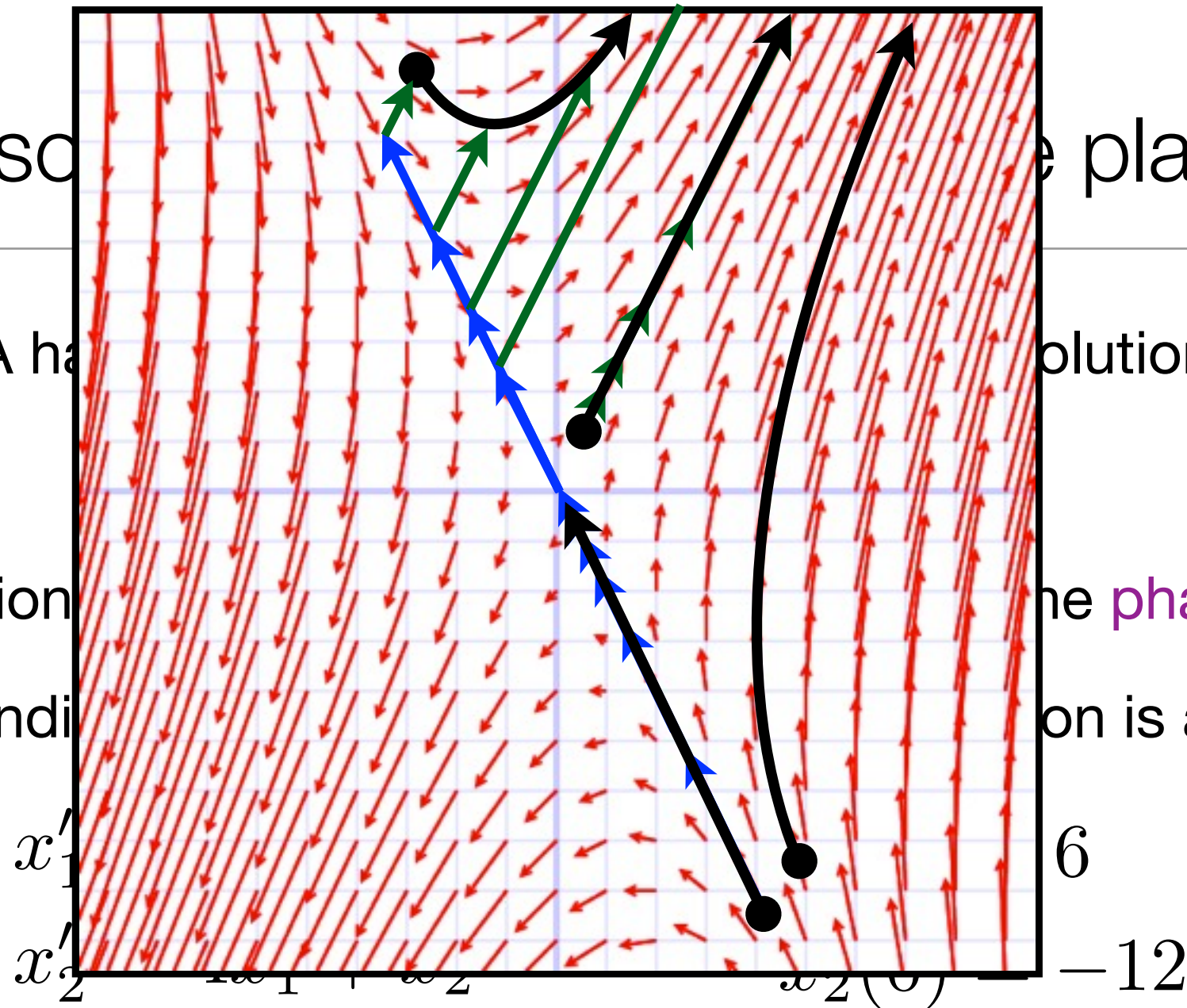
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Shapes of solutions

- When matrix A has...
- What do solutions...
- If the initial condition is a straight line. Example:



the plane
 solution to $\mathbf{x}' = A\mathbf{x}$ is
 the **phase plane**?
 on is a straight line.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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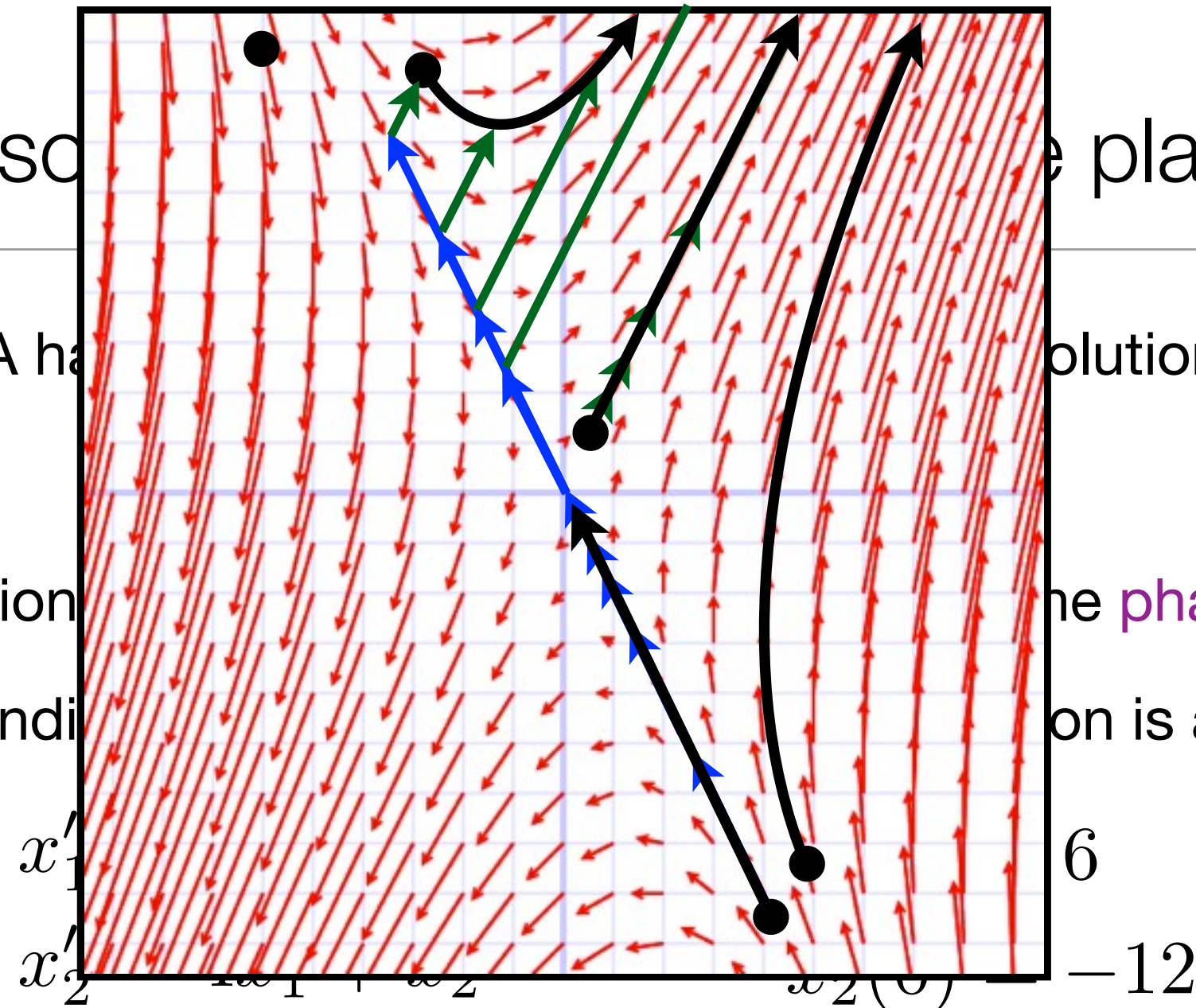
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Shapes of so

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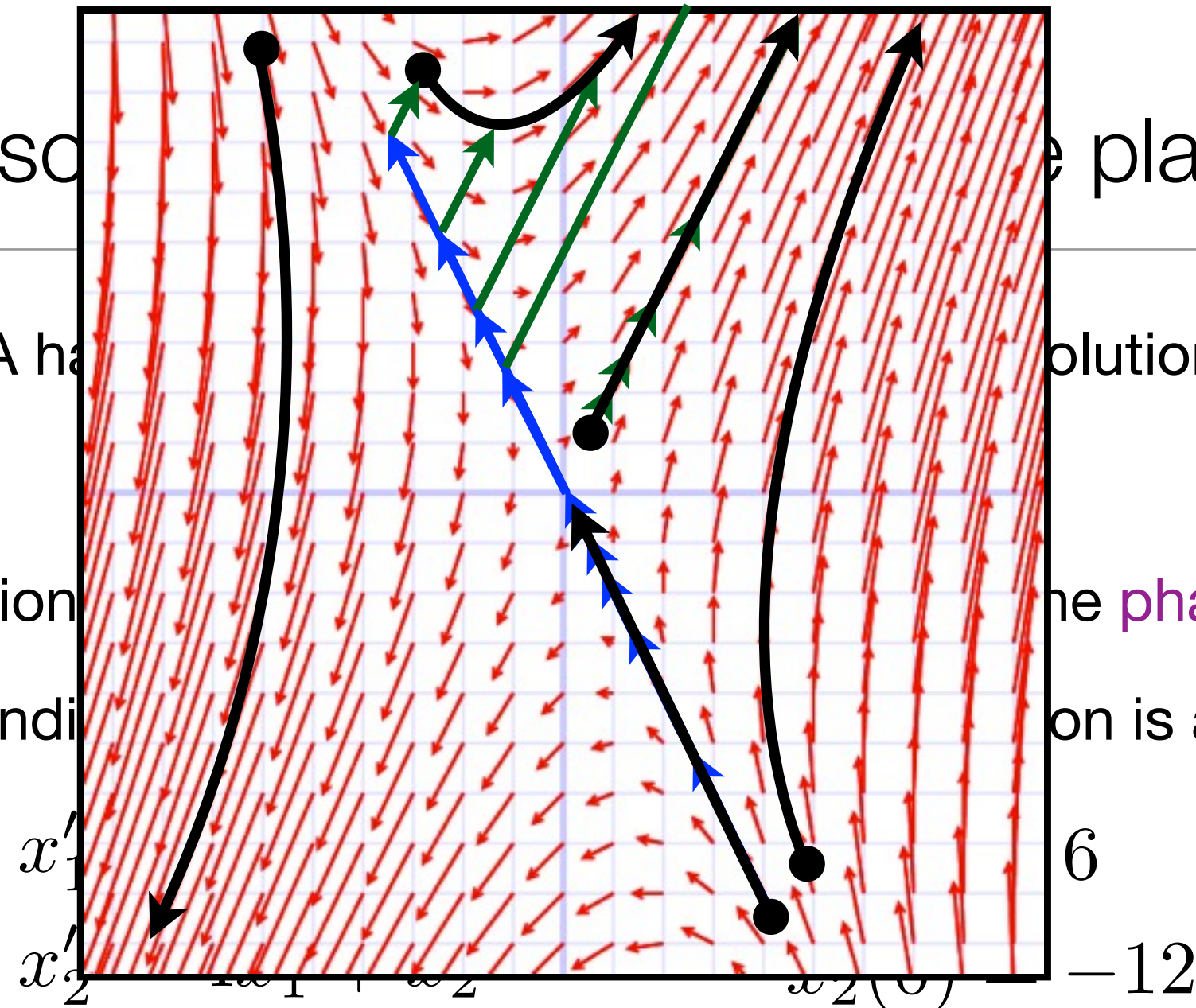
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Shapes of solutions

