Today

- Intro to Laplace transforms
- A bunch of examples
- Solving ODEs (that we already know how to solve) using Laplace transforms

Laplace transforms - intro (6.1)

- Motivation for Laplace transforms:
  - We know how to solve $ay'' + by' + cy = g(t)$ when $g(t)$ is polynomial, exponential, trig.
  - In applications, $g(t)$ is often "piece-wise continuous" meaning that it consists of a finite number of pieces with jump discontinuities in between. For example,

$$g(t) = \begin{cases} \sin(\omega t) & 0 < t < 10, \\ 0 & t \geq 10. \end{cases}$$

  - These can be handled by previous techniques (modified) but it isn't pretty (solve from t=0 to t=10, use y(10) as the IC for a new problem starting at t=10).

Include physics example
(LRC with on/off switch)

Laplace transforms - intro (6.1)

- Instead, we use Laplace transforms.

- Idea:

$$\begin{tikzcd}
\text{Unknown } y(t) \text{ that satisfies some ODE} \arrow[r, "\text{solve ODE}"] \arrow[d, "\text{Transform } y(t) \text{ and the ODE}"'] & \text{Found } y(t) \\
\text{Unknown } Y(s) \text{ that satisfies an algebraic equation} \arrow[r, "\text{solve algebraic eq}"] & \text{Found } Y(s) \arrow[u, "\text{Invert the transform}"]
\end{tikzcd}$$

- Laplace transform of y(t): $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\ dt$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = 3$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} 3 \; dt$$

$$= -\frac{3}{s} e^{-st}\Big|_0^\infty$$

$$= \lim_{A\to\infty} -\frac{3}{s} e^{-st}\Big|_0^A$$

$$= -\frac{3}{s}\left(\lim_{A\to\infty} e^{-sA} - 1\right)$$

$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

Laplace transforms - examples (6.1)

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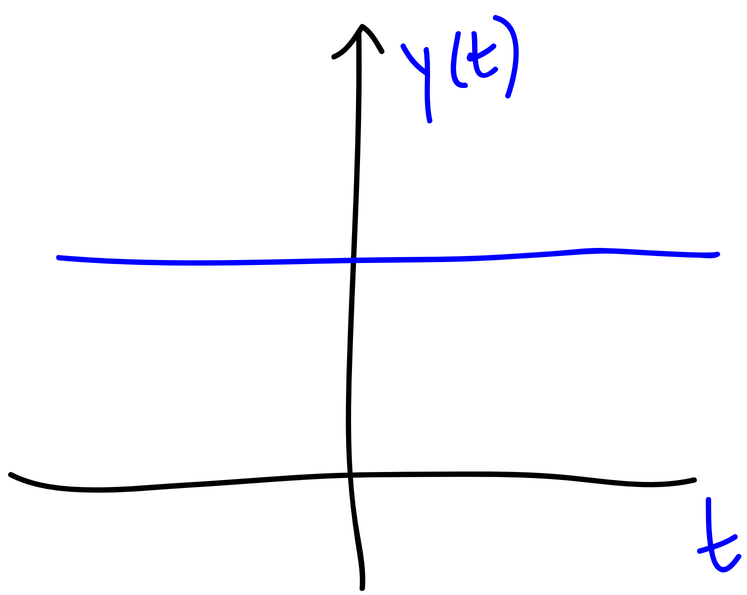
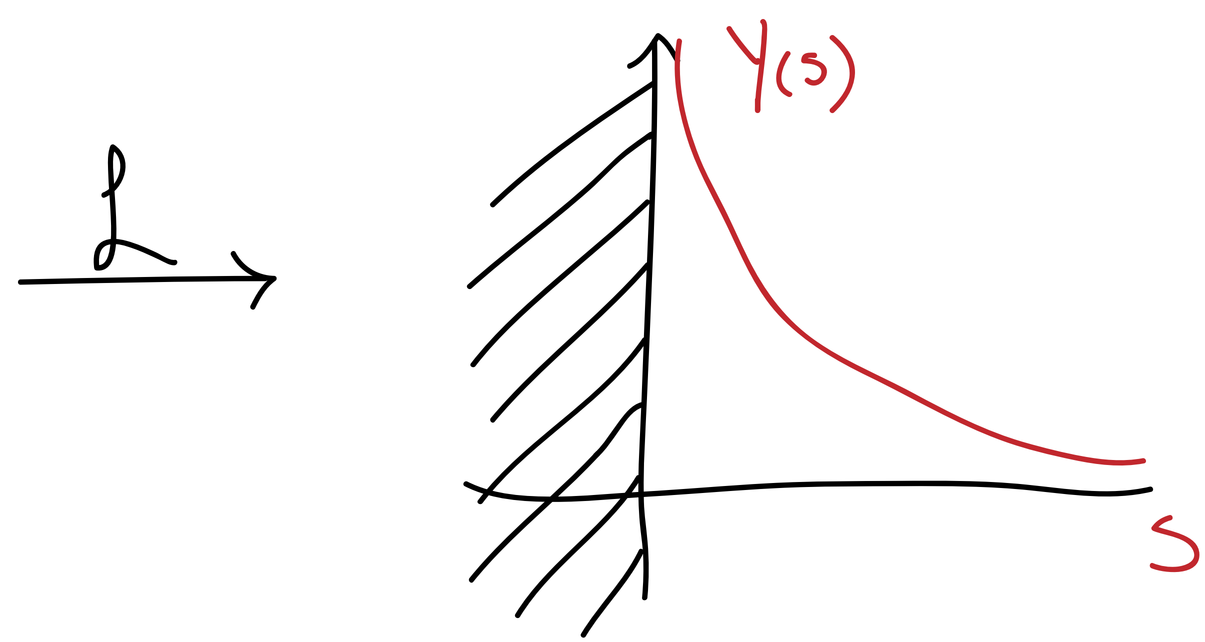


