Today

- Summary of steps for solving the Diffusion Equation with homogeneous Dirichlet or Neumann BCs using Fourier Series.
- Nonhomogeneous BCs
- Mixed Dirichlet/Neumann BCs
- End-of-term info:
  - Don’t forget to complete the online teaching evaluation survey.
  - Next Thursday, two-stage review (optionally for 2/50 exam points).
  - Office hours during exams TBA but sometime Apr 15/16/27.

Using Fourier Series to solve the Diffusion Equation

- Steps to solving the PDE:
  - Determine the eigenfunctions for the problem (look at BCs).
  - Represent the IC u(x,0)=f(x) by a sum of eigenfunctions (Fourier series).
  - Write down the solution by inserting $e^{\lambda t}$ into each term of the FS.

$$u_t = D u_{xx}$$ → PDE determines all possible eigenfunctions.

$$\left.\frac{du}{dx}\right|_{x=0,L} = 0$$ → BCs select a subset of the eigenfunctions.

$$u(x,0) = f(x)$$ → IC is satisfied by adding up eigenfunctions.

Using Fourier Series to solve the Diffusion Equation

$u_t = Du_{xx}$ ⟶ PDE determines all possible eigenfunctions.

Let's look for all possible eigenfunctions:

$Dv_{xx}(x) = \lambda v(x)$

Case I: λ<0. $v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

For each value of λ<0, these are both eigenfunctions.

Case II: λ=0. $v_{xx} = 0 \Rightarrow v_x = C_1 \Rightarrow v(x) = C_1 x + C_2$

The λ=0 eigenfunctions are therefore $v(x) = 1$ and $v(x) = x$.

These do not decay with time so they form the steady state.

Case III: λ>0. $v_\lambda(x) = e^{\sqrt{\frac{\lambda}{D}}x}$ and $w_\lambda(x) = e^{-\sqrt{\frac{\lambda}{D}}x}$

These don't satisfy any BCs so we'll drop this case.

Using Fourier Series to solve the Diffusion Equation

$u_t = D u_{xx}$ → PDE determines all possible eigenfunctions.

$\left.\dfrac{du}{dx}\right|_{x=0,L} = 0$ → BCs select a subset of the eigenfunctions.

Case I: λ<0. $v_\lambda(x) = \cos\left(\sqrt{\dfrac{-\lambda}{D}}\,x\right)$ and $w_\lambda(x) = \sin\left(\sqrt{\dfrac{-\lambda}{D}}\,x\right)$

The BC at x=0 only works for $v_\lambda(x)$ and the BC at x=L only works for certain λ, in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: λ=0. $v(x) = 1$ and $v(x) = x$

Represent IC u(x,0) = f(x) by $u(x,0) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

e^{0t}

<-- Steady state is constant term, that is the average value of f(x)!

Using Fourier Series to solve the Diffusion Equation

$u_t = Du_{xx}$ → PDE determines all possible eigenfunctions.

$u(0,t) = u(2,t) = 0$ → BCs select a subset of the eigenfunctions.

Case I: λ<0. $v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

The BC at x=0 only works for $w_\lambda(x)$ and the BC at x=L only works for certain λ, in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: λ=0. $v(x) = 1$ and $v(x) = x$

Represent IC u(x,0) = f(x) by $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = 0$$

$$u(x,0) = \sin\frac{3\pi x}{2}$$

Write down the solution to this IVP.

(A) $u(x,t) = e^{-9\pi^2 t}\cos\frac{3\pi x}{2}$ ← doesn't satisfy IC.

(B) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2}$ ← don't satisfy BCs.