in the phase plane

- When matrix A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$, the general solution to $\mathbf{x}' = A\mathbf{x}$ is
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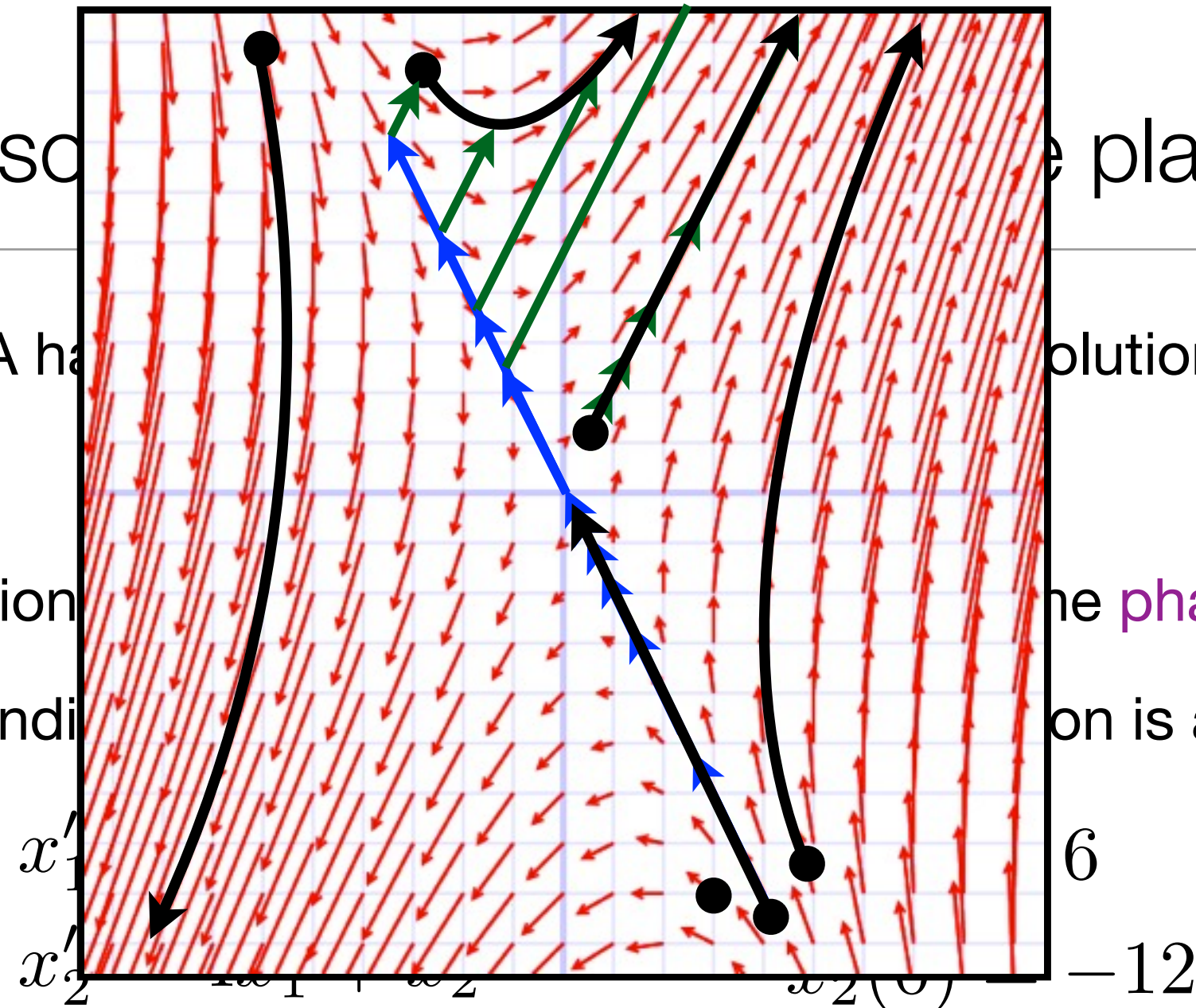
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Shapes of solutions

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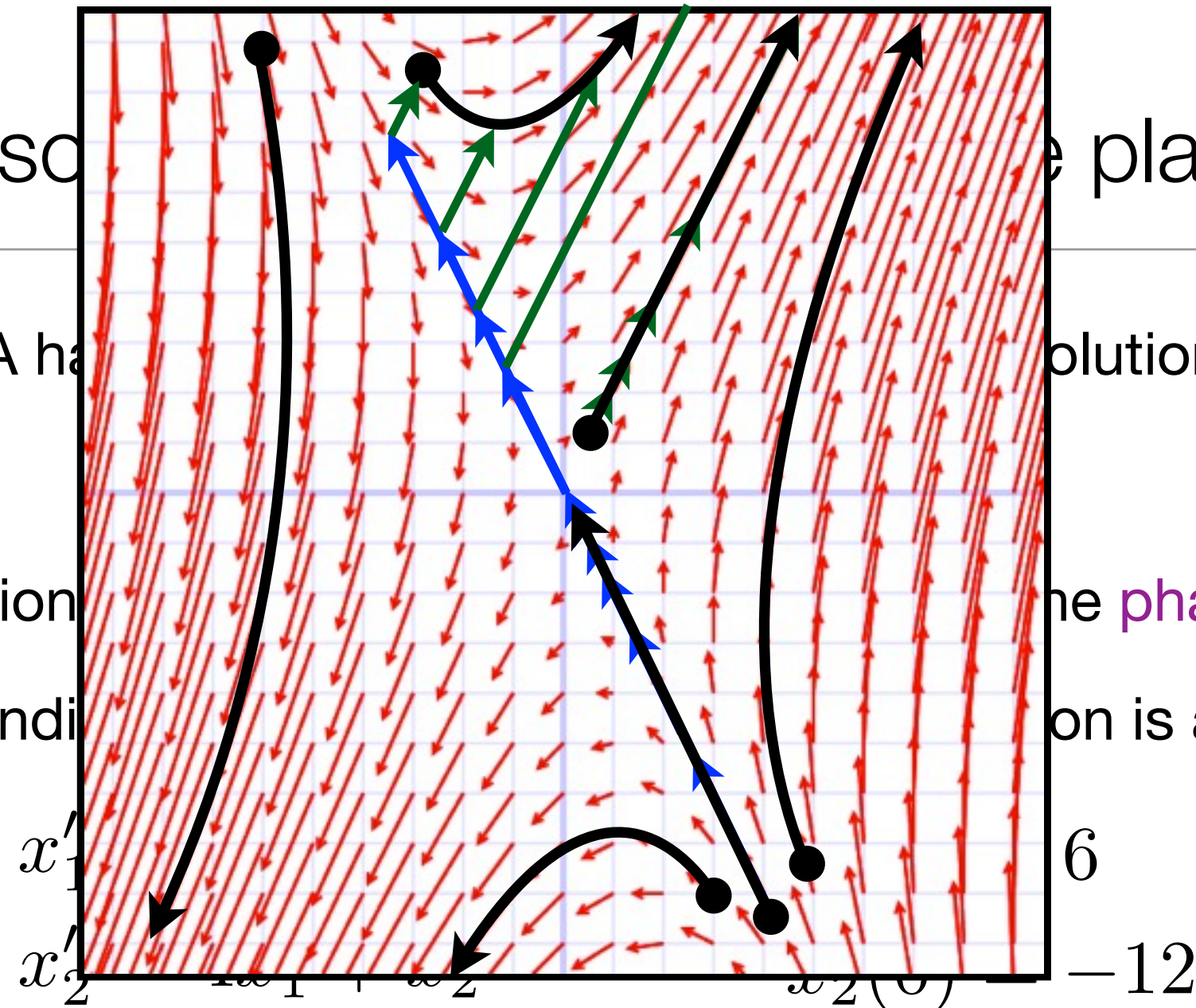
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Shapes of solution curves in the phase plane

- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

Shapes of solution curves in the phase plane

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Shapes of solution curves in the phase plane

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$$\mathbf{x} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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- Can we plot solutions in x_1 - x_2 plane by graphing x_2 versus x_1 ?

Shapes of solution curves in the phase plane

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$$x_1(t) = C_1 e^{\lambda_1 t} \quad t = \frac{1}{\lambda_1} \ln \left(\frac{x_1}{C_1} \right)$$

$$x_2(t) = C_2 e^{\lambda_2 t} \quad t = \frac{1}{\lambda_2} \ln \left(\frac{x_2}{C_2} \right)$$

$$\frac{1}{\lambda_2} \ln \left(\frac{x_2}{C_2} \right) = \frac{1}{\lambda_1} \ln \left(\frac{x_1}{C_1} \right)$$

$$\ln \left(\frac{x_2}{C_2} \right) = \frac{\lambda_2}{\lambda_1} \ln \left(\frac{x_1}{C_1} \right)$$

$$\ln \left(\frac{x_2}{C_2} \right) = \ln \left(\frac{x_1}{C_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

$$x_2 = C_2 \left(\frac{x_1}{C_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

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Shapes of solution curves in the phase plane

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- For the shape of solutions, we need to know the sign and size of $\frac{\lambda_2}{\lambda_1}$.

Shapes of solution curves in the phase plane

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$$\lambda_2 = -3\lambda_1$$

Shapes of solution curves in the phase plane

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$$\lambda_2 = -3\lambda_1$$

$$x_2 = \frac{C}{x_1^3}$$

Shapes of solution curves in the phase plane

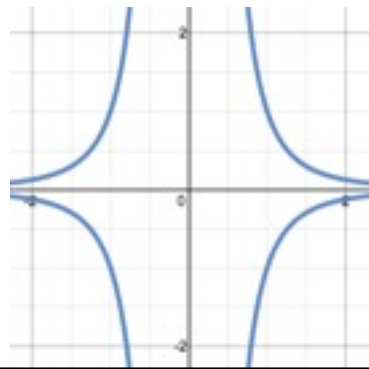
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Shapes of solution curves in the phase plane

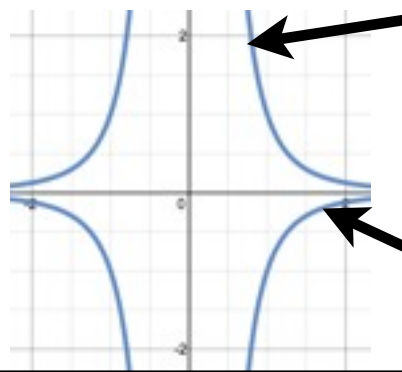
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- For the shape of solutions, we far from x_2 axis know the sign and size of $\frac{\lambda_2}{\lambda_1}$.

$$\lambda_2 = -3\lambda_1$$

$$x_2 = \frac{C}{x_1^3}$$



far from x_2 axis

close to x_1 axis

Shapes of solution curves in the phase plane

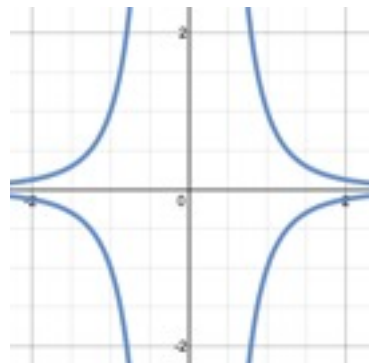
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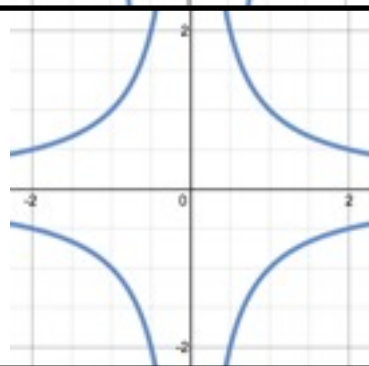
$$\lambda_2 = -3\lambda_1$$

