Today

- Finish up undetermined coefficients
- Physics applications - mass springs
- Undamped, over/under/critically damped oscillations

Method of undetermined coefficients (3.5)

- Example. Find the general solution to $y'' + 2y' = e^{2t} + t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt$

(B) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt + E$

(C) $y_p(t) = Ae^{2t} + (Bt^4 + Ct^3 + Dt^2 + Et)$

(D) $y_p(t) = Ae^{2t} + Be^{-2t} + Ct^3 + Dt^2 + Et + F$

(E) Don’t know / still thinking.

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For each wrong answer, for what DE is it the correct form?

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Can't find A that works! Need 3 unknowns to match all 3 terms.

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$$\begin{aligned} y_p(x) &= Ax^3 e^{-5x} + Bx^2 e^{-5x} + Cxe^{-5x} \\ &= x(Ax^2 e^{-5x} + Bxe^{-5x} + Ce^{-5x}) \end{aligned}$$

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    - Never include a solution to the h-problem as it won't survive L[ ]. Just make sure you aren't missing another term somewhere.

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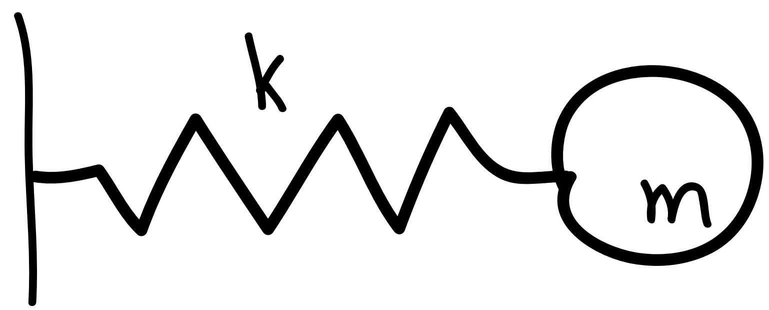
  - If you try a form and you can make LHS=RHS with some choice for the coefficients then you're done.
  - If you can't, your guess is most likely missing a term(s).

Applications - vibrations (3.7)

Mass-spring systems



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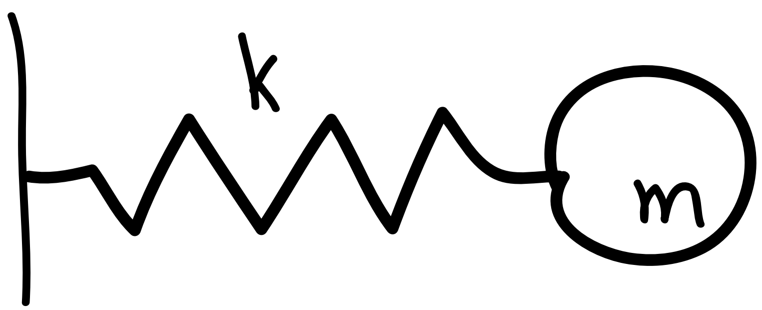

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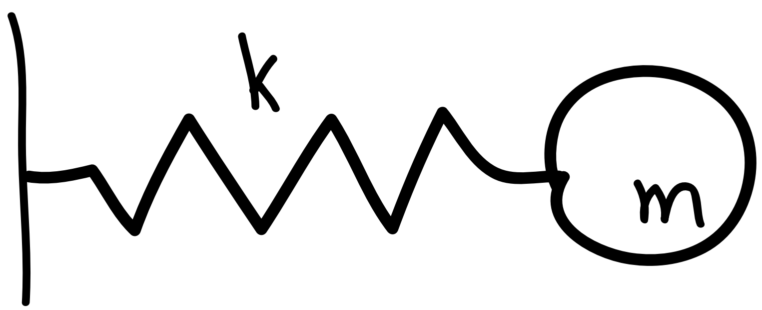

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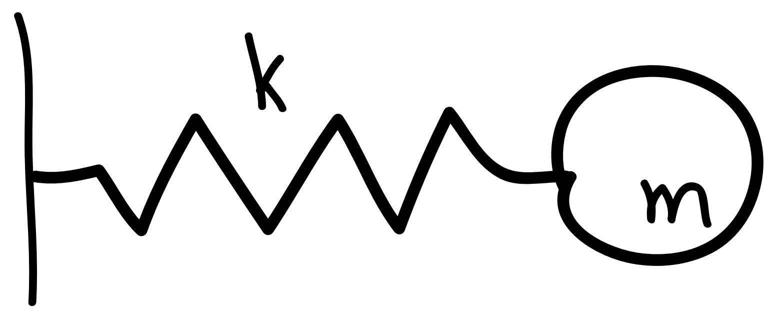

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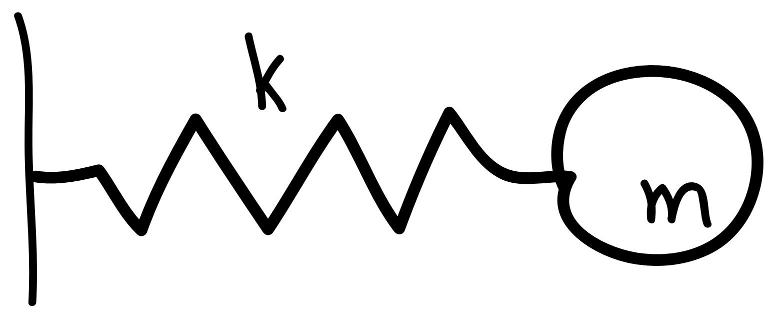

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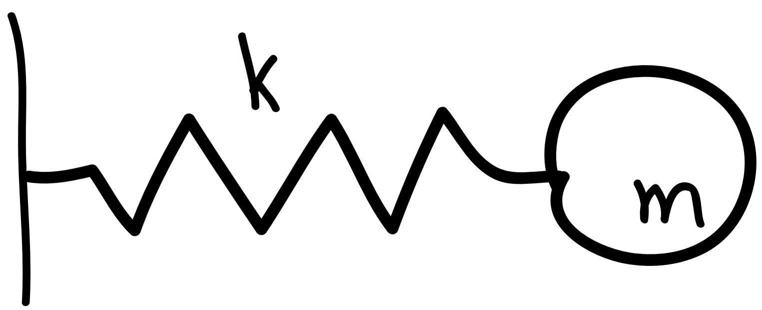

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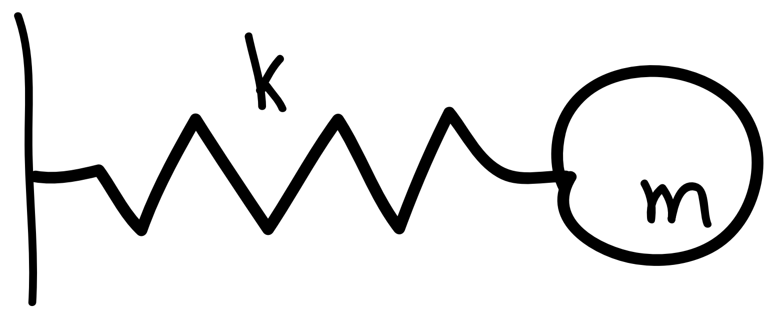

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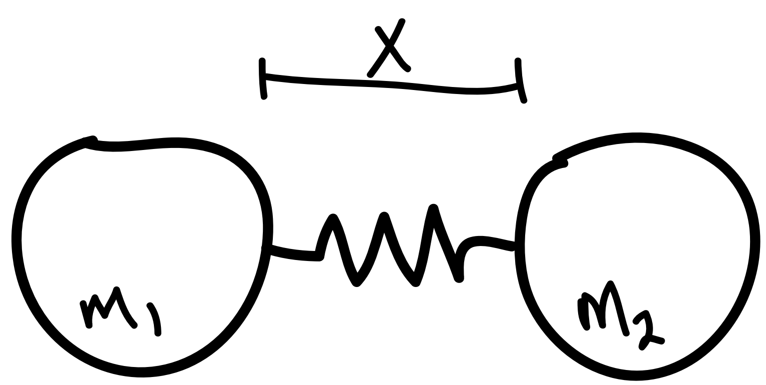