(C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$ $\qquad b_n = \int_0^2 \sin\frac{3\pi x}{2}\sin\frac{n\pi x}{2}\,dx$

★ (D) $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos\frac{n\pi x}{2}$ $\qquad a_n = \int_0^2 \sin\frac{3\pi x}{2}\cos\frac{n\pi x}{2}\,dx$

...with nonhomogeneous boundary conditions

$u_t = Du_{xx}$

$u(0,t) = 0$

$u(2,t) = 4$ → Nonhomogeneous BCs

an eigenfunction for the homogeneous BCs

Case I: λ<0. ~~$v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$~~ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

The BC at x=0 only works for $w_\lambda(x)$ and the BC at x=L "almost" works for certain λ, in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: λ=0. ~~$v(x) = 1$~~ and $v(x) = 2x$ ← a particular eigenfunction for the inhomogeneous BCs

Ultimately, we want $u(x,t) = 2x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$

What function do we use to calculate the Fourier series $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$?

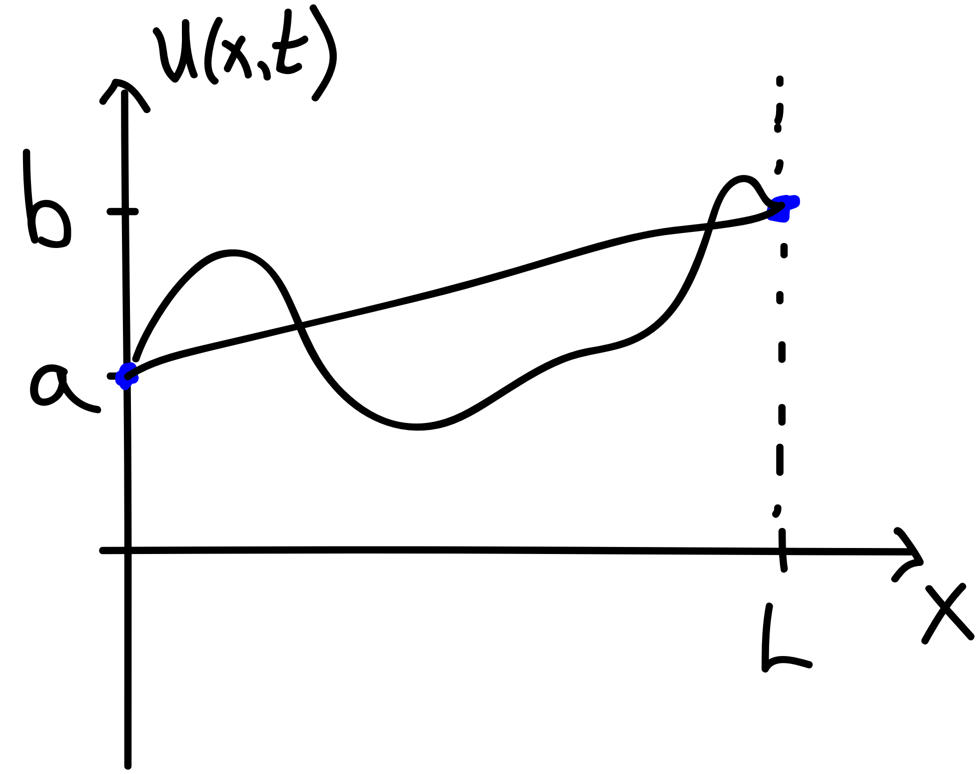
(A) u(x,0) (B) u(x,0) - 2 (C) u(x,0) - 2x ★ (D) u(x,0) + 2x

...with nonhomogeneous boundary conditions

- Solve the Diffusion Equation with nonhomogeneous BCs:

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

- Recall - rate of change is proportional to curvature so bumps get ironed out.



$$v(x,t) = u(x,t) - \left(a + \frac{b-a}{L}x\right)$$

$$\left.\begin{aligned} v_t &= u_t \\ v_{xx} &= u_{xx} \end{aligned}\right\} \Rightarrow v_t = Dv_{xx}$$

$$v(0,t) = u(0,t) - a = 0$$
$$v(L,t) = u(L,t) - b = 0$$
$$v(x,0) = u(x,0) - \left(a + \frac{b-a}{L}x\right)$$

- v(x,t) satisfies the Diffusion Eq with homogeneous Dirichlet BCs and a new IC.
- General trick: define v=u-SS and find v as before.