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$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



Shapes of solution curves in the phase plane

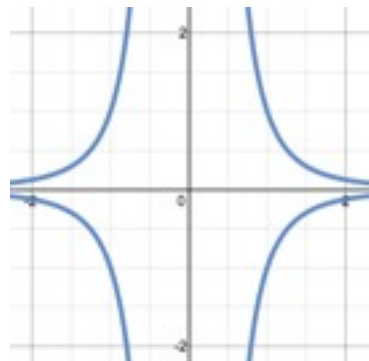
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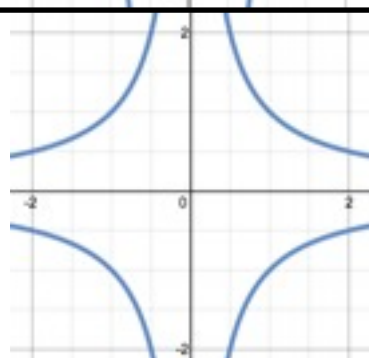
$$\lambda_2 = -3\lambda_1$$

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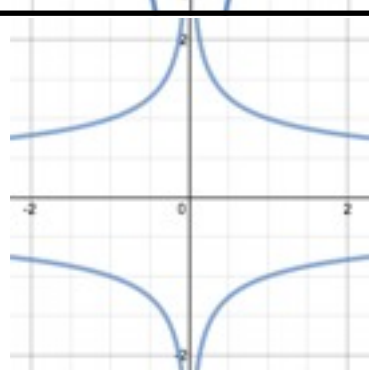
$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



$$\lambda_2 = -\frac{1}{3}\lambda_1$$

$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



Shapes of solution curves in the phase plane

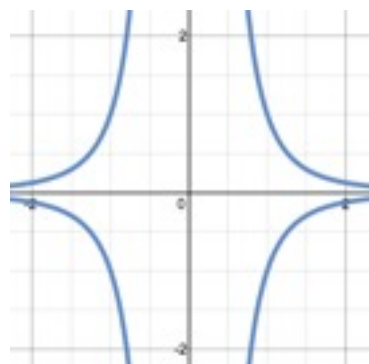
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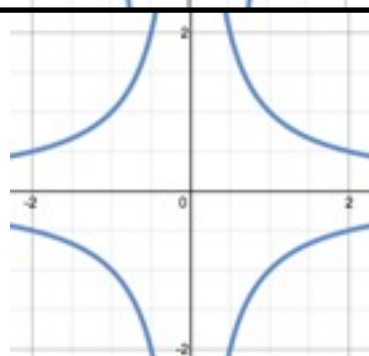
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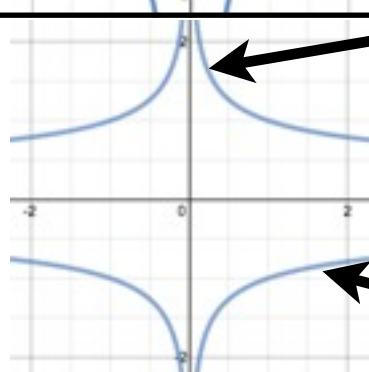


close to
x₂ axis

far from
x₁ axis

$$\lambda_2 = -\frac{1}{3}\lambda_1$$

$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



Shapes of solution curves in the phase plane

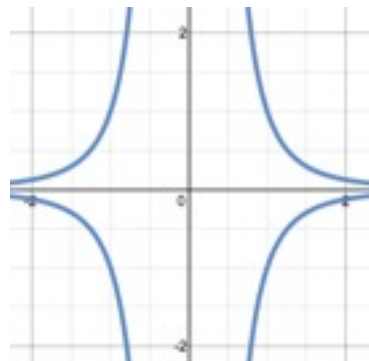
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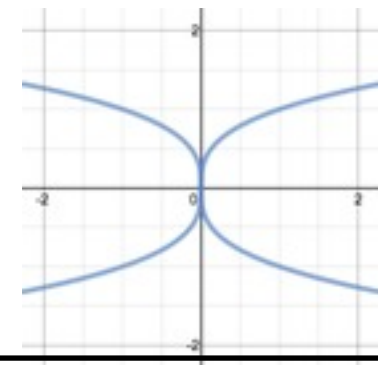
$$\lambda_2 = -3\lambda_1$$

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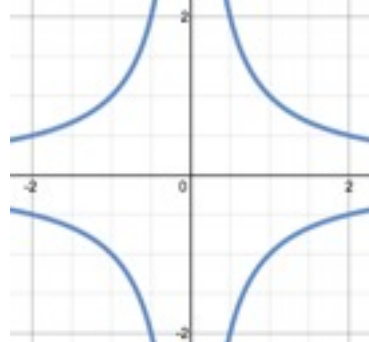
$$\lambda_2 = \frac{1}{3}\lambda_1$$

$$x_2 = C \sqrt[3]{x_1}$$



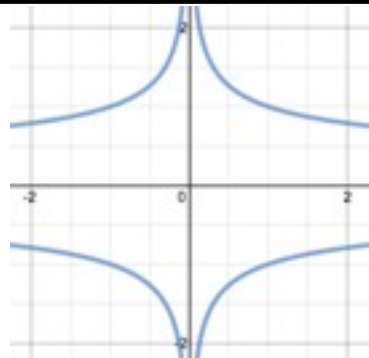
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Shapes of solution curves in the phase plane

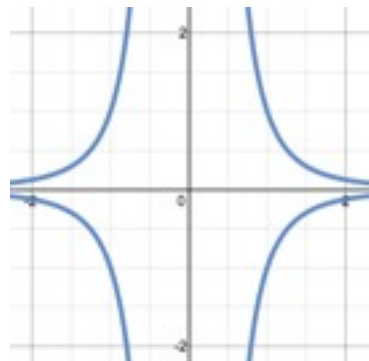
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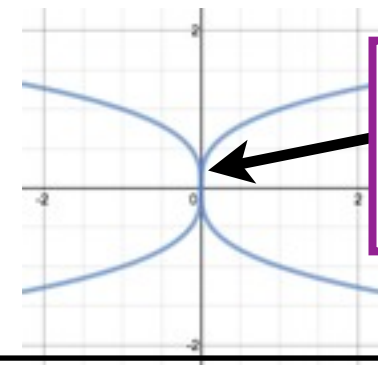
$$\lambda_2 = -3\lambda_1$$

$$x_2 = \frac{C}{x_1^3}$$



$$\lambda_2 = \frac{1}{3}\lambda_1$$

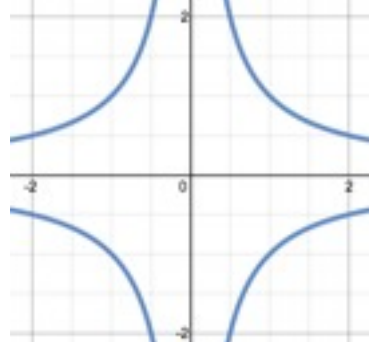
$$x_2 = C \sqrt[3]{x_1}$$



stays near
x₂ axis

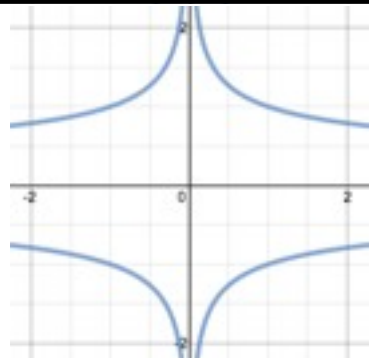
$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



$$\lambda_2 = -\frac{1}{3}\lambda_1$$

$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



Shapes of solution curves in the phase plane

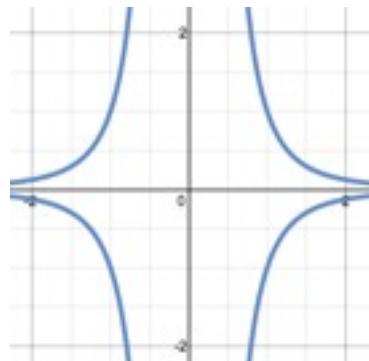
- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

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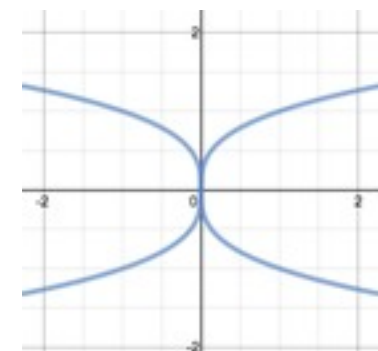
$$\lambda_2 = -3\lambda_1$$

$$x_2 = \frac{C}{x_1^3}$$



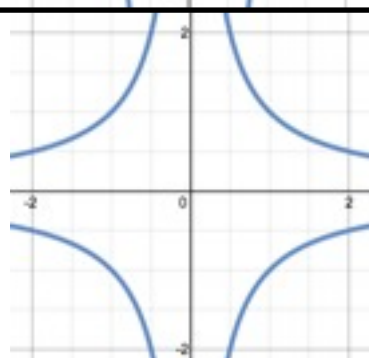
$$\lambda_2 = \frac{1}{3}\lambda_1$$

$$x_2 = C \sqrt[3]{x_1}$$



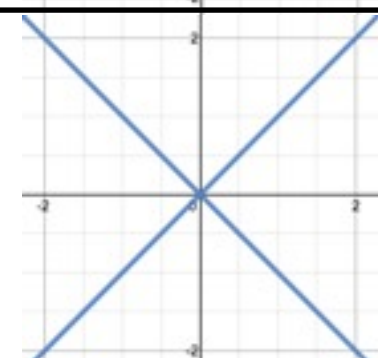
$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



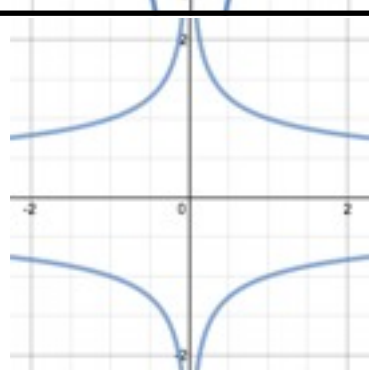
$$\lambda_2 = \lambda_1$$

$$x_2 = C x_1$$



$$\lambda_2 = -\frac{1}{3}\lambda_1$$

$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



Shapes of solution curves in the phase plane

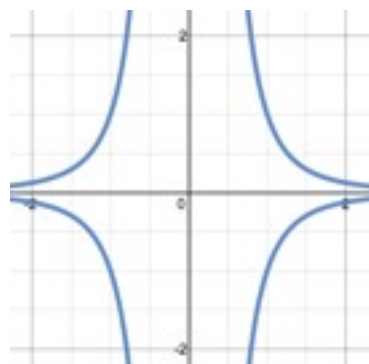
- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

