MATH 256 – Midterm 2 – March 20, 2014.

Last name: ________________ First name: ________________ Student #: ________________

I attend the tutorial in room: MATH 105 MATH 203 (circle one)

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. [**3 pts**] Calculate the Laplace transform of the function $g(t) = 3\delta(t-2) + 3u_3(t) + te^{4t}$.

2. [**3 pts**] Find the inverse Laplace transform of $Y(s) = e^{-2s}\dfrac{s}{s^2+5s+6}$.

3. [**3 pts**] Find the inverse Laplace transform of $Y(s) = \dfrac{12}{s^2+4s+40}$.

4. **[3 pts]** Find the general solution to the equation

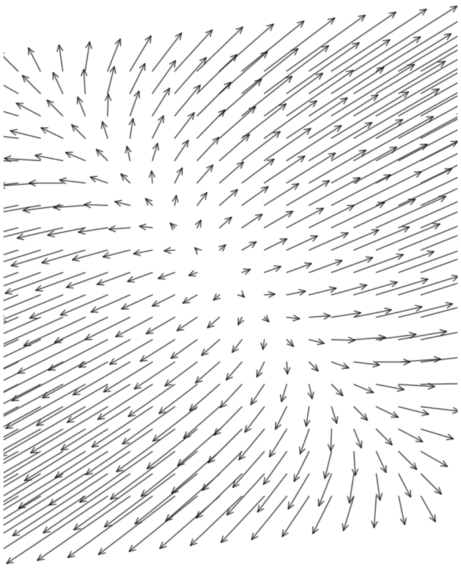
$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

Hint: one of the eigenvalues is $2 + 3i$.

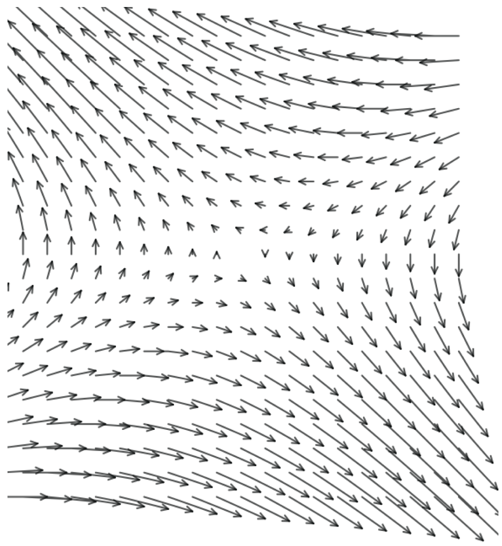
5. **[4 pts]** Morphine is cleared from the body at a rate proportional to the amount present with a rate constant $k = 1/3$ hour^{-1}. A patient is given an injection of 20 mg morphine immediately after surgery ($t = 0$) and again at 6 and 12 hours after surgery. Treating each injection as an instantaneous event, write down a differential equation to model the quantity of morphine in the patient's body as a function of time.

6. **[4 pts]** Match each solution to one of the vector fields.

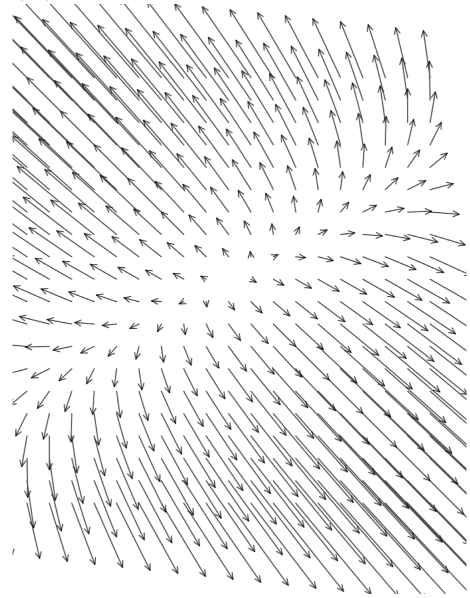
(A)