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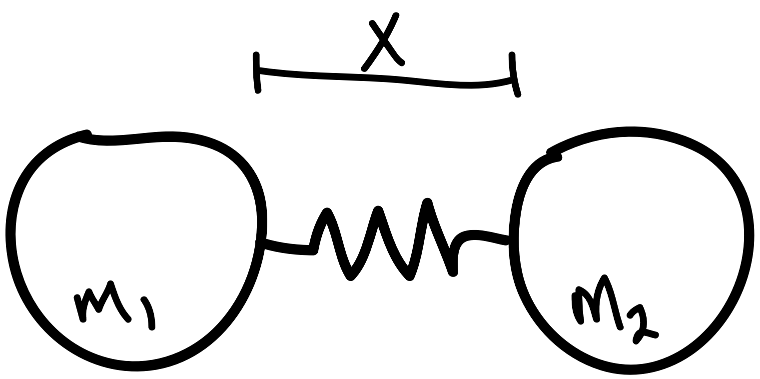

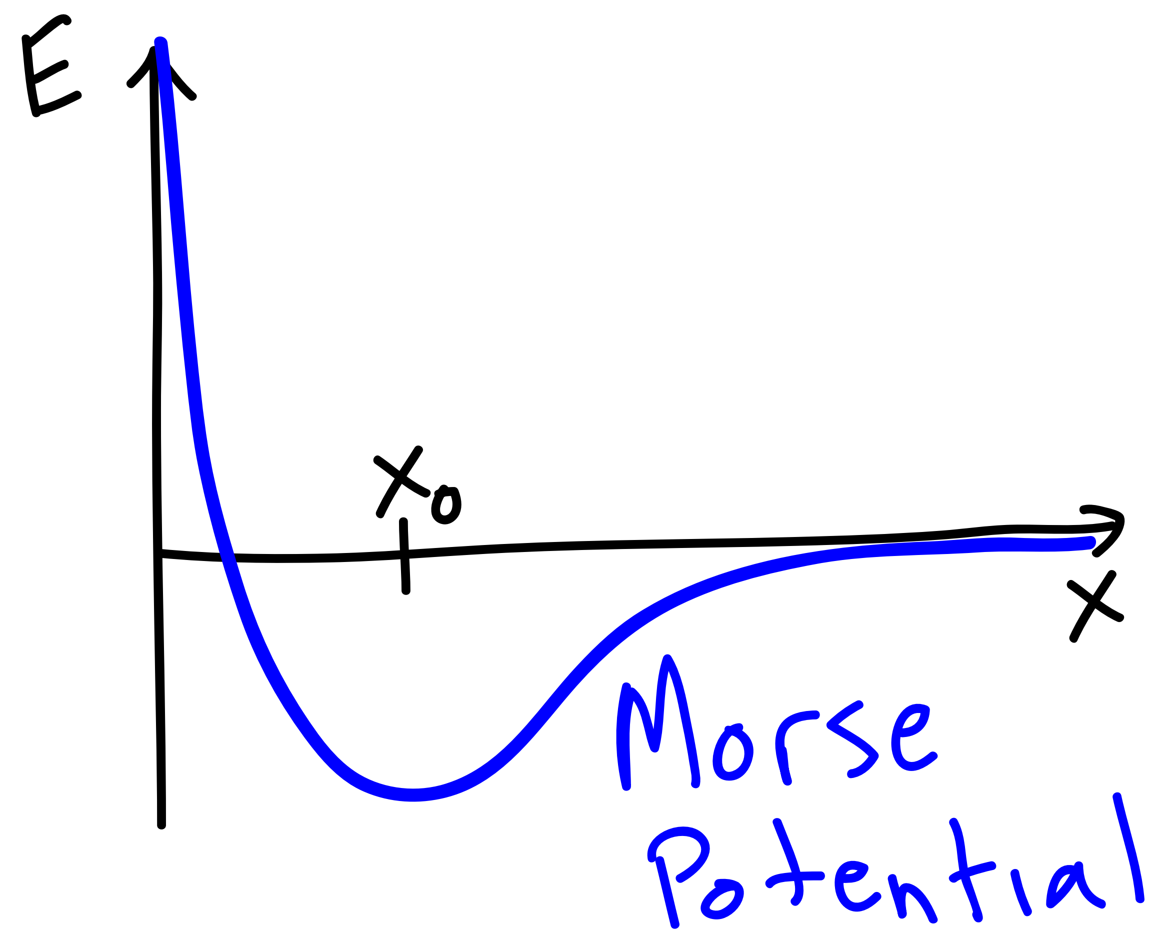

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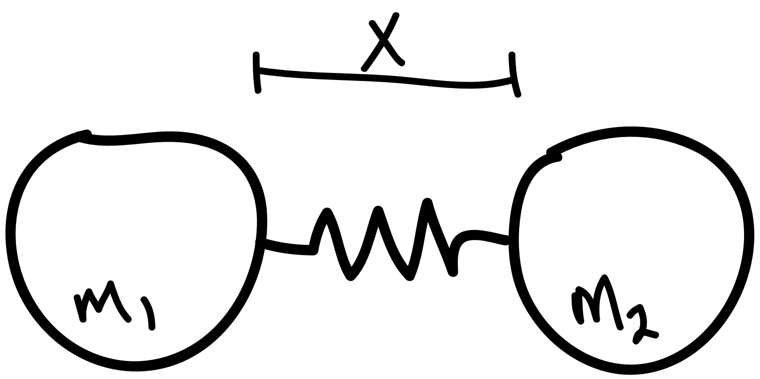