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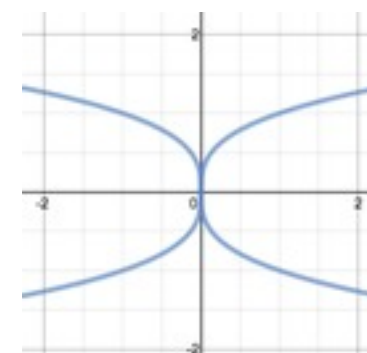
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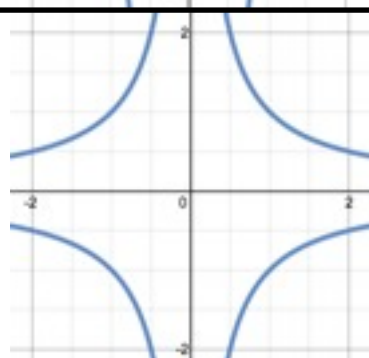
$$\lambda_2 = \frac{1}{3}\lambda_1$$

$$x_2 = C \sqrt[3]{x_1}$$



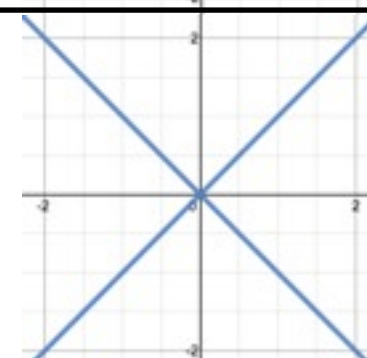
$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



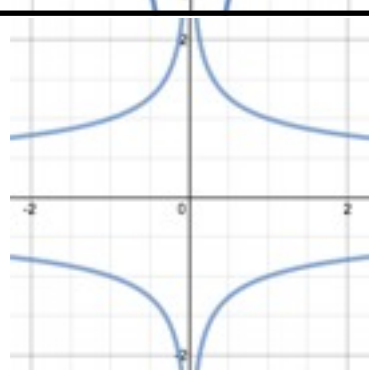
$$\lambda_2 = \lambda_1$$

$$x_2 = Cx_1$$



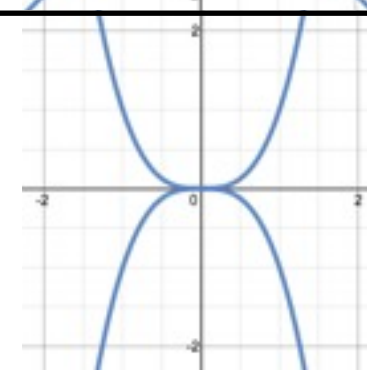
$$\lambda_2 = -\frac{1}{3}\lambda_1$$

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$$\lambda_2 = 3\lambda_1$$

$$x_2 = Cx_1^3$$



Shapes of solution curves in the phase plane

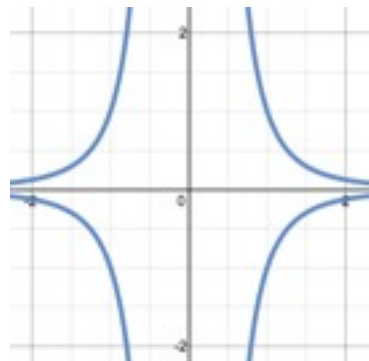
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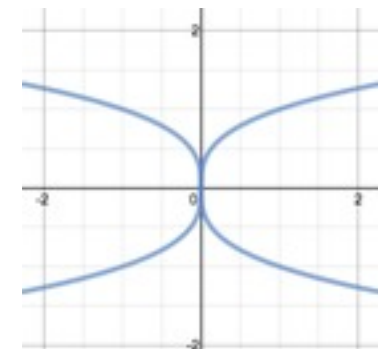
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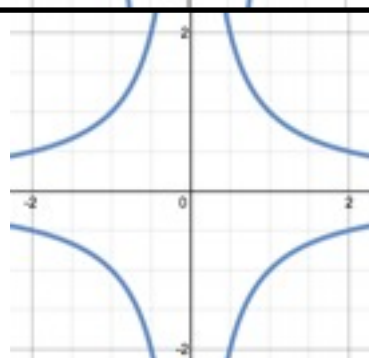
$$\lambda_2 = \frac{1}{3}\lambda_1$$

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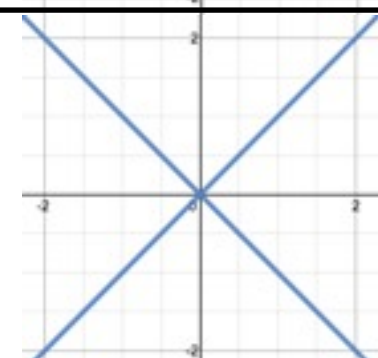
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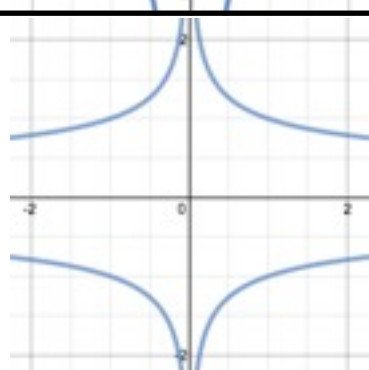
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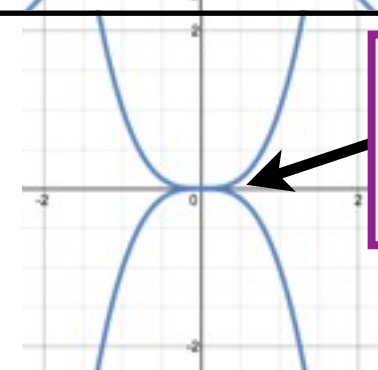
$$\lambda_2 = -\frac{1}{3}\lambda_1$$

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$$\lambda_2 = 3\lambda_1$$

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stays near
x₁ axis

Shapes of solution curves in the phase plane

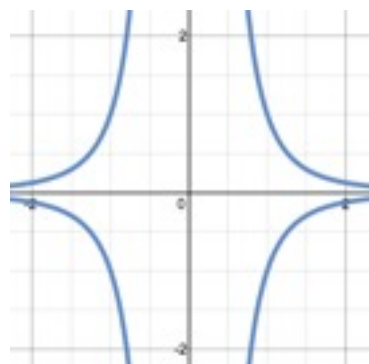
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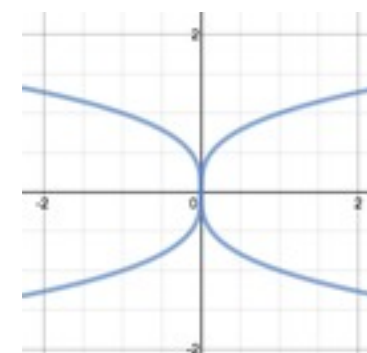
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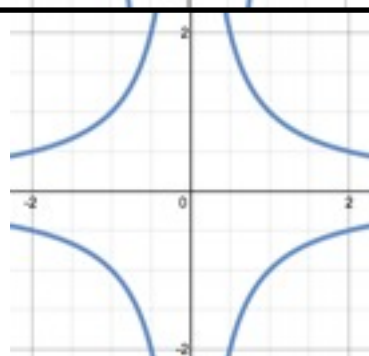
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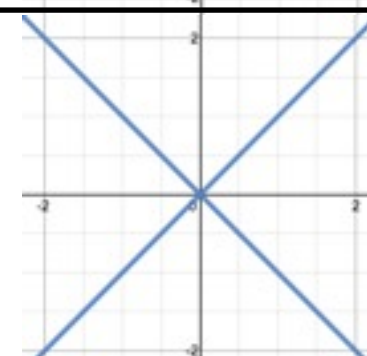
$$\lambda_2 = -\lambda_1$$

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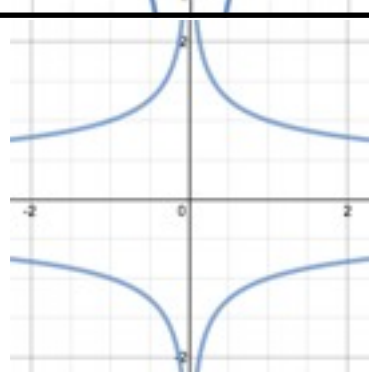
$$\lambda_2 = \lambda_1$$

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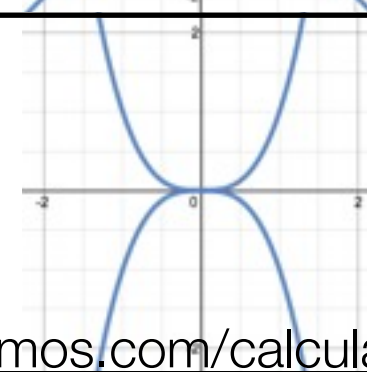
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$$x_2 = Cx_1^3$$



<https://www.desmos.com/calculator/c4rhrgotmo>

Shapes of solution curves in the phase plane

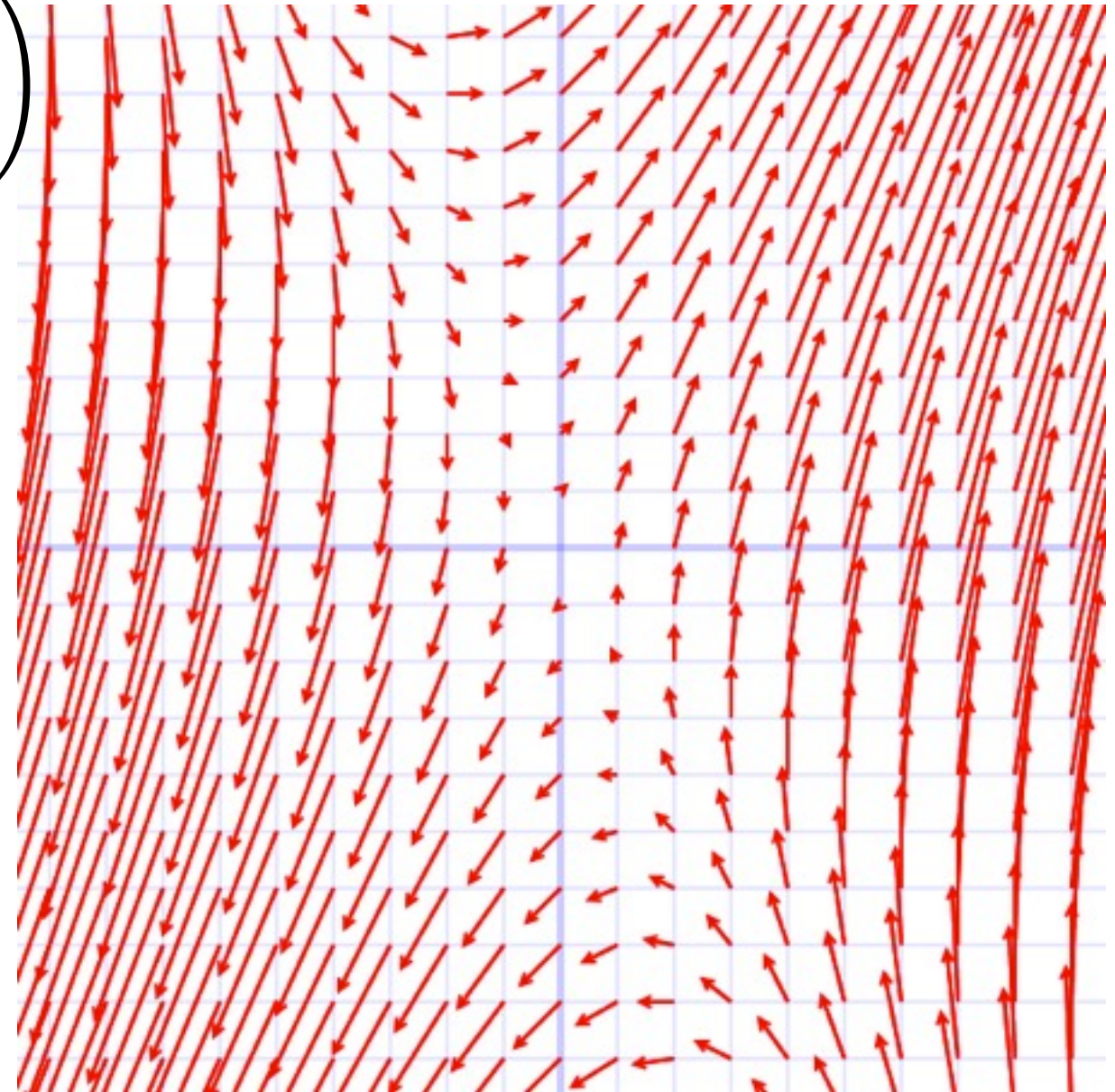
- With more complicated solutions (eigenvectors off-axis), tilt shapes accordingly.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Shapes of solution curves in the phase plane

