Today

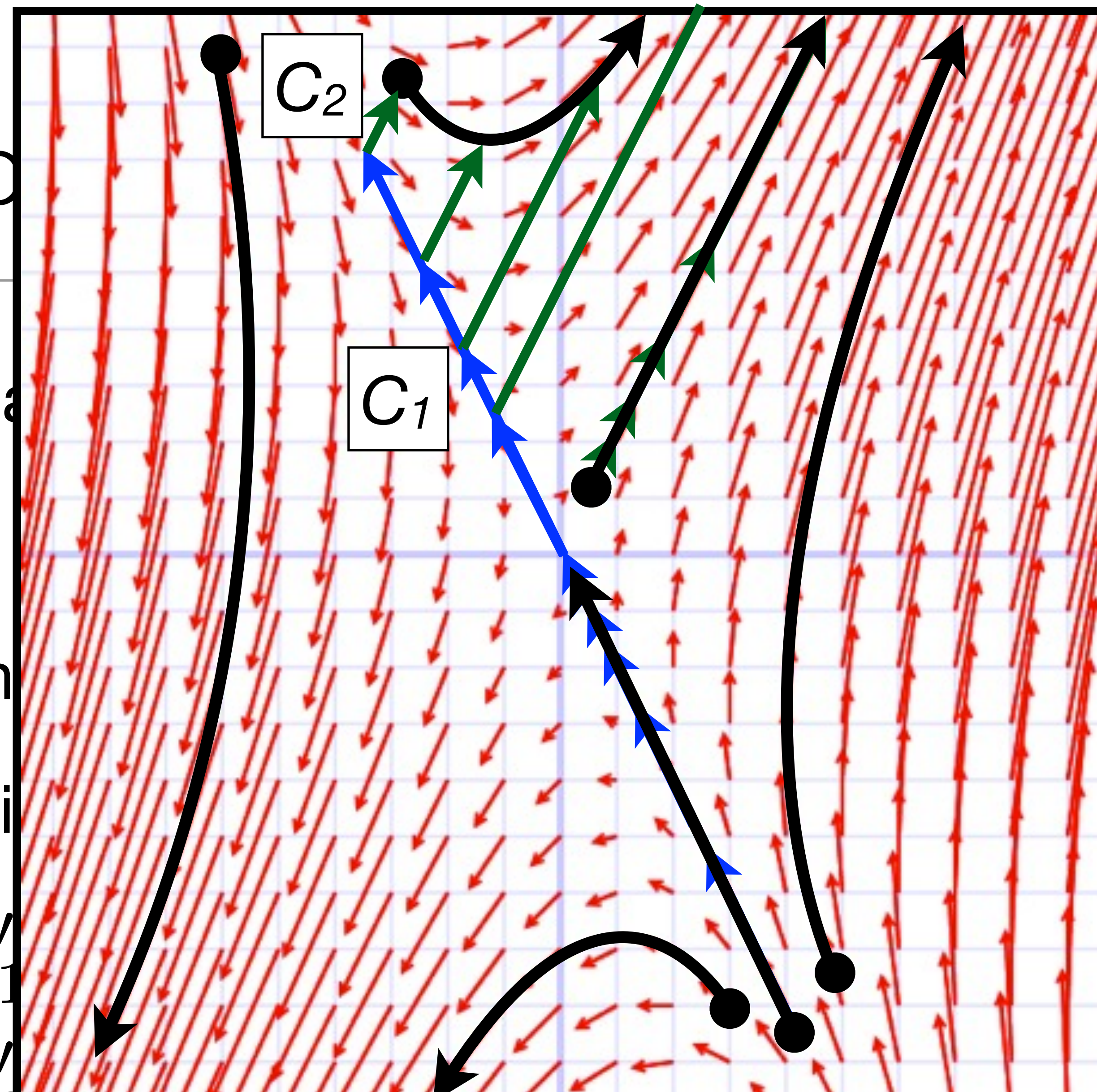
- Shapes of solutions for distinct eigenvalues case.
- General solution for complex eigenvalues case.
- Shapes of solutions for complex eigenvalues case.
- Office hours: Friday 1-2 pm, Monday 1-3 pm (to be confirmed)

Shapes of so… e plane

- When matrix A ha… olution to **x**'=A**x** is
- What do solution… he phase plane)?
- If the initial condi… on is a straight line. Example:

x_1' … 6

x_2' … $x_1 + x_2$ … $x_2(0) = -12$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$C_1 = 6, \; C_2 = 0$$

- IVP solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

Shapes of solution curves in the phase plane

- Simple example to show general idea.

$$\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1(t) = C_1 e^{\lambda_1 t} \qquad t = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$x_2(t) = C_2 e^{\lambda_2 t} \qquad t = \frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right)$$

$$\frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right) = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \frac{\lambda_2}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \ln\left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

$$x_2 = C_2 \left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

- Can we plot solutions in x_1-x_2 plane by graphing x_2 versus x_1?