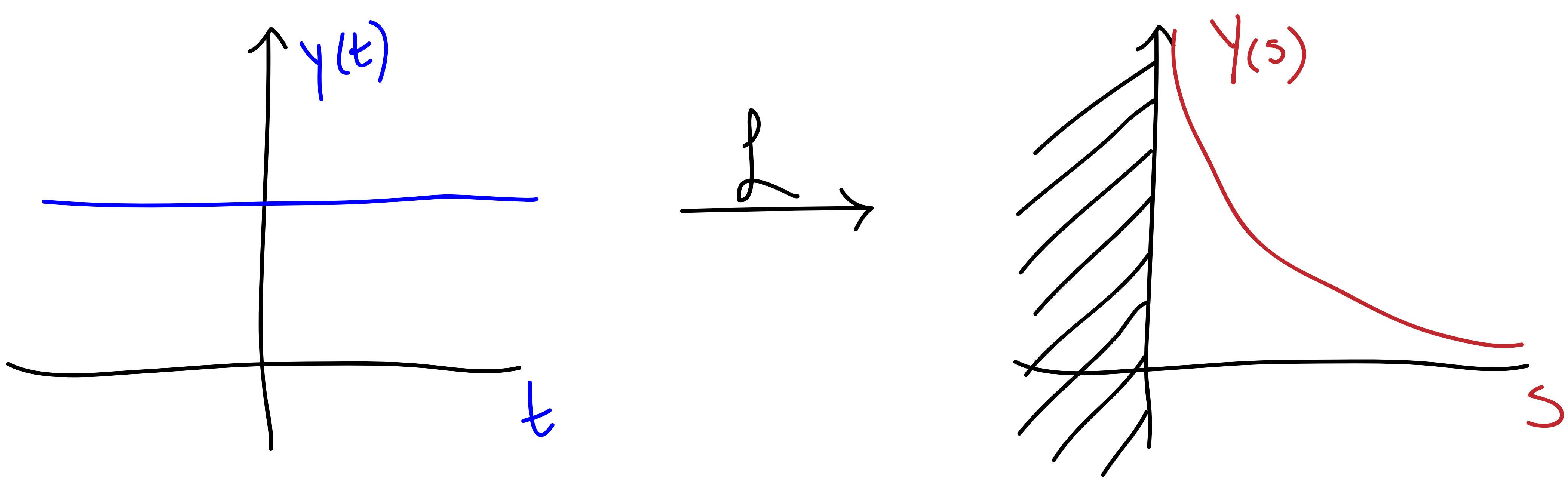


Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = C$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} C \; dt$$

