MATH 256 – Midterm 1 SOLUTIONS – February 6, 2014.

Last name: ______________ First name: ______________ Student #: ______________

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. [**5 pts**] The solution to the equation $y''+by'+cy=g(t)$ is given by $y(t)=C_1\cos(3t)+C_2\sin(3t)+t\sin(3t)$. What are the constants b and c and what is the function $g(t)$?

$y_h = C_1\cos 3t + C_2 \sin 3t$

$\Rightarrow r = \pm 3i$

$\Rightarrow r^2 = -9$

$\Rightarrow y''+9y$ ①

$\Rightarrow \boxed{b=0,\ c=9}$ ①

$y_p = t\sin 3t$

① $y_p' = \sin 3t + 3t\cos 3t$

① $y_p'' = 3\cos 3t + 3\cos 3t - 9t\sin 3t$

$y_p''+9y_p = 6\cos 3t$ ①

$\boxed{g(t) = 6\cos 3t}$

2. [**5 pts**] A tank initially contains m_0 kg of salt and a volume V litres of water. Saltwater with a concentration of c_0 kg/litre enters a tank at the rate r litres/minute. The solution is mixed and drains from the tank at the same rate (r litres/minute). Write down an Initial Value Problem (that is, **a differential equation** and **an initial condition**) for the mass of salt $m(t)$ in the tank as a function of time. You DO NOT need to solve it.

$$\frac{dm}{dt} = \underset{①}{c_0 r} - \underset{②}{\frac{r}{V}m}$$

$m(0) = m_0$ ①

① for +mass in − mass out

3. [**3 pts**] For each of the following equations, circle all the terms that render the equation nonlinear.

(a) $y'' + y'y + \sin(t)y = t^2$

(b) $h' = -k\sqrt{h} + 5e^{-2t}$

(c) $\theta'' + \frac{g}{L}\sin(\theta) = 0$

+1 for each correct

−1 for each incorrect

4. **[5 pts]** State the most suitable form for the particular solution as required for the Method of Undetermined Coefficients for the equation $y'' - y' - 30y = 7x^2e^{-5x}$. You DO NOT need to determine the coefficients.

$$r^2 - r - 30 = 0$$

$$r = -5, 6 \quad (1)$$

$\Rightarrow e^{-5x}$ solves homog. eq. so

$$y_p = \underset{(1)}{x}\,(\underset{(1)}{Ax^2} + \underbrace{Bx + C}_{(1)})\,\underset{(1)}{e^{-5x}}$$

5. (a) **[2 pts]** What does the Wronskian tell you about two solutions to a second order linear differential equation?

If $W(y_1, y_2) \neq 0$ (1) then y_1 and y_2 are independent and form a fundamental set of solutions which can be used to solve any IC. (1) (for any mention of ~)

(b) **[2 pts]** Calculate the Wronskian of $x(t) = \cos(\omega t)$ and $y(t) = \sin(\omega t)$.

$$\det\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{pmatrix} = \omega\cos^2\omega t + \omega\sin^2\omega t \quad (1)$$

$$= \omega \quad (1)$$

6. **[4 pts]** Find the general solution to the equation $y' + ty = t$.

$$y' + ty = t$$

$$\mu = e^{t^2/2} \quad (1)$$

$$(e^{t^2/2}y)' = te^{t^2/2}$$

$$e^{t^2/2}y = \underset{(1)}{e^{t^2/2}} + C \quad (1)$$

$$y = 1 + Ce^{-t^2/2} \quad (1)$$

7. An ectotherm is an animal that does not regulate its own temperature but instead depends on the environmental temperature to determine its body temperature. The rate of change of an ectotherm's body temperature satisfies Newton's Law of Cooling:

$$B' = k(E(t) - B)$$

where $B(t)$ is the animal's body temperature, $E(t)$ is the environmental temperature and k is a positive constant that is determined by how well insulated the animal is - better insulation means a smaller k value. The environmental temperature varies daily as $E(t) = 20 + 10\cos(t)$ in degrees Celsius.

(a) **[8 pts]** Find the general solution $B(t)$. Hint: This can be done using an integrating factor or the Method of Undetermined Coefficients. The latter approach is the simpler one in this case.