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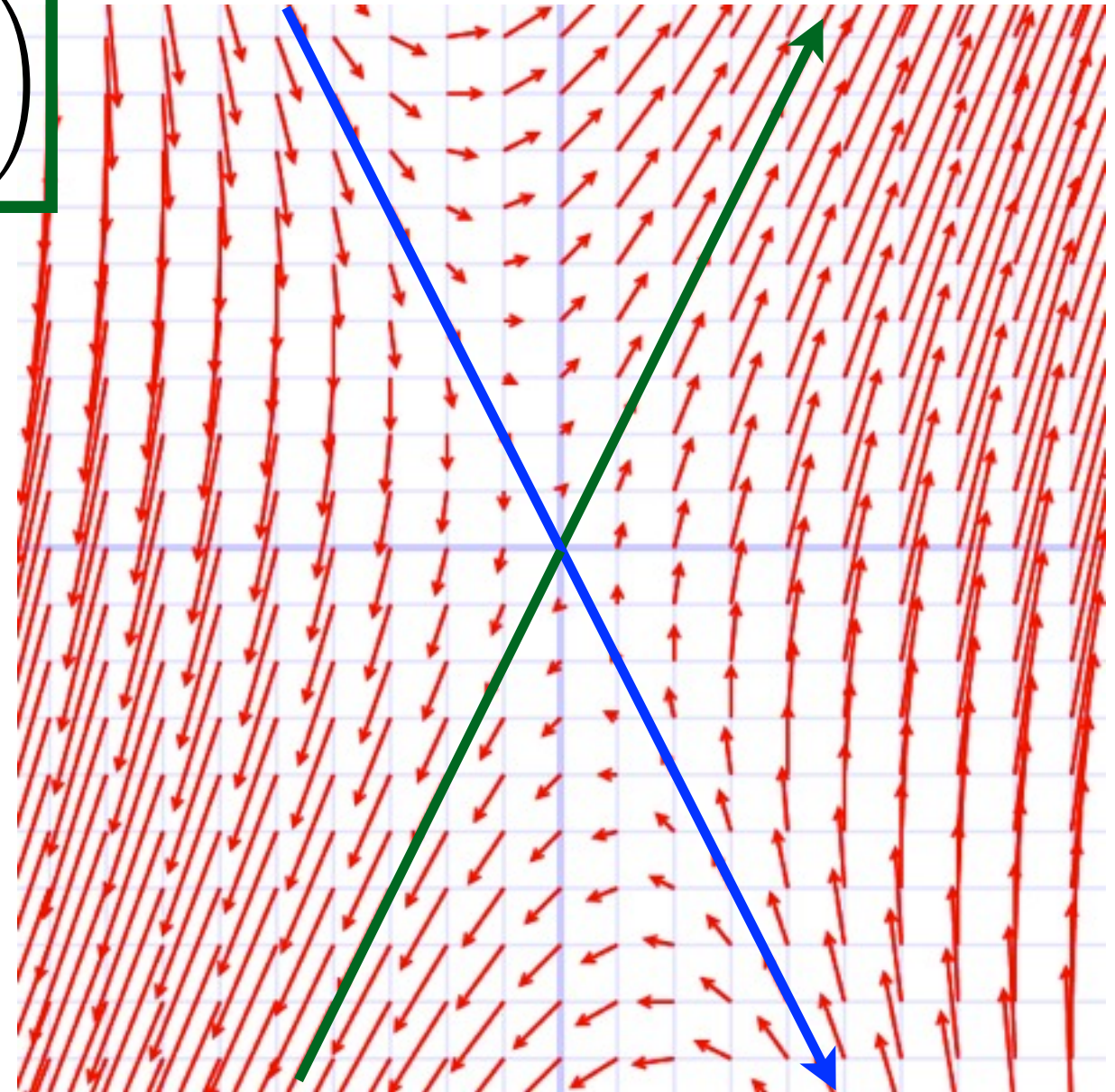
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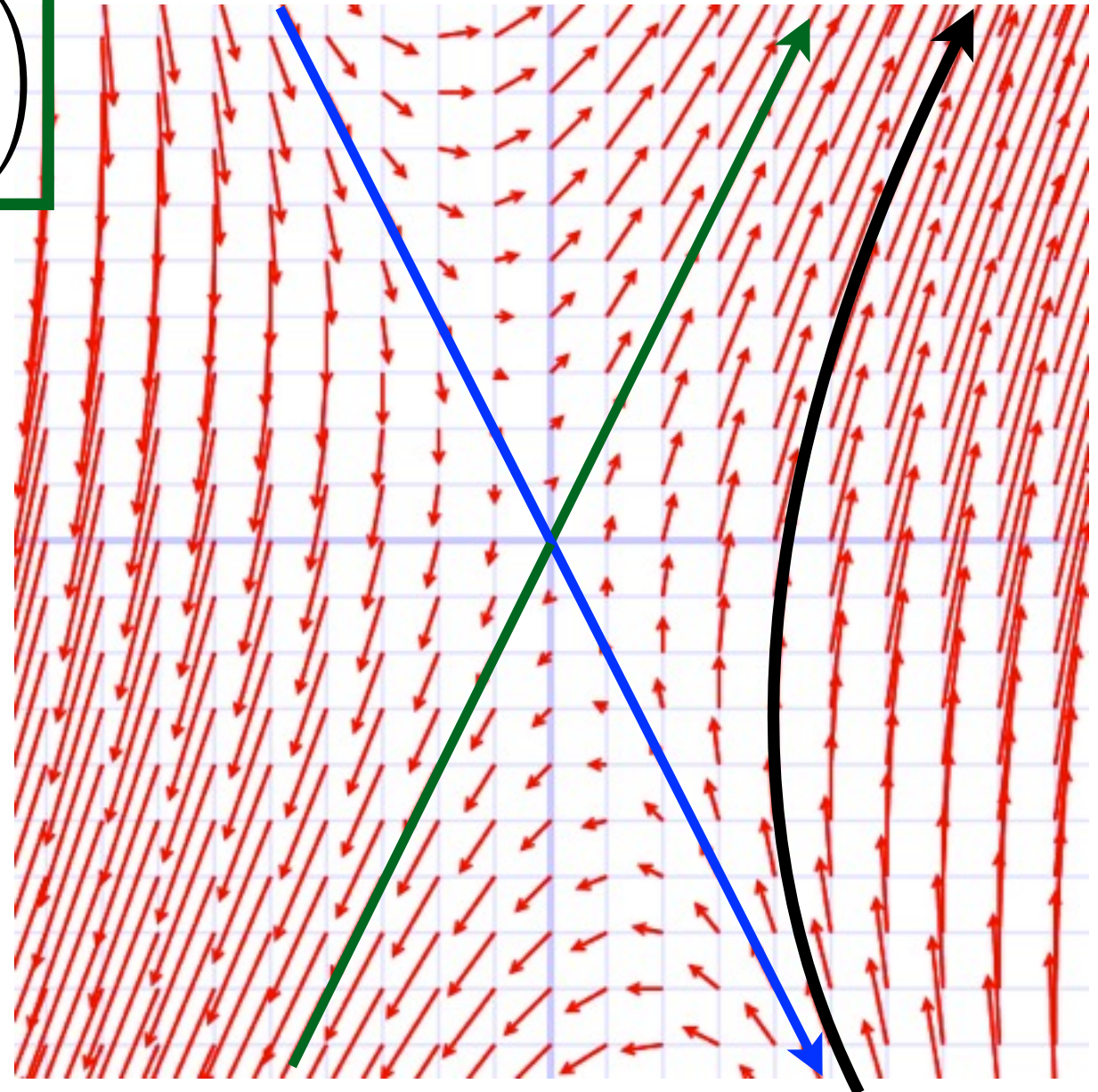
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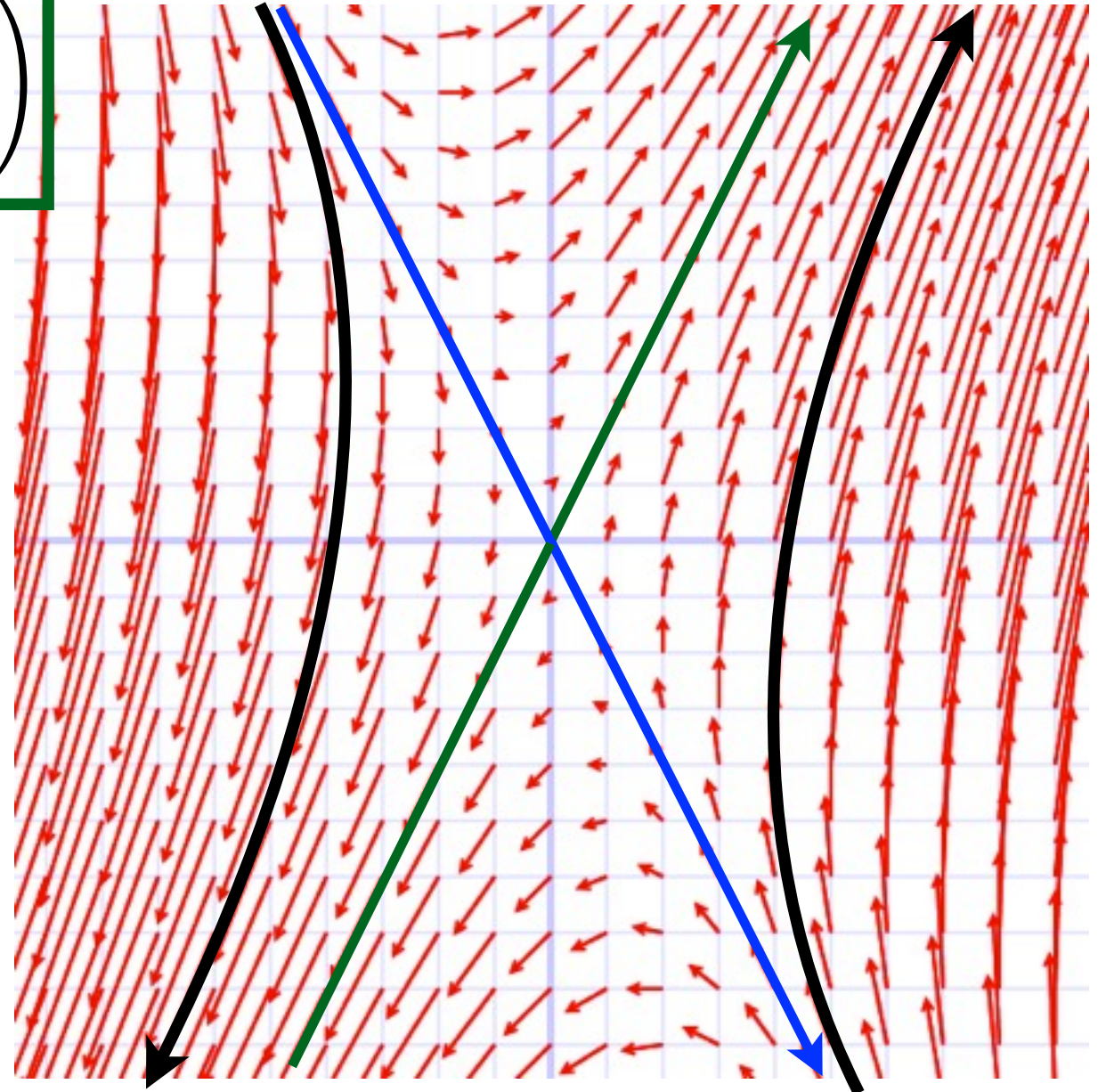
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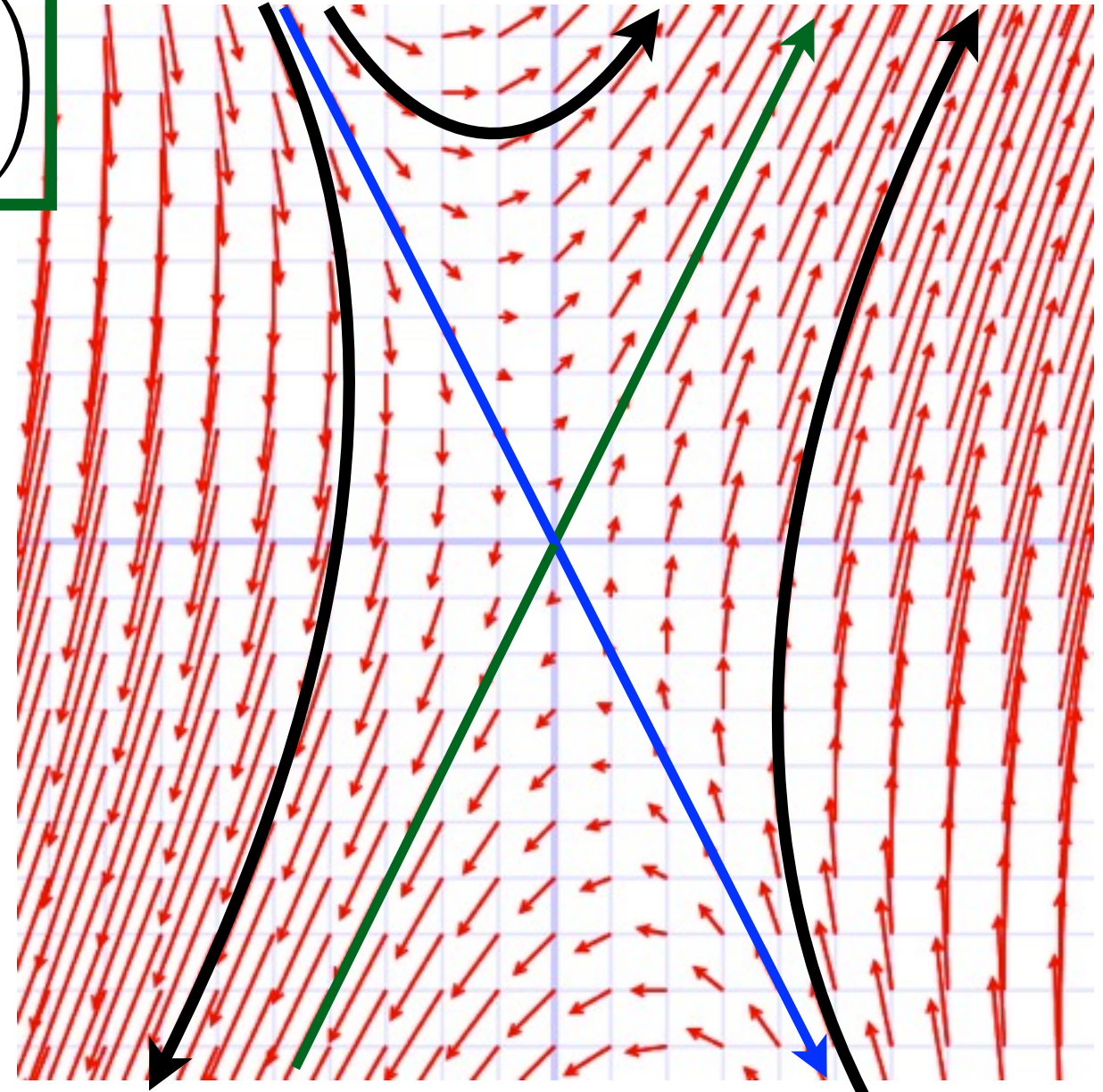