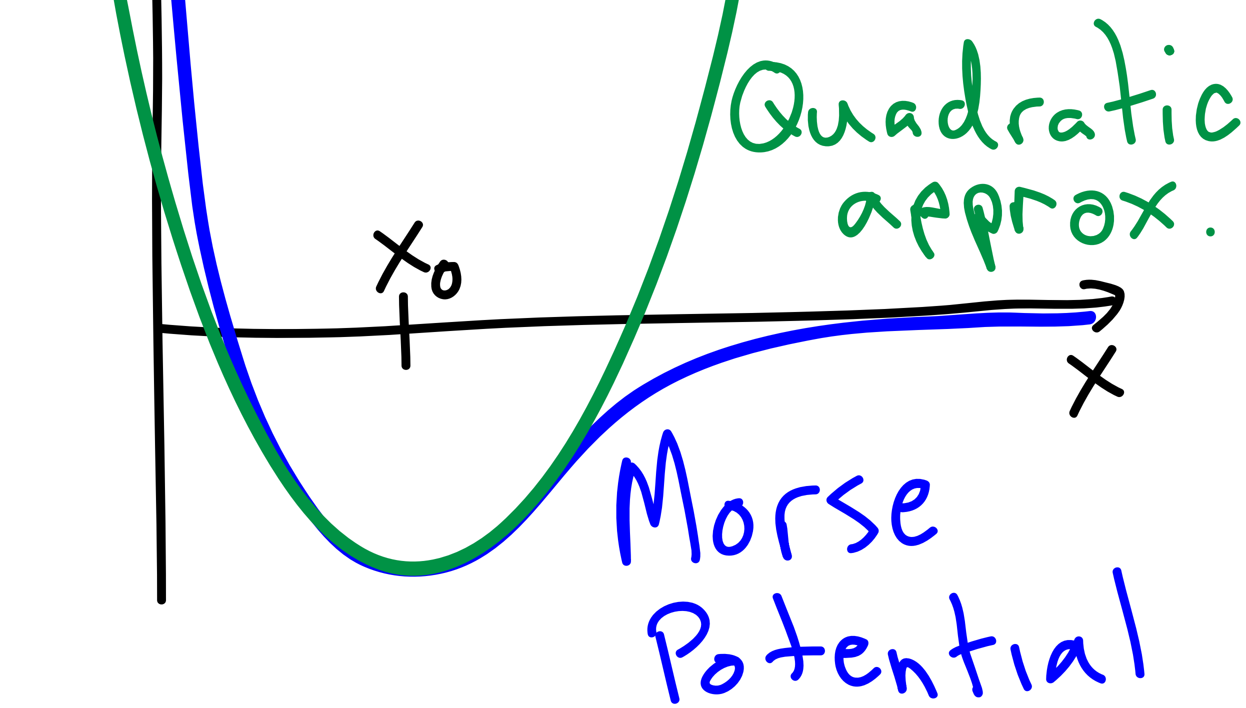

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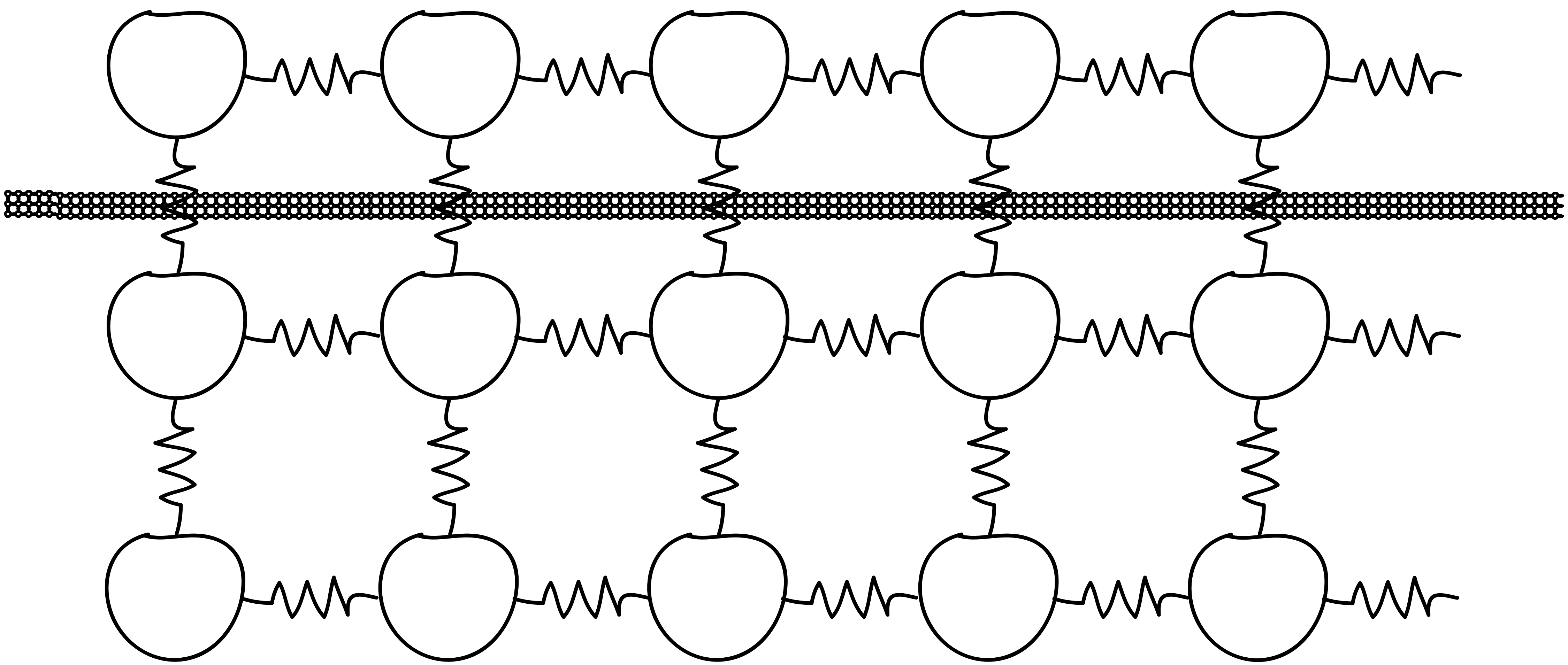
e.g. tuning fork, bridges, buildings



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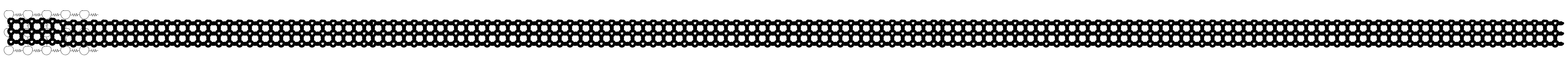

Applications - vibrations (3.7)

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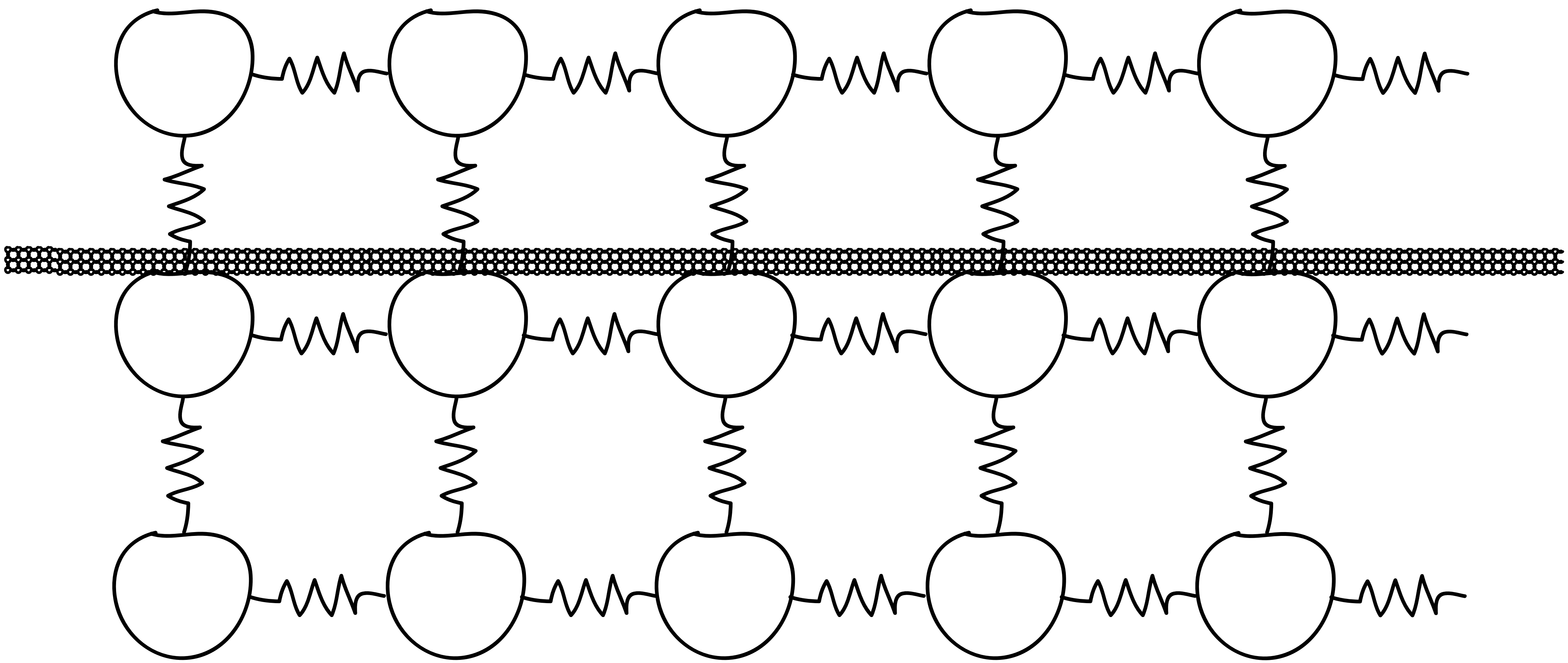