Using undetermined coefficients:

$$B' + kB = k(20 + 10\cos t)$$

$$B_h(t) = Ae^{-kt} \quad ①$$

$$B_p(t) = \mathcal{B} + C\cos t + D\sin t \quad ③$$

$$B_p'(t) = -C\sin t + D\cos t$$

$$B_p' + kB_p = k\mathcal{B} + (kC + D)\cos t + (kD - C)\sin t$$

$$k\mathcal{B} = 20k \qquad kC + D = 10k \qquad kD - C = 0$$

$$\mathcal{B} = 20 \quad ① \qquad k(kD) + D = 10k \qquad C = kD$$

$$D = \frac{10k}{1+k^2} \quad ① \qquad C = \frac{10k^2}{1+k^2} \quad ①$$

$$B(t) = Ae^{-kt} + 20 + \frac{10k^2}{1+k^2}\cos t + \frac{10k}{1+k^2}\sin t \quad ①$$

Or using an integrating factor:

$$B' + kB = k(20 + 10\cos t)$$

$$① \quad (e^{kt}B)' = ke^{kt}(20 + 10\cos t)$$

$$e^{kt}B = 20e^{kt} ① + 10k\int e^{kt}\cos t\,dt$$

$$I = \int e^{kt}\cos t\,dt \overset{*}{=} e^{kt}\sin t - \int ke^{kt}\sin t\,dt$$

$$\overset{*}{=} e^{kt}\sin t + ke^{kt}\cos t - k^2\int e^{kt}\cos t\,dt$$

$$= e^{kt}\sin t + ke^{kt}\cos t - k^2 I$$

$$(1+k^2)I = e^{kt}\sin t + ke^{kt}\cos t + C'$$

$$I = \frac{e^{kt}\sin t}{1+k^2} ① + \frac{ke^{kt}\cos t}{1+k^2} + C \quad ①$$

① for the "I" approach

① for each IBPs

$$B(t) = 20 + \frac{10k}{1+k^2}\sin t + \frac{10k^2}{1+k^2}\cos t + Ce^{-kt} \quad ①$$

* by integration by parts

(b) **[1 pt]** What is the amplitude of the oscillatory part of the general solution? Your answer should depend on k.

$$\text{Amp} = \sqrt{\left(\frac{10k^2}{1+k^2}\right)^2 + \left(\frac{10k}{1+k^2}\right)^2}$$

$$= \frac{10k}{1+k^2}\sqrt{k^2+1} = \frac{10k}{\sqrt{1+k^2}} \quad 1$$

(c) **[1 pt]** Give an approximation (independent of k) for the phase difference between the oscillatory part of the animal's temperature and the environmental temperature when the animal is very poorly insulated (k very large)?

$$\cos\phi = \frac{k}{\sqrt{1+k^2}} \longrightarrow 1 \text{ as } k \to \infty$$

$$\sin\phi = \frac{1}{\sqrt{1+k^2}} \longrightarrow 0 \text{ as } k \to \infty$$

$$\text{so } \phi \to 0 \quad 1$$

(d) **[1 pt]** Give an approximation (independent of k) for the phase difference between the oscillatory part of the animal's temperature and the environmental temperature when the animal is very well insulated (k very small)?

$$\cos\phi = \frac{k}{\sqrt{1+k^2}} \longrightarrow 0 \text{ as } k \to 0$$

$$\sin\phi = \frac{1}{\sqrt{1+k^2}} \longrightarrow 1 \text{ as } k \to 0$$

$$\text{so } \phi \to \frac{\pi}{2} \quad 1$$

Anything on this page will not be marked. It is for rough work.

Formulae

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$C_1\cos(\omega t) + C_2\sin(\omega t) = A\cos(\omega t - \phi)$$

$$A = \sqrt{C_1^2 + C_2^2}, \quad \cos(\phi) = \frac{C_1}{A}, \quad \sin(\phi) = \frac{C_2}{A}$$

$$\omega_0^2 = \frac{k}{m}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Critical damping: $\gamma^2 = 4mk$

$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$W(y_1, y_2)(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$