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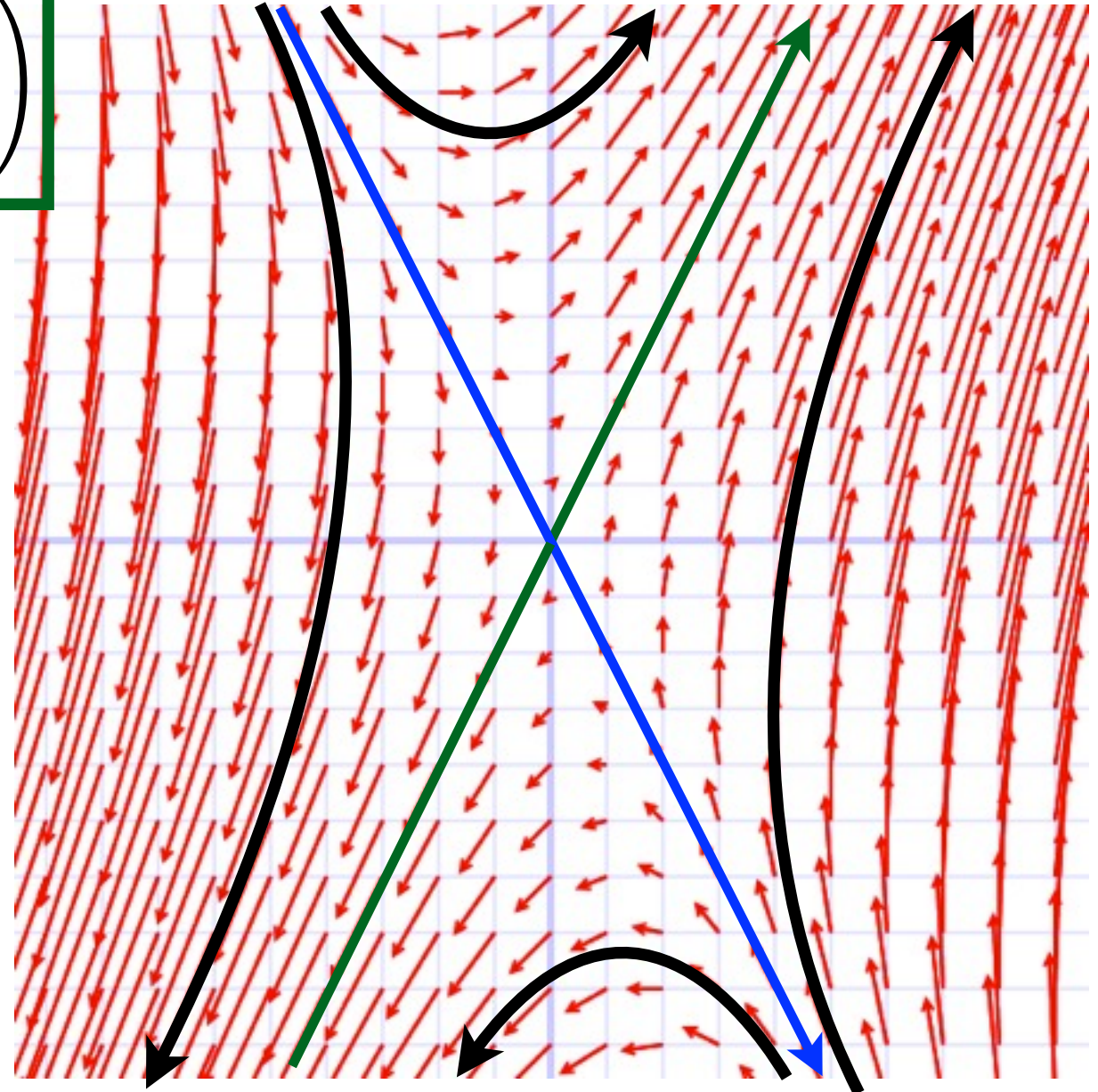
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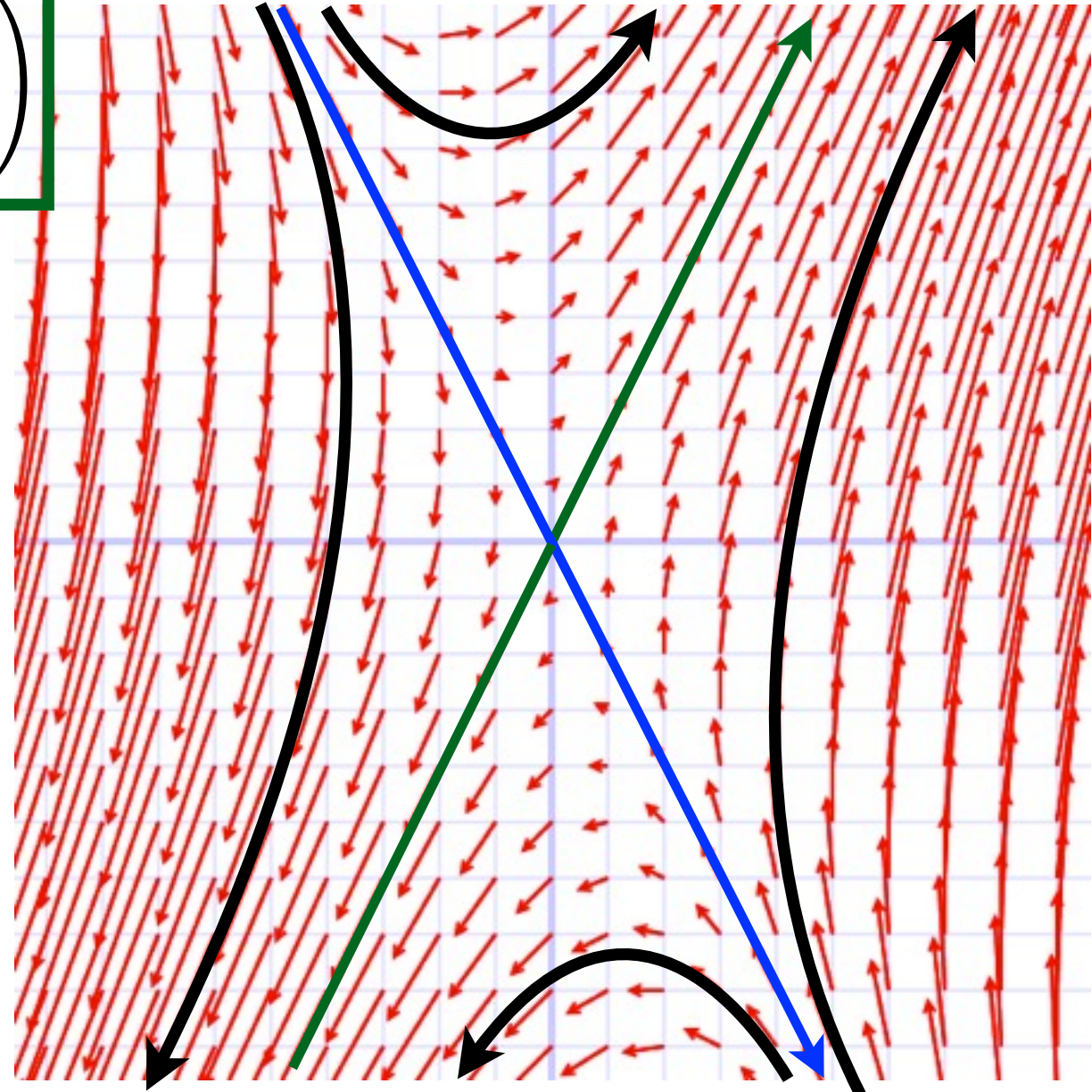
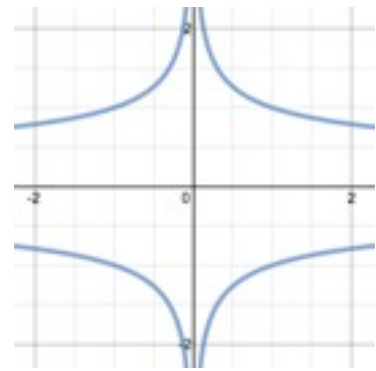
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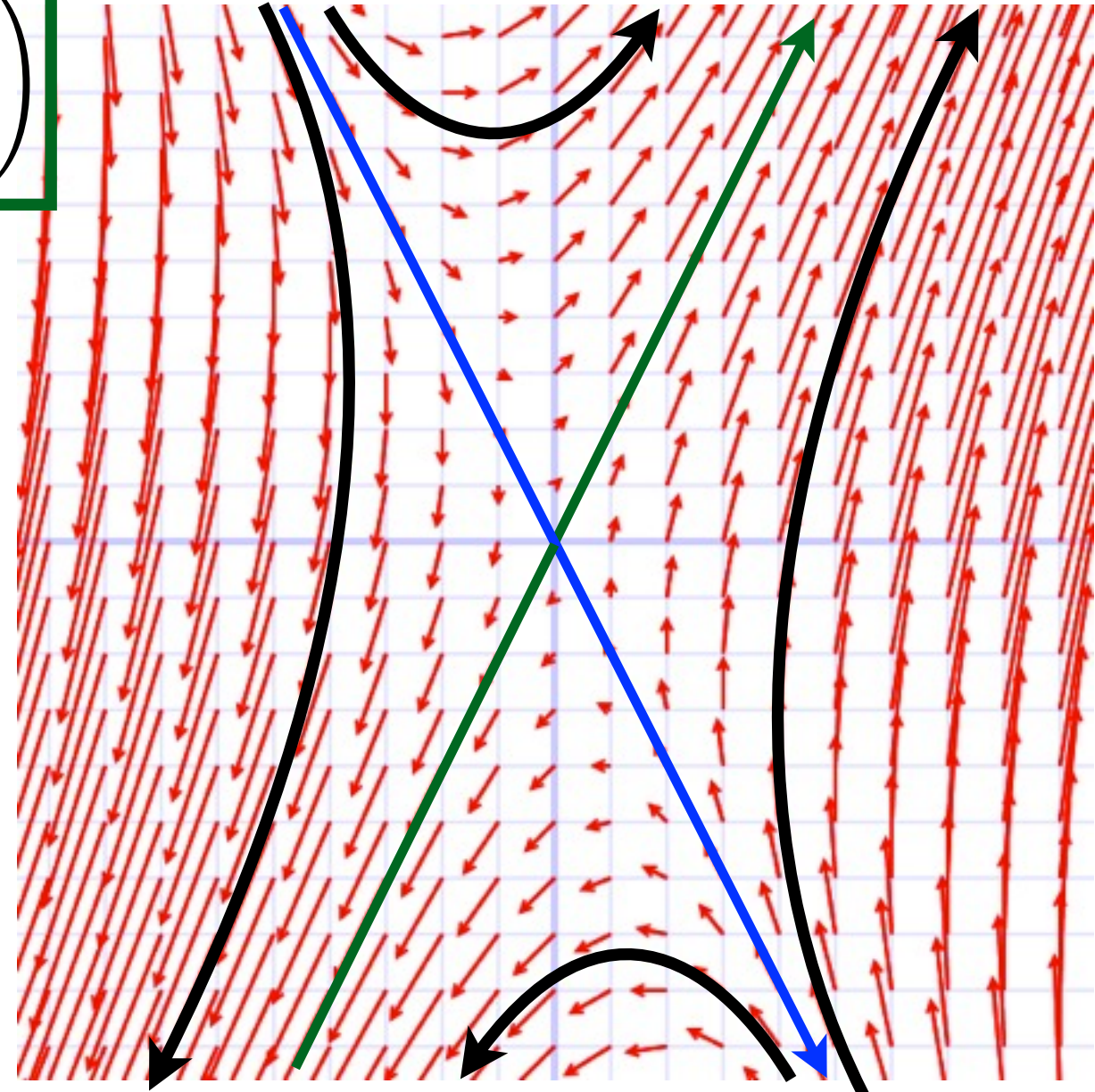
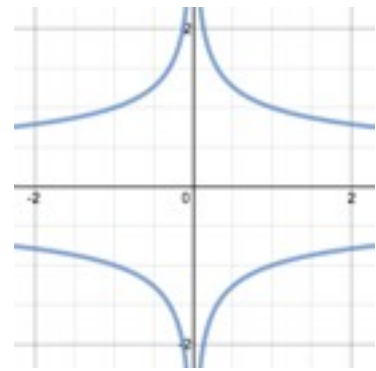