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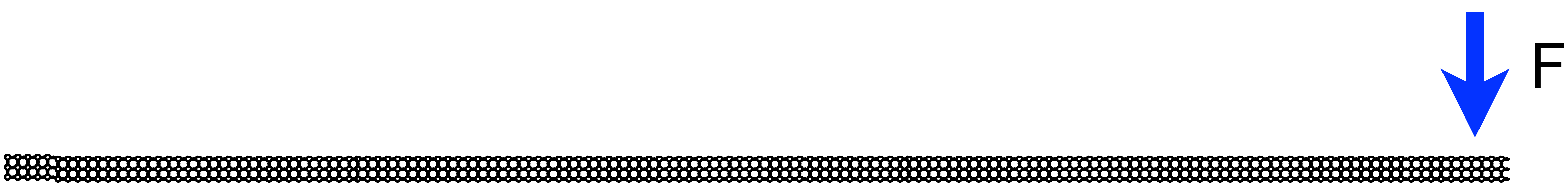

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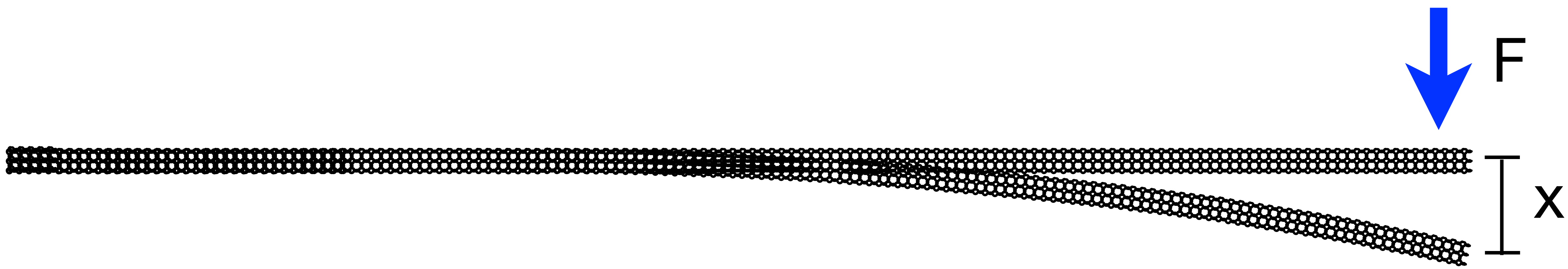

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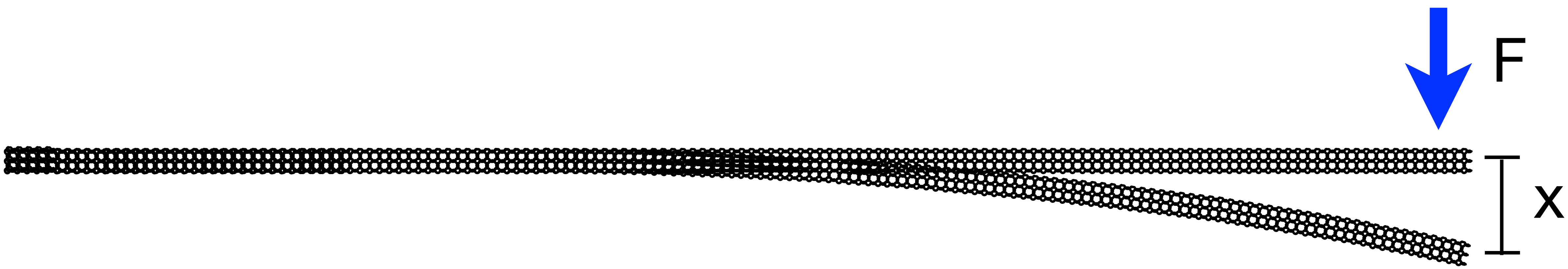

$$x'' = -Kx$$

where K depends on the molecular details of the material and the cross-sectional geometry of the rod.

Applications - vibrations (3.7)

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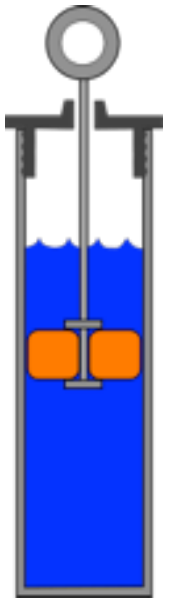

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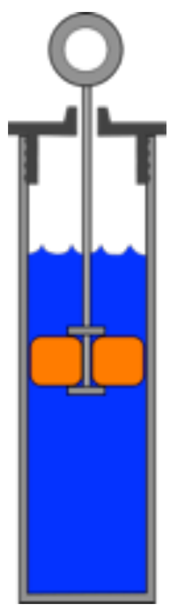

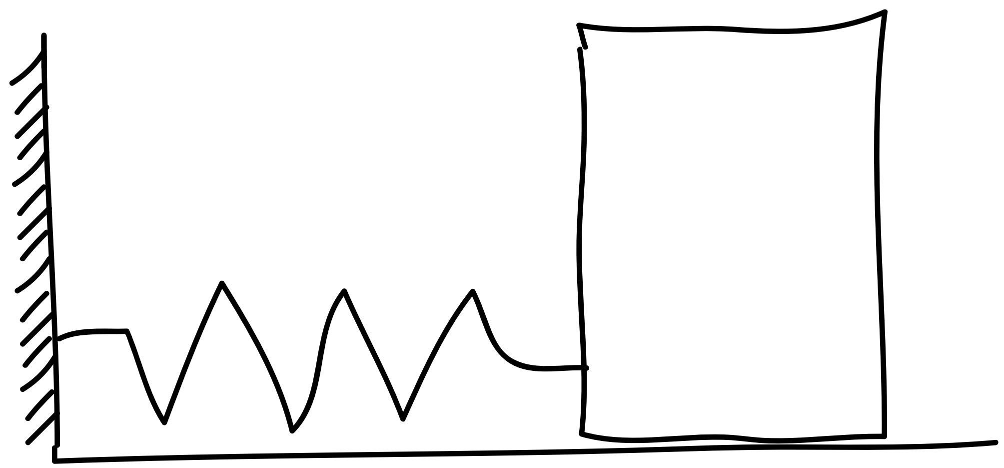

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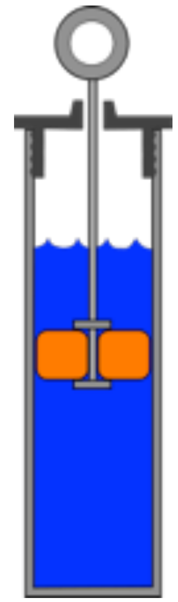

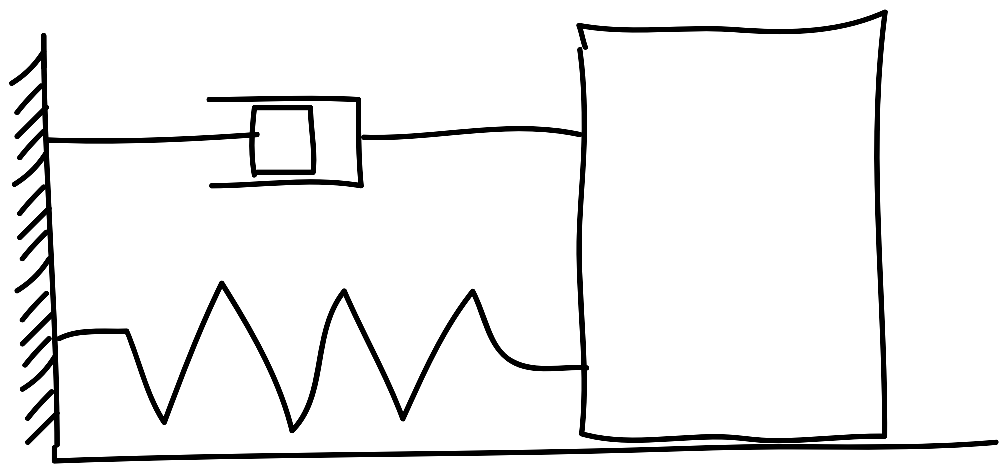

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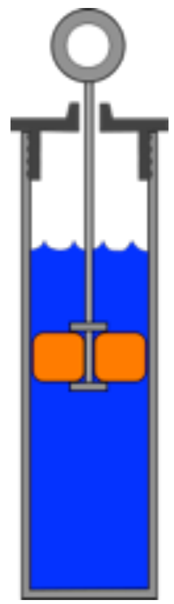