Shapes of solution curves in the phase plane

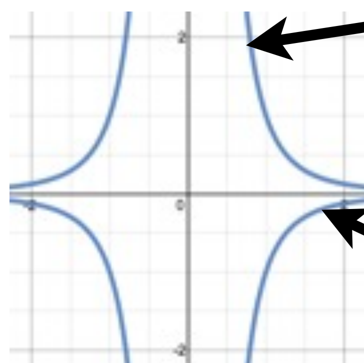
- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

$$x_2 = C_2 \left(\frac{x_1}{C_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

- For the shape of solutions, w… know the sign and size of $\frac{\lambda_2}{\lambda_1}$.

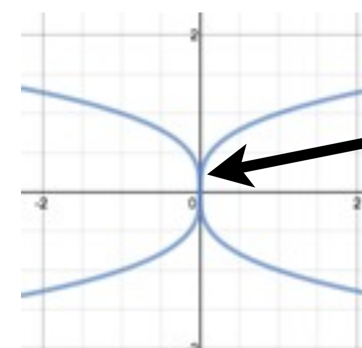
$$\lambda_2 = -3\lambda_1$$

$$x_2 = \frac{C}{x_1^3}$$



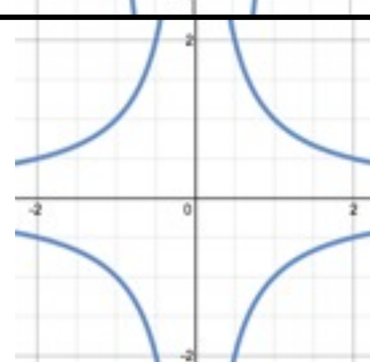
$$\lambda_2 = \frac{1}{3}\lambda_1$$

$$x_2 = C\sqrt[3]{x_1}$$



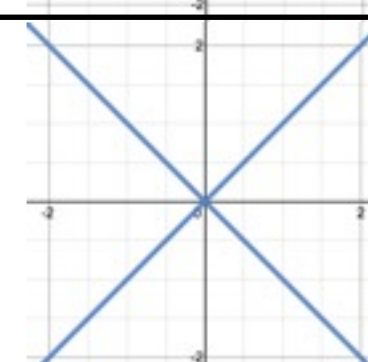
$$\lambda_2 = -\lambda_1$$

$$x_2 = \frac{C}{x_1}$$