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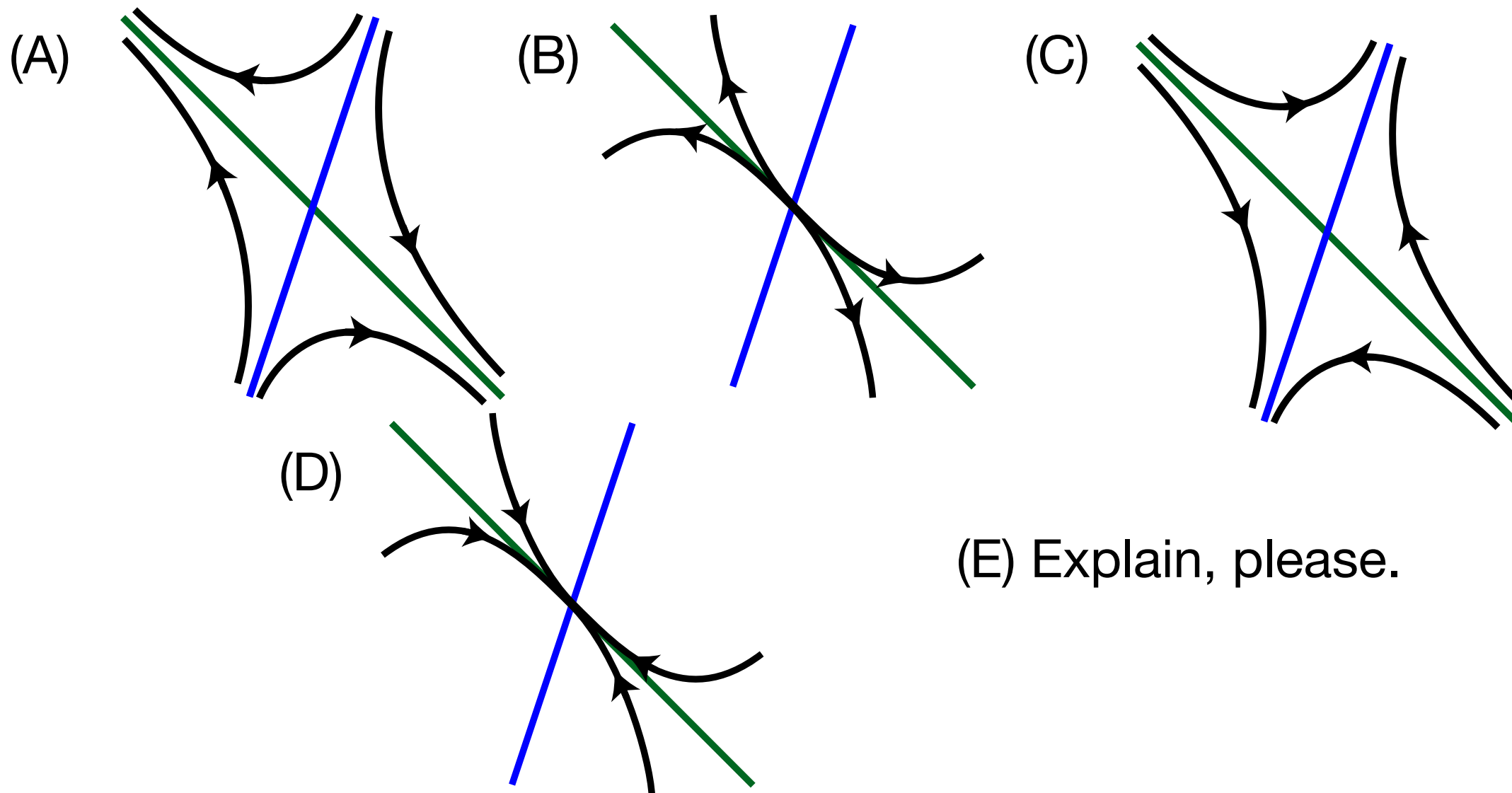
- Going forward in time, the **blue component** shrinks slower than the **green component** grows so solutions appear closer to **blue** “axis” than to **green** “axis”