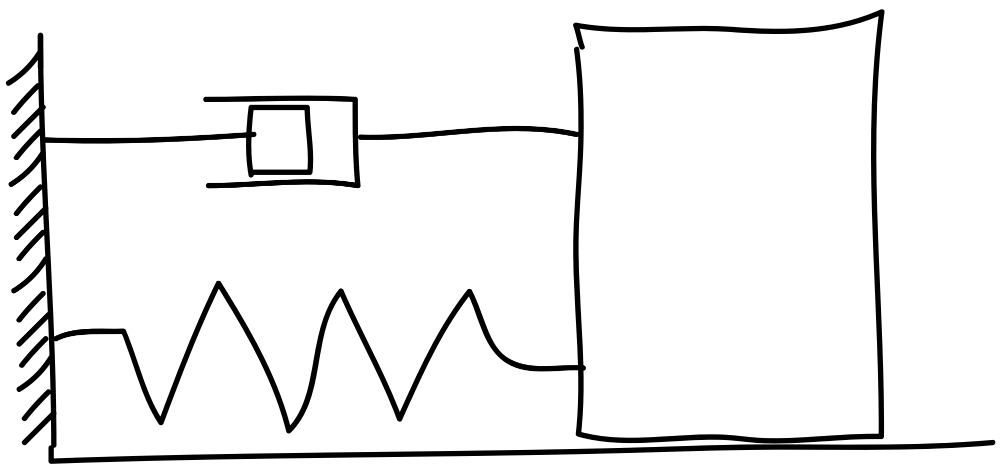

Kelvin-Voigt model

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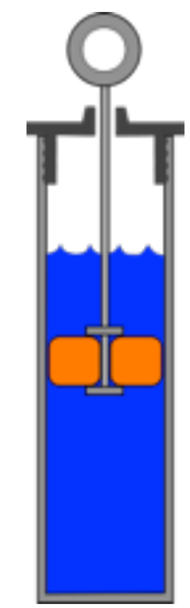

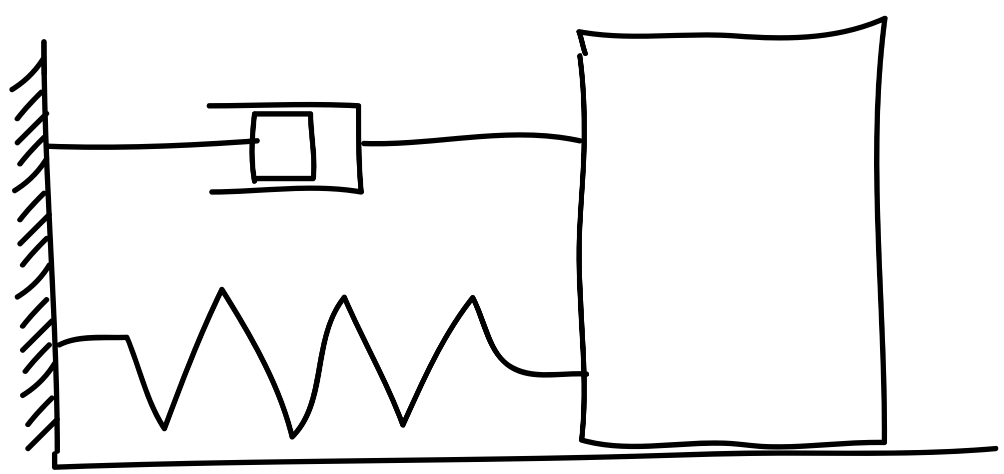

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shock absorber

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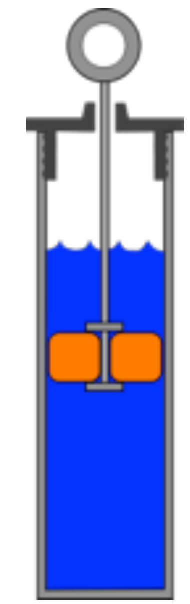
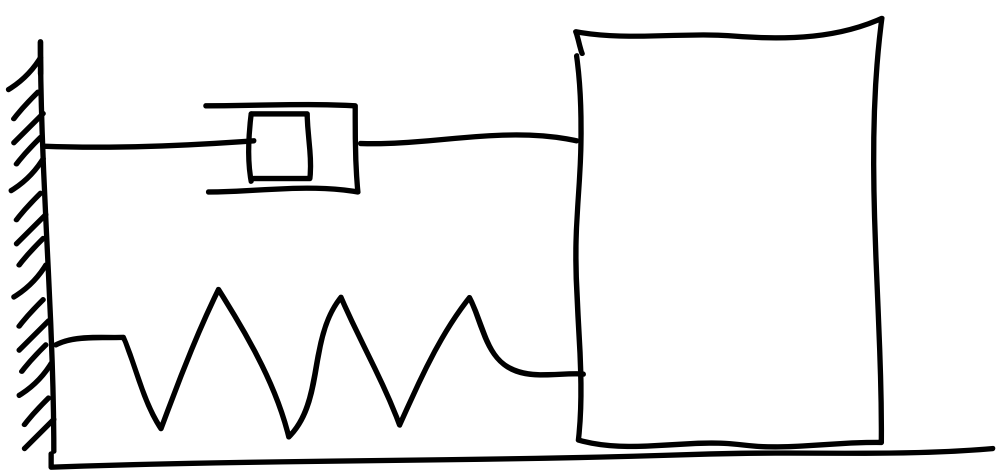


$$ma = -k(x - x_0) - \gamma v$$



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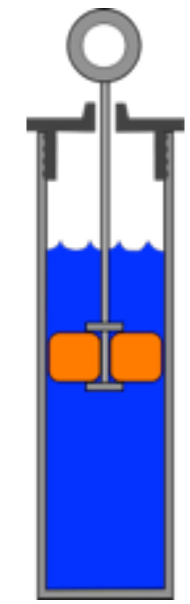

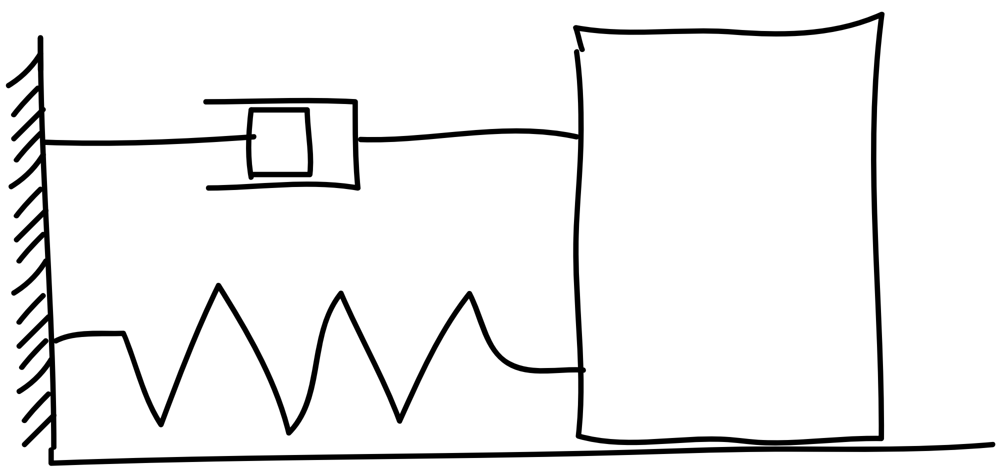

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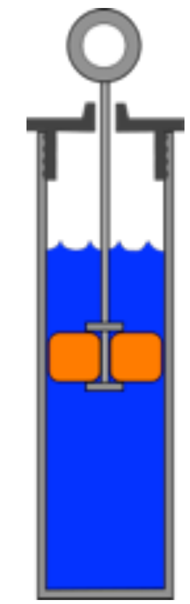