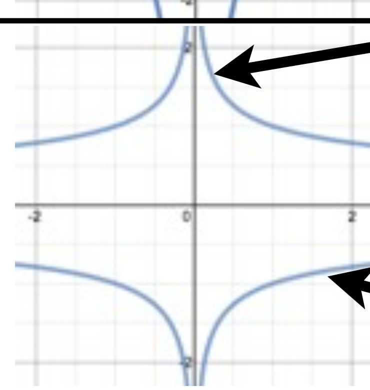
$$\lambda_2 = \lambda_1$$

$$x_2 = Cx_1$$



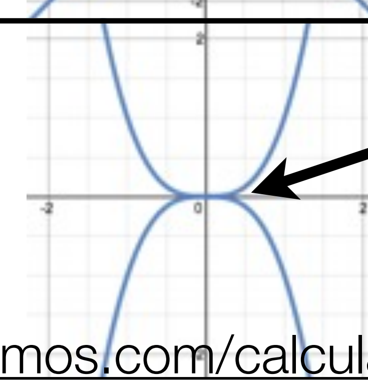
$$\lambda_2 = -\frac{1}{3}\lambda_1$$

$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



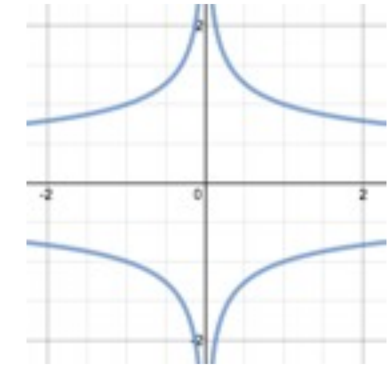
$$\lambda_2 = 3\lambda_1$$

$$x_2 = Cx_1^3$$



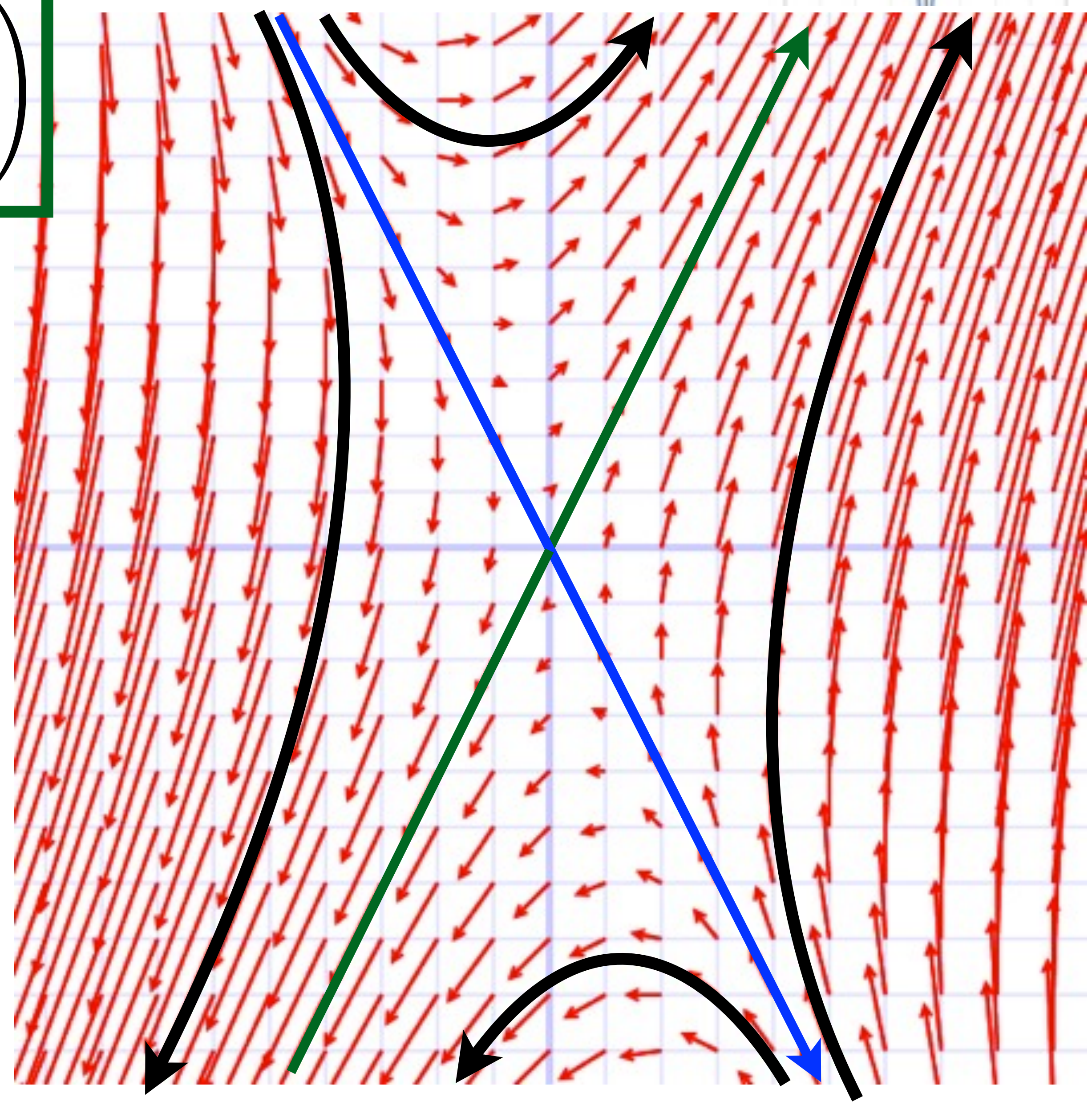
https://www.desmos.com/calculator/c4rhrgotmo

Shapes of solution curves in the phase plane



- With more complicated solutions (evectors off-axis), tilt shapes accordingly.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} + \boxed{C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$