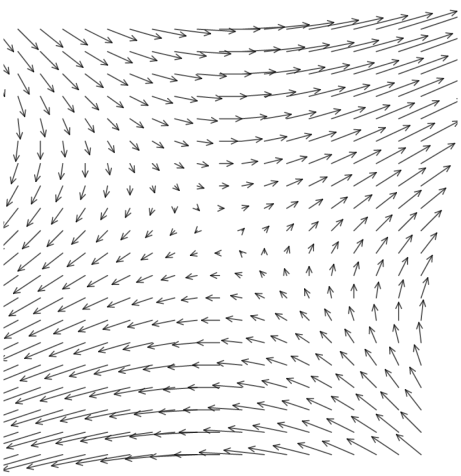
(B)



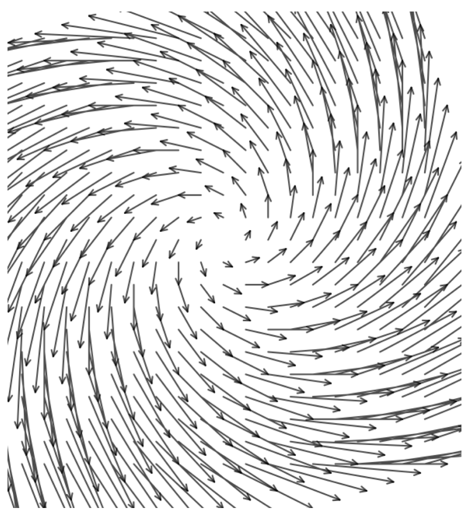
(C)



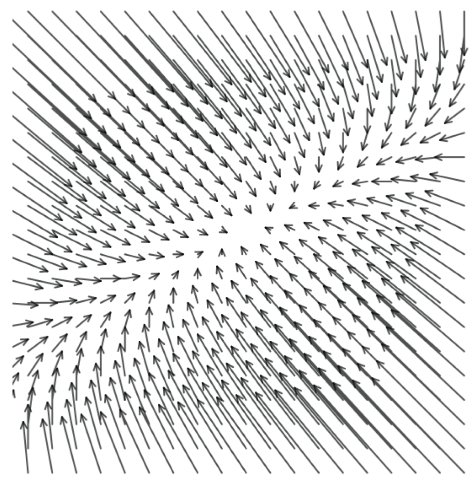
(D)



(E)

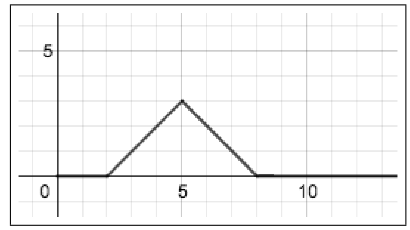


(F)



Solution	Vector field (enter A,B,C,D,E or F)
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$	
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$	
$\mathbf{x}(\mathbf{t}) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$	
$\mathbf{x}(\mathbf{t}) = e^{2t} \left[C_1 \begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix} \right]$	

7. [**4 pts**] Write down an expression for the function $g(t)$ shown in the figure below using Heavside functions in the form $u_c(t)$.



8. [**4 pts**] Consider the equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} \alpha & \beta \\ 1 & \alpha \end{pmatrix}.$$

In each row of the table below, give inequalities involving α and β which ensure that the steady state is of the given type. The first row provides an example.

Type	Condtion(s) on α and β
unstable node	$0 < \beta < \alpha^2$, $\alpha > 0$
stable node	
unstable spiral	
stable spiral	
saddle	

Anything on this page will not be marked. It is for rough work.

Laplace transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$u_c(t)f(t-c)$	$e^{-sc}F(s)$
$\delta(t-c)$	e^{-sc}
$\int_0^t f(t-w)g(w)\ dw$	$F(s)G(s)$

Equations

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)]$$