Shapes of solution curves in the phase plane

- Which phase plane matches the general solution

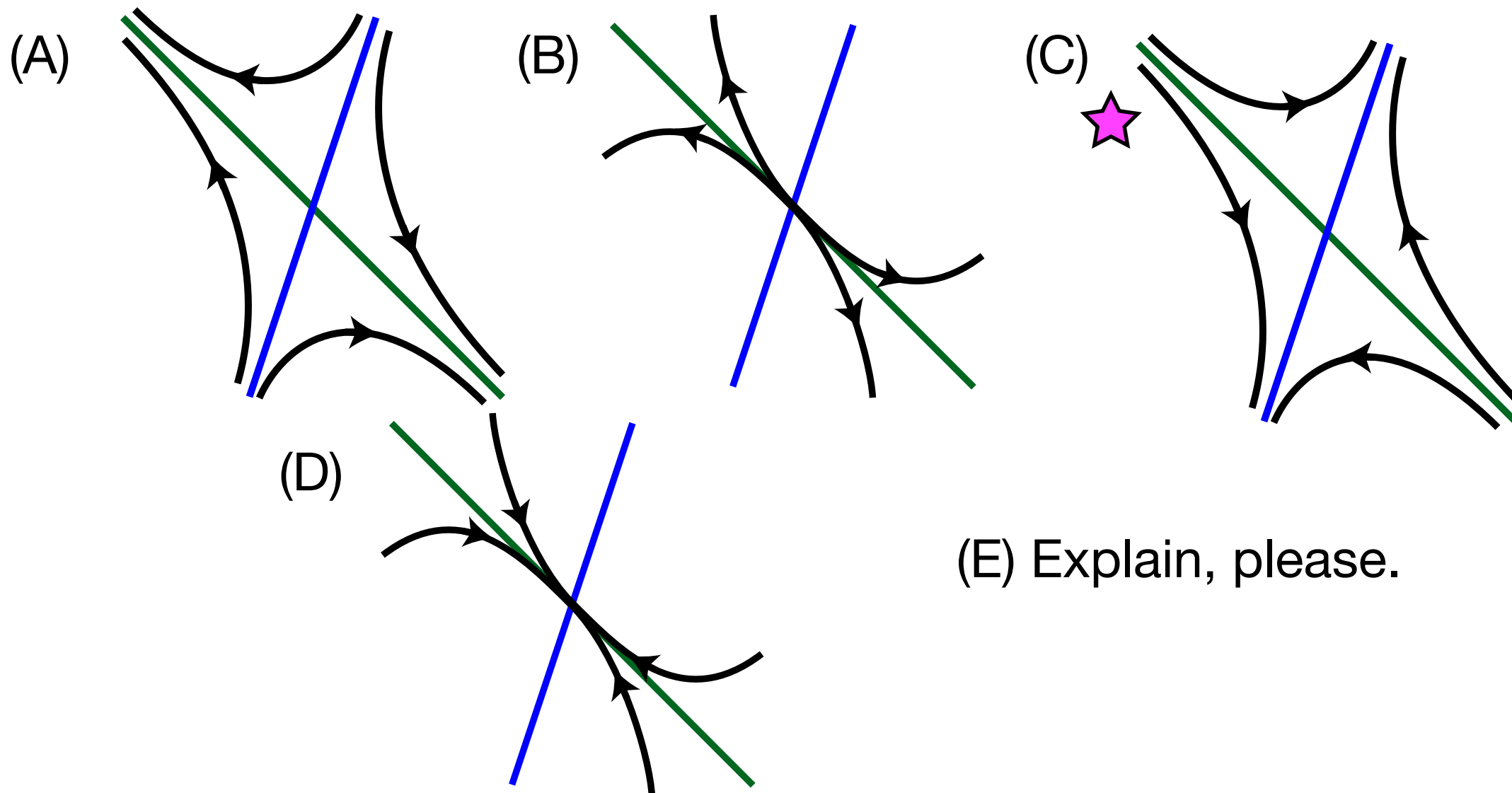
$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} ?$$



Shapes of solution curves in the phase plane

- Which phase plane matches the general solution

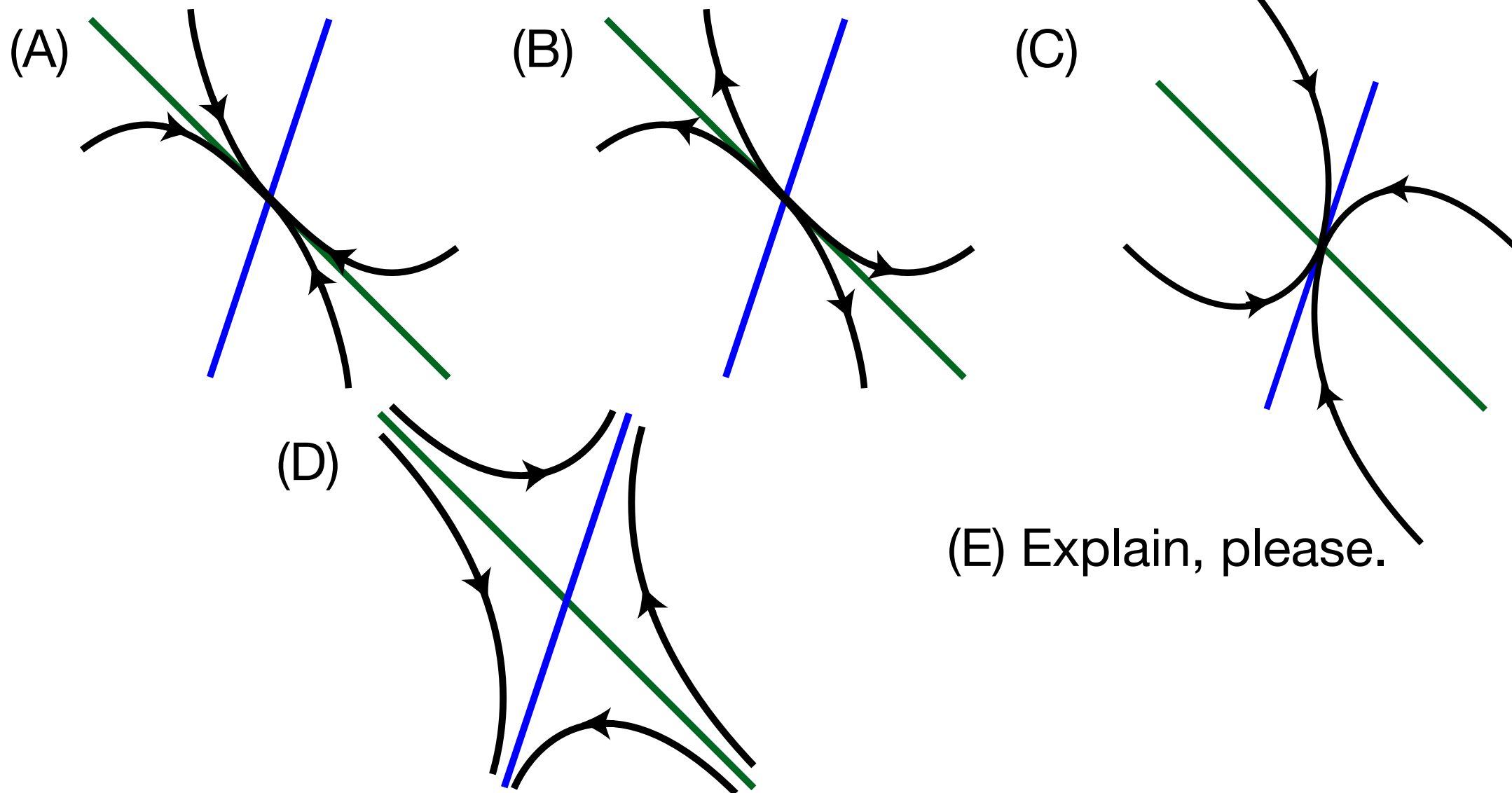
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Shapes of solution curves in the phase plane

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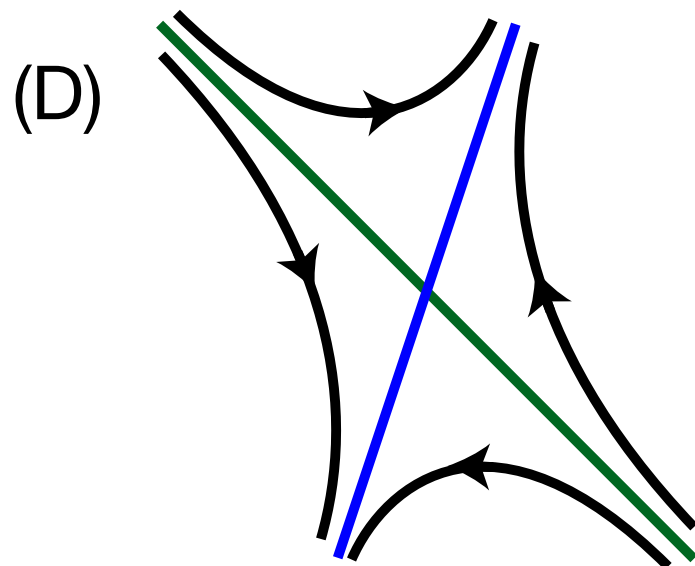
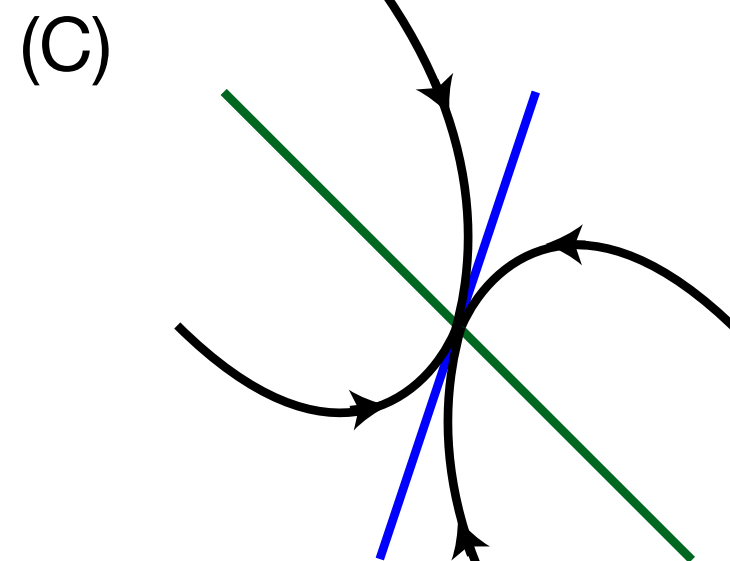
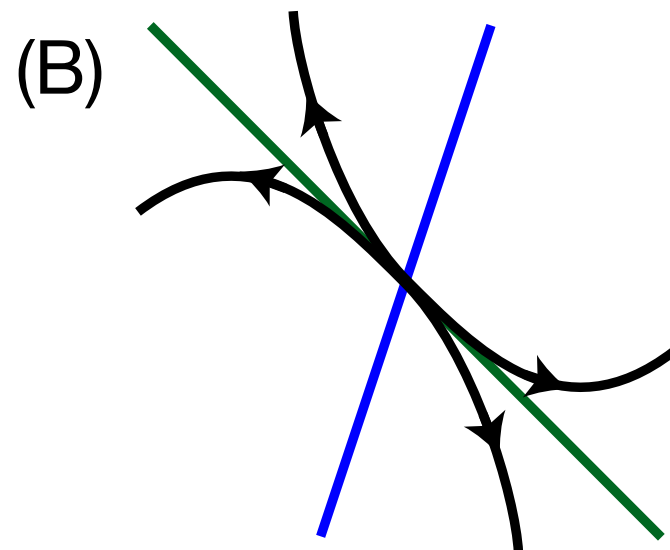
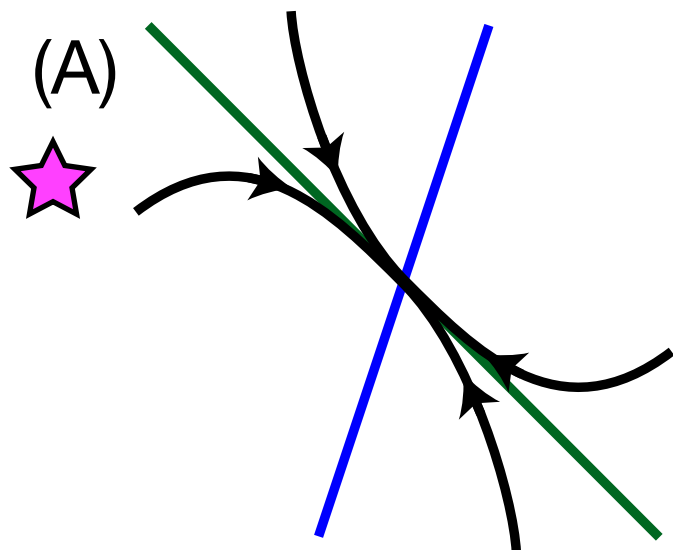
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Shapes of solution curves in the phase plane

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(E) Explain, please.