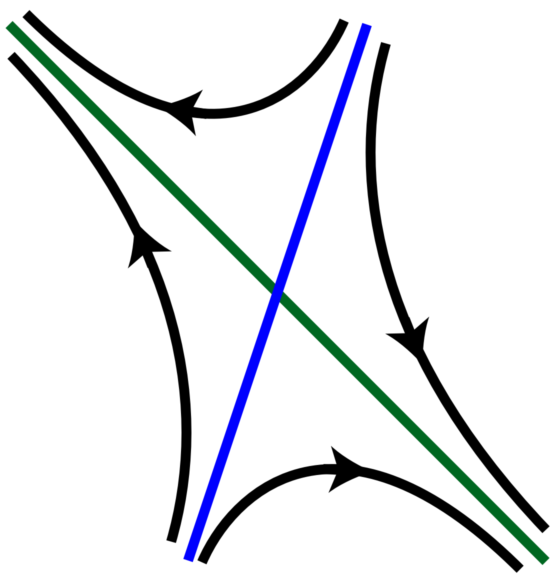
- Going forward in time, the blue component shrinks slower than the green component grows so solutions appear closer to blue "axis" than to green "axis"

Shapes of solution curves in the phase plane

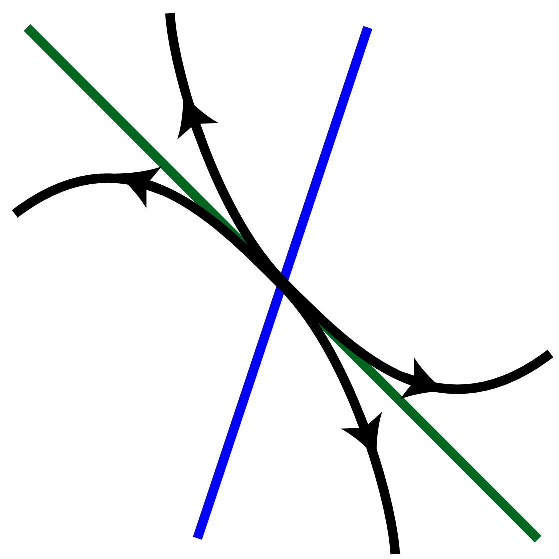
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{3t}\begin{pmatrix}1\\3\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}\ ?$$

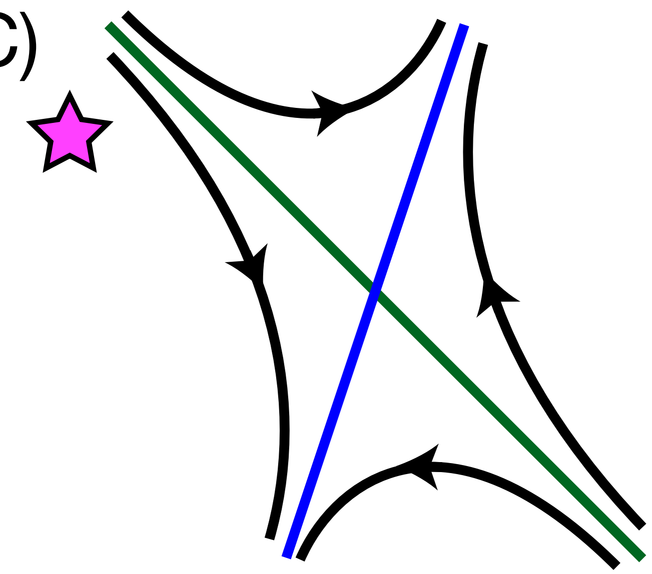
(A)



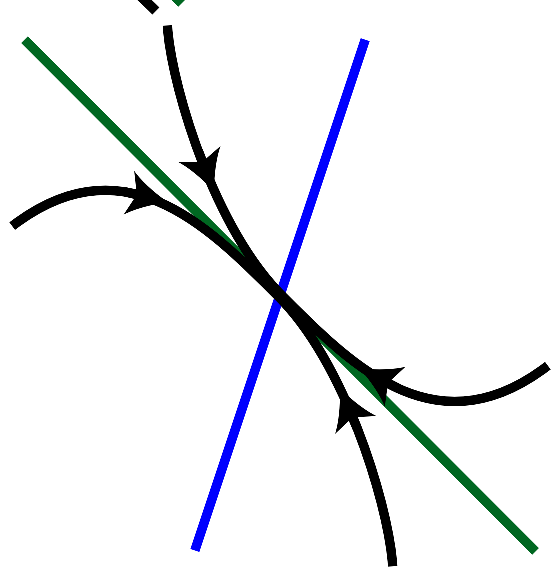
(B)



(C)



(D)



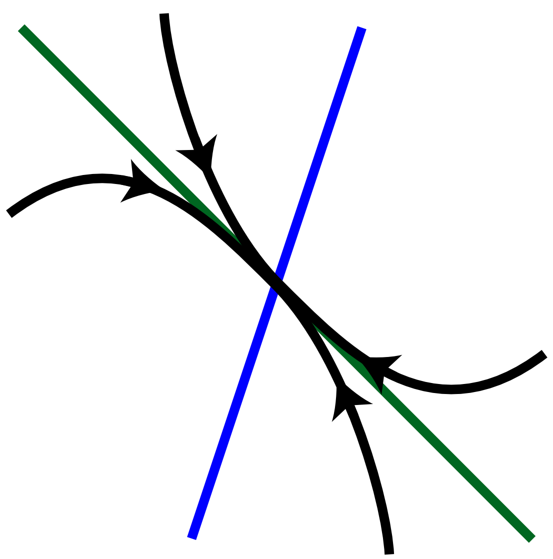
(E) Explain, please.