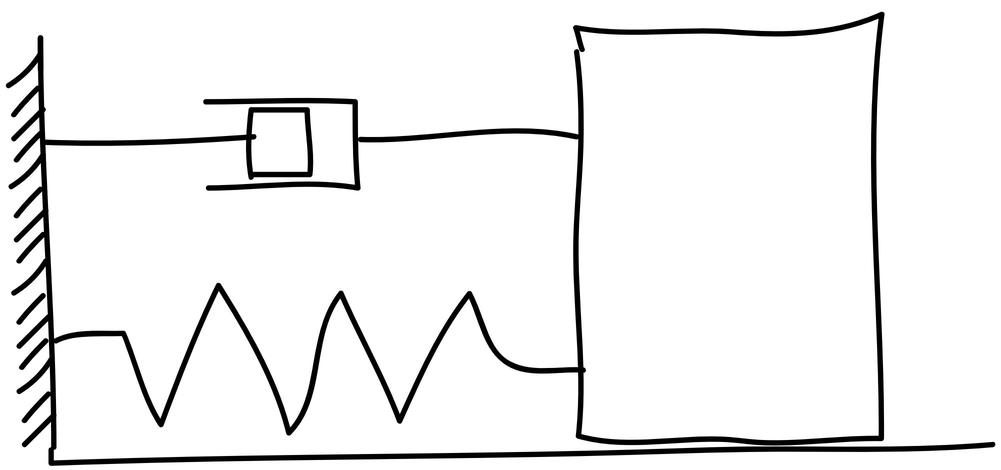

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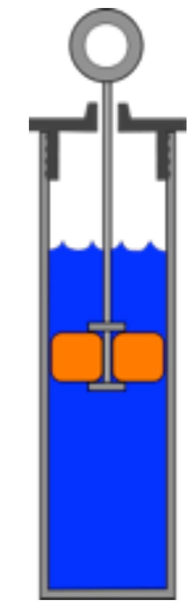

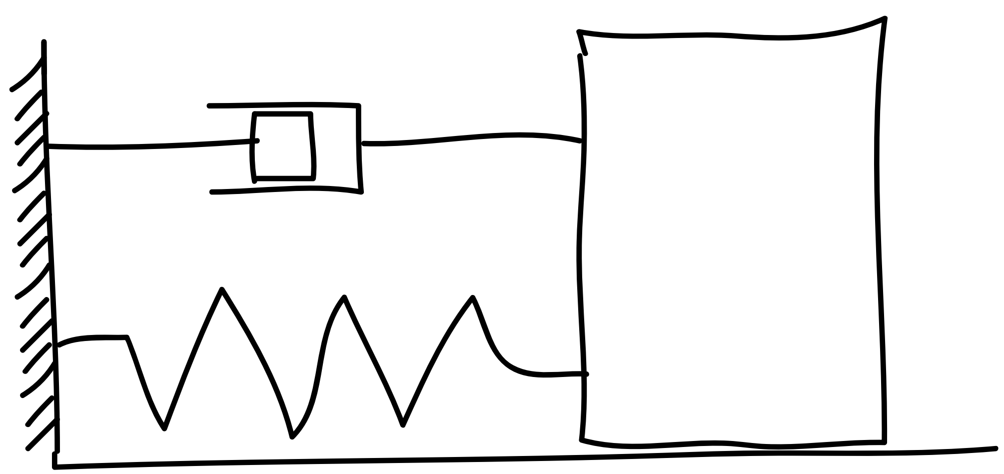

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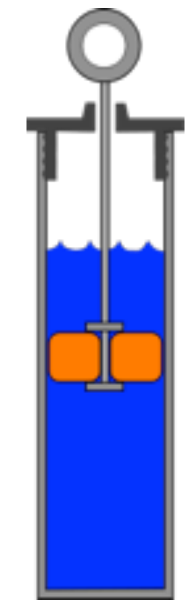

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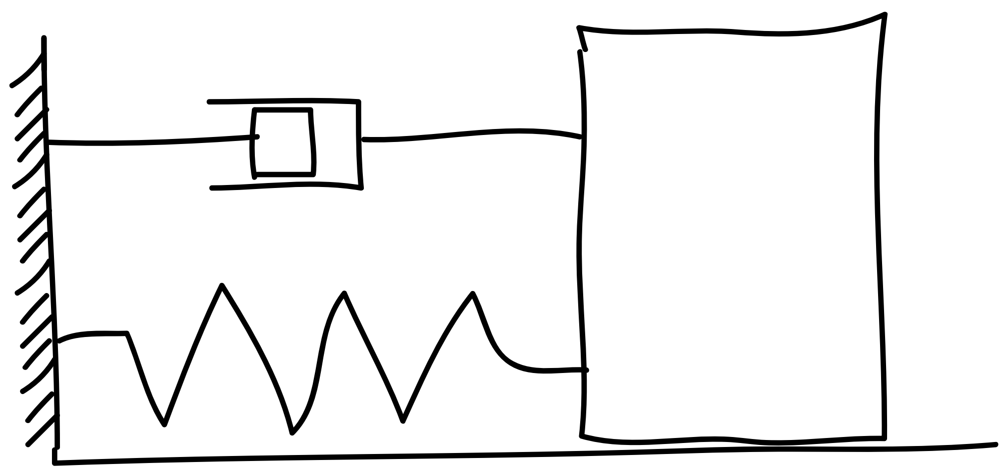

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$$my'' + \gamma y' + ky = 0$$

Applications - forced vibrations (3.8)

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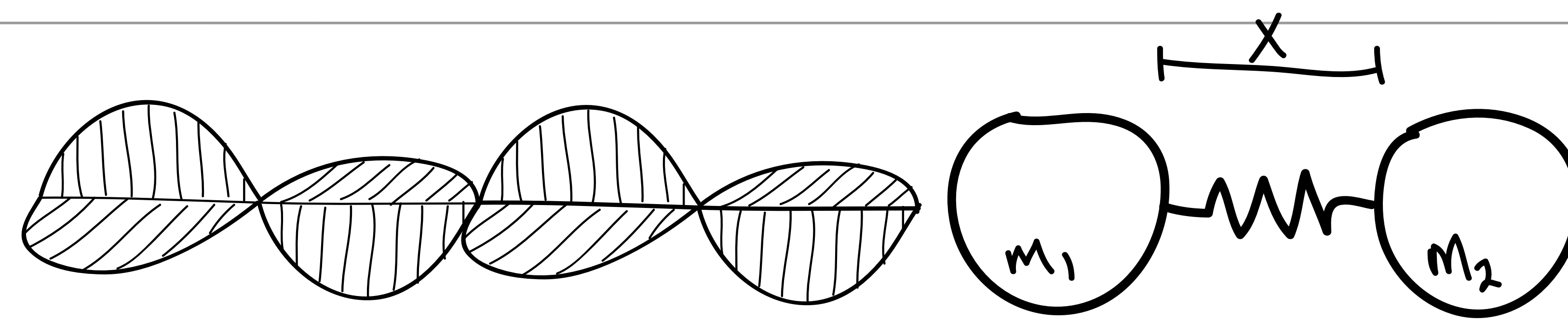
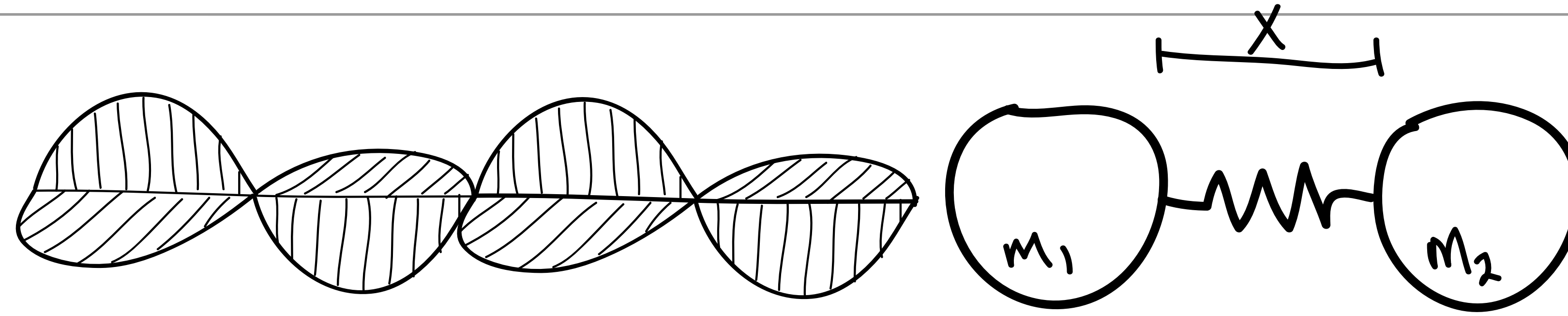