Shapes of solution curves in the phase plane

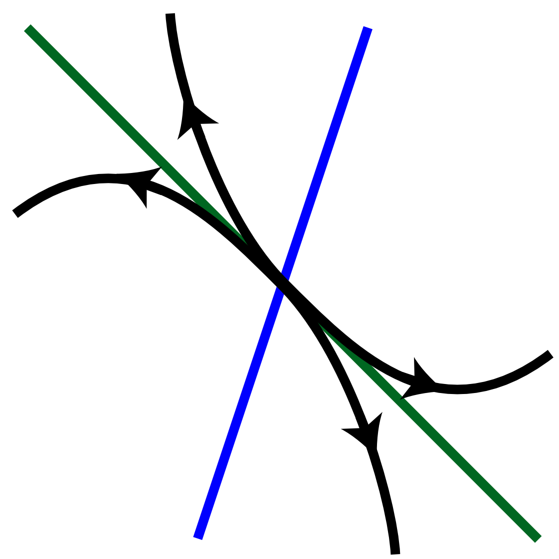
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ ?$$

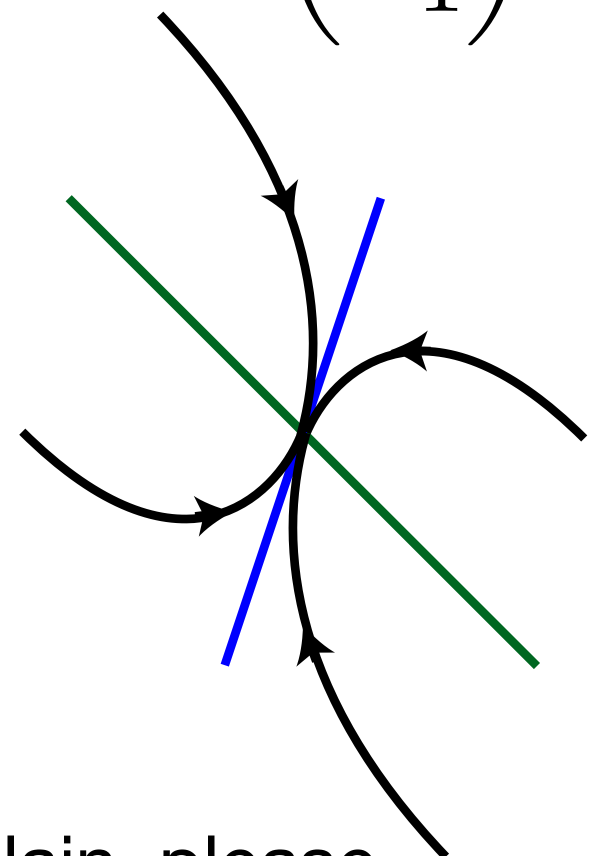
(A)



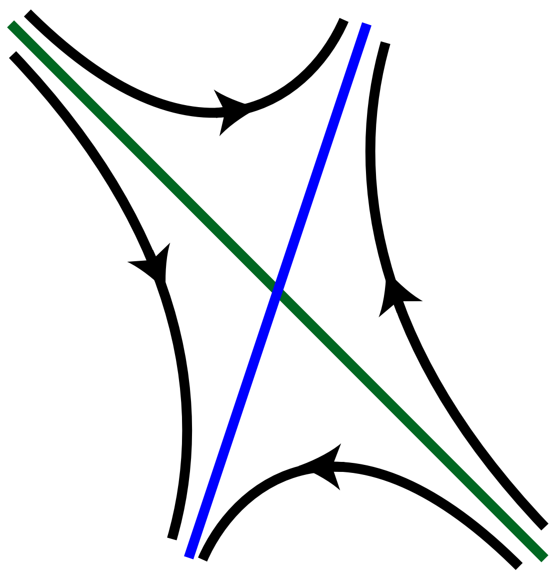
(B)



(C)



(D)



(E) Explain, please.