- light hitting a molecular bond

Applications - forced vibrations (3.8)



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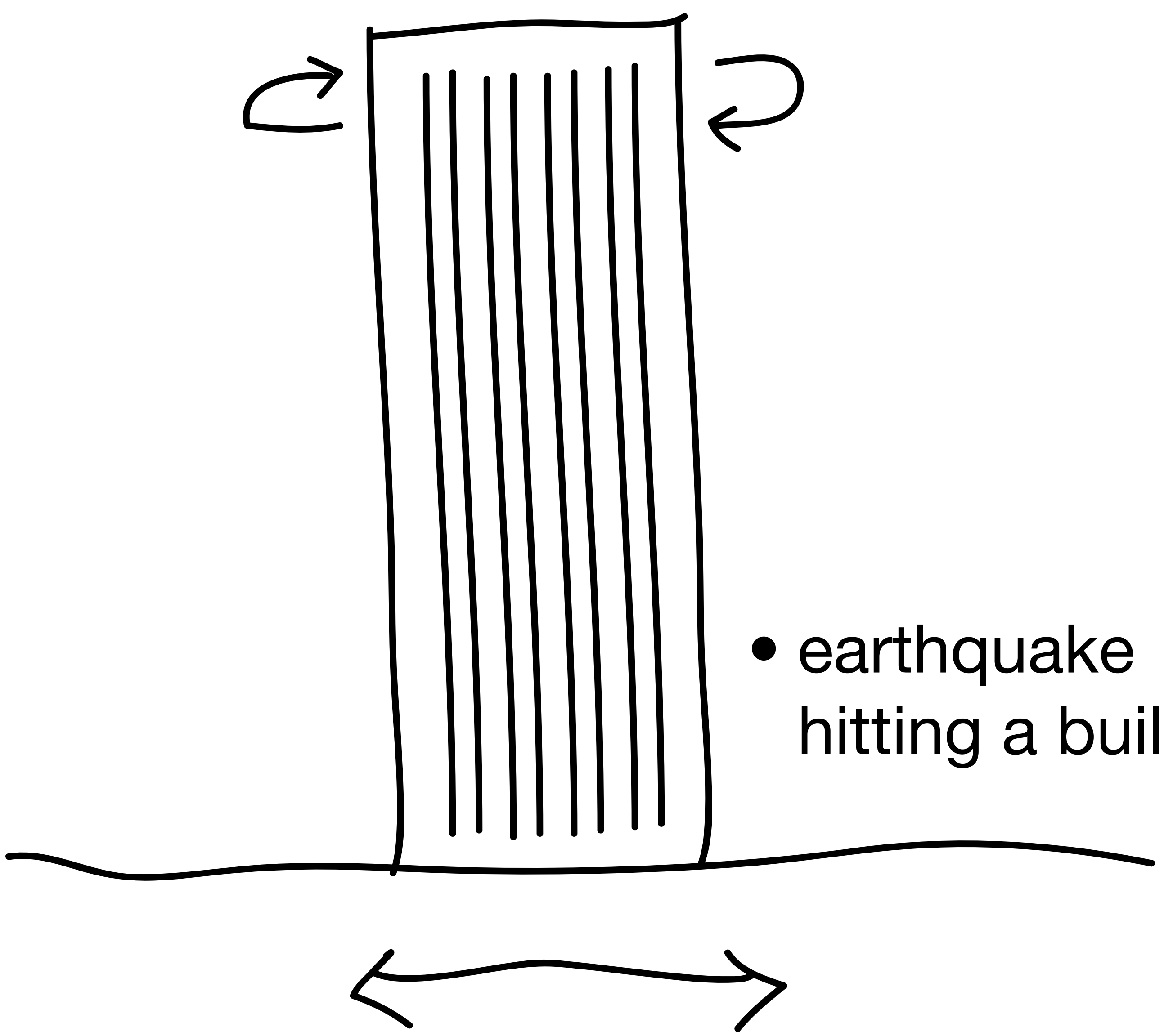

- earthquake hitting a building

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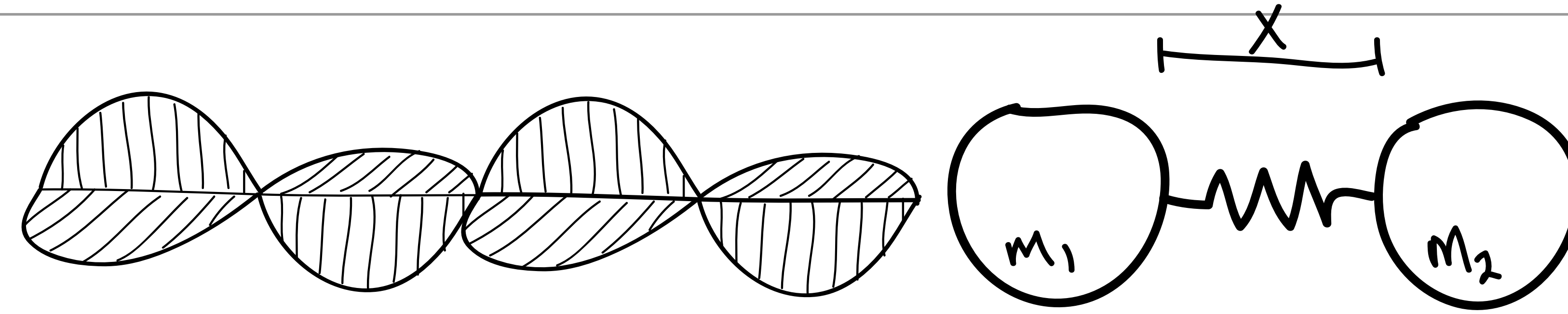

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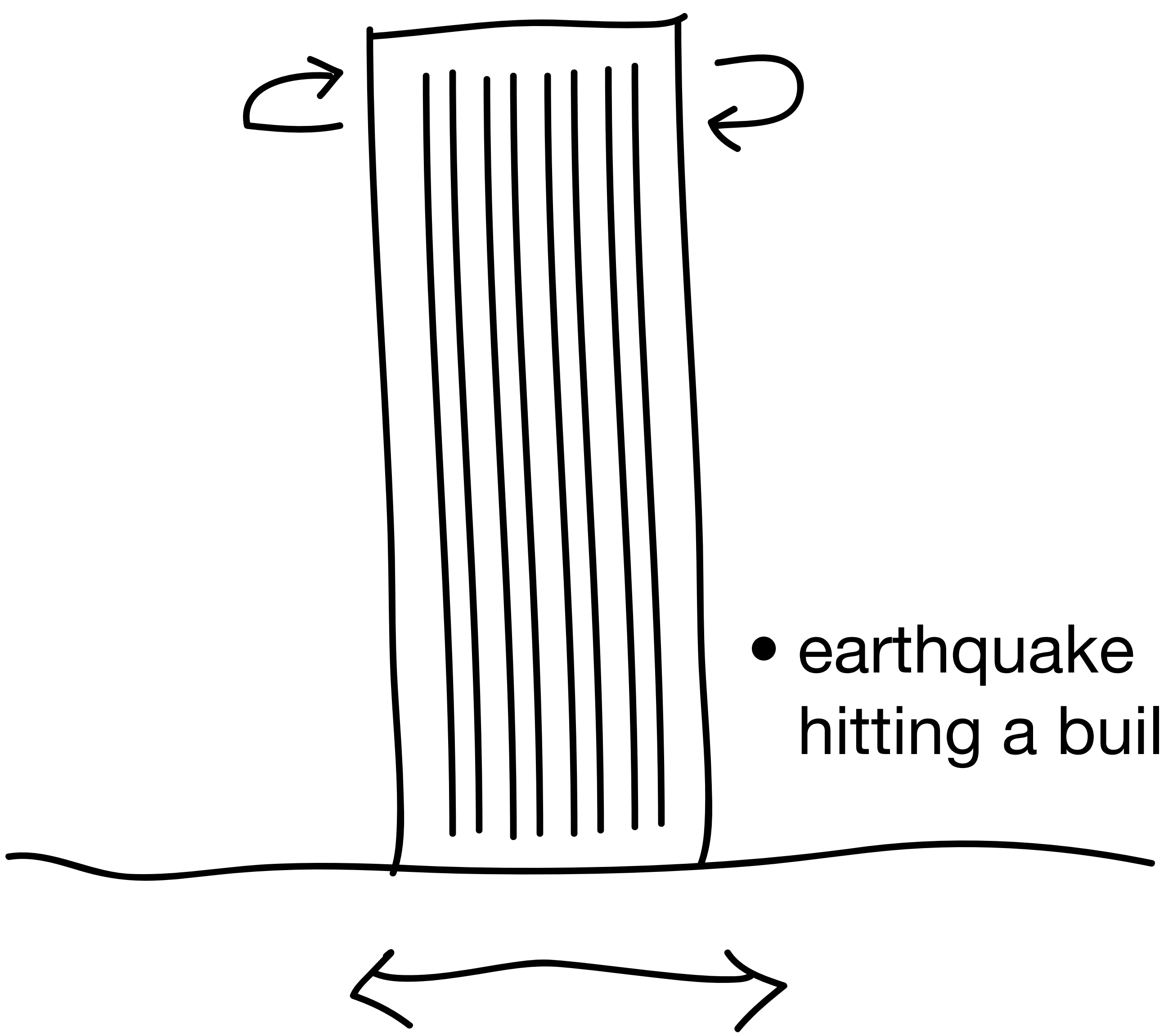

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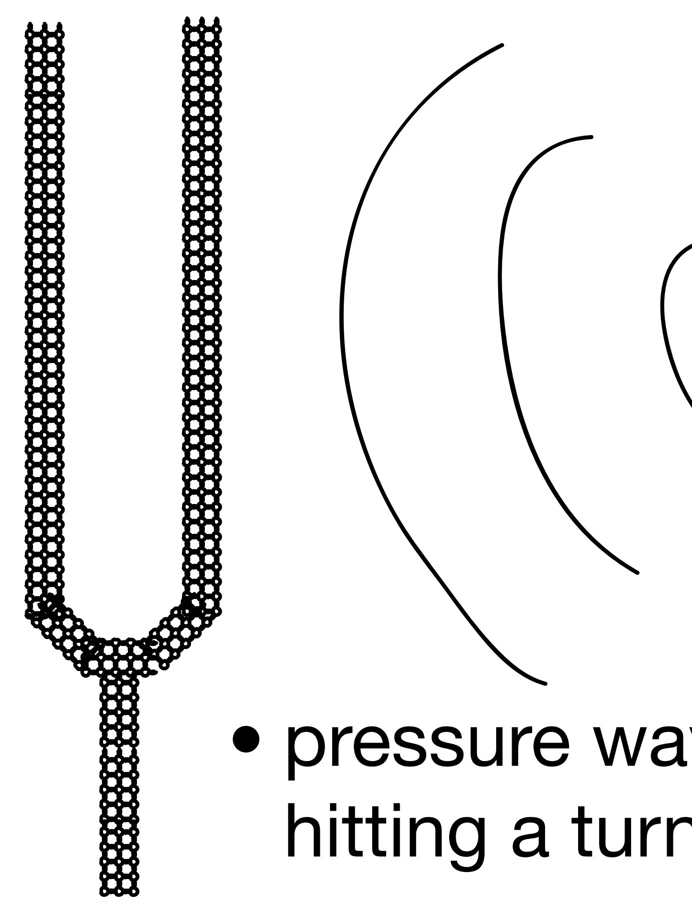

- pressure waves (sound) hitting a turning fork.

Applications - vibrations (3.7)

- Undamped mass spring

$$mx'' + kx = 0$$

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- frequency
  - increases with stiffness
  - decreases with mass

Applications - vibrations (3.7)

Trig identity reminders

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

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$$2\cos(3t+\pi/3) =$$

(A) $2\sin(\pi/3)\cos(3t) - 2\sin(\pi/3)\cos(3t)$

(B) $2\sin(\pi/3)\cos(3t) + 2\sin(\pi/3)\cos(3t)$

(C) $2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$

(D) $2\cos(\pi/3)\cos(3t) + 2\sin(\pi/3)\sin(3t)$

(E) Don’t know / still thinking.

Applications - vibrations (3.7)

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$$2\cos(3t + \pi/3) =$$

$$2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$$
$$= \cos(3t) - \sqrt{3}\sin(3t)$$

Applications - vibrations (3.7)

- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$4\cos(2t) + 3\sin(2t)$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

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4 3

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\end{aligned}
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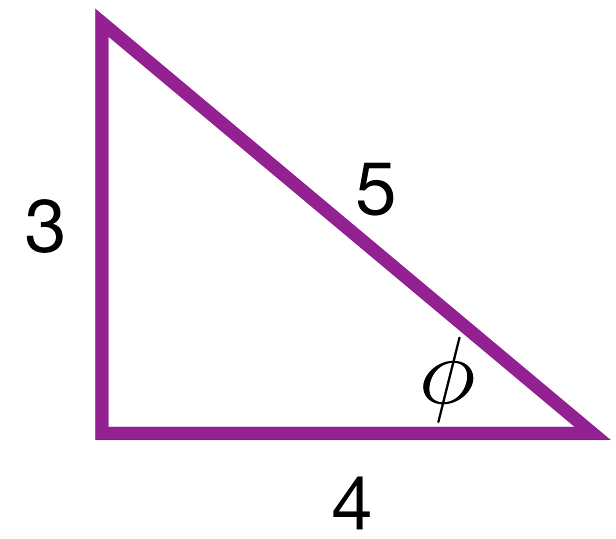

~~4~~ ~~3~~

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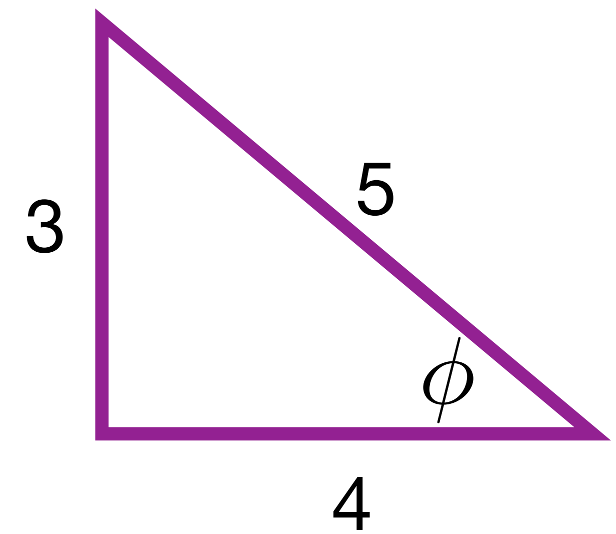

$$\phi = 0.9273$$

~~4~~ ~~3~~

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  - Step 3 - Rewrite the solution as $y(t) = A\cos(\omega_0 t - \phi)$.

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$$mx'' + \gamma x' + kx = 0 \qquad\qquad m, \gamma, k > 0$$

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smaller than 1
or complex

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We have the usual three cases...

Applications - vibrations (3.7)

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

(i) $\frac{4km}{\gamma^2} < 1$

(ii) $\frac{4km}{\gamma^2} = 1$

(iii) $\frac{4km}{\gamma^2} > 1$

Applications - vibrations (3.7)

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

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(over damped - γ large)

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For graphs, see:

https://www.desmos.com/calculator/psy5r8hpln