$$= \frac{C}{s}$$ provided $s > 0$ and does not exist otherwise.



Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = e^{6t}$?

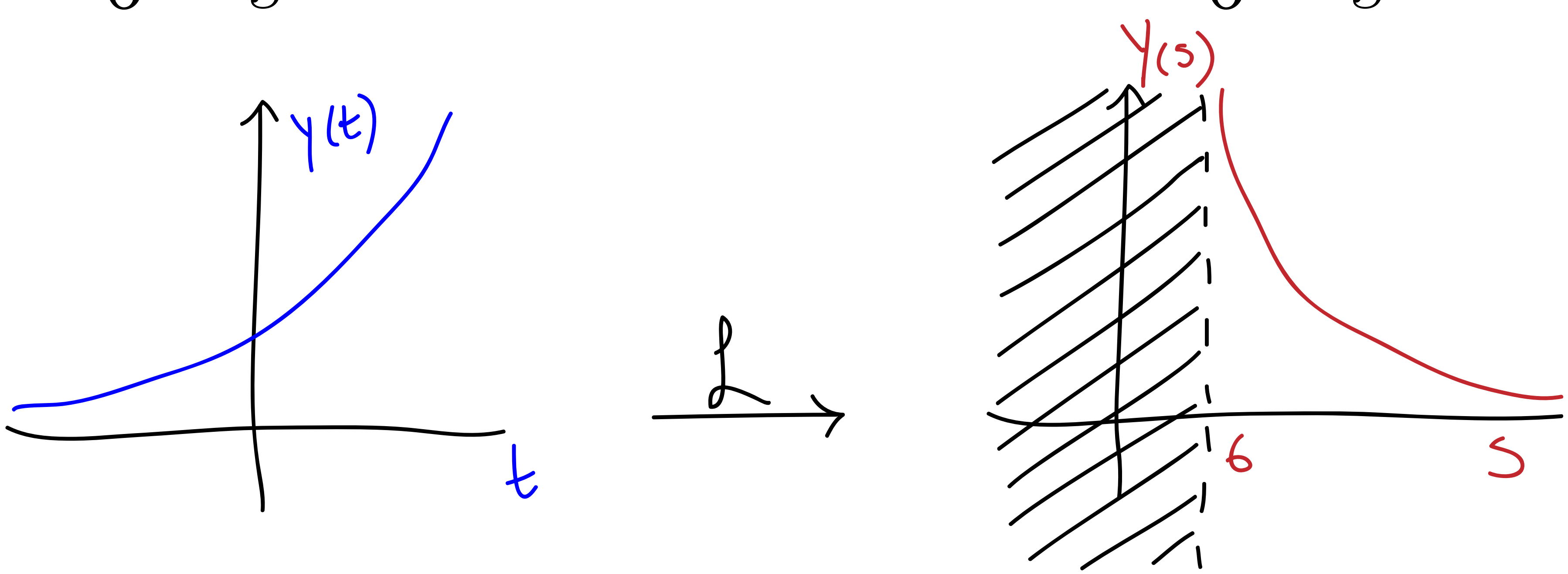
$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} e^{6t} \; dt$$

(A) $Y(s) = \dfrac{1}{s-6} \qquad s > 0$

★(C) $Y(s) = \dfrac{1}{s-6} \qquad s > 6$

(B) $Y(s) = \dfrac{1}{6-s} \qquad s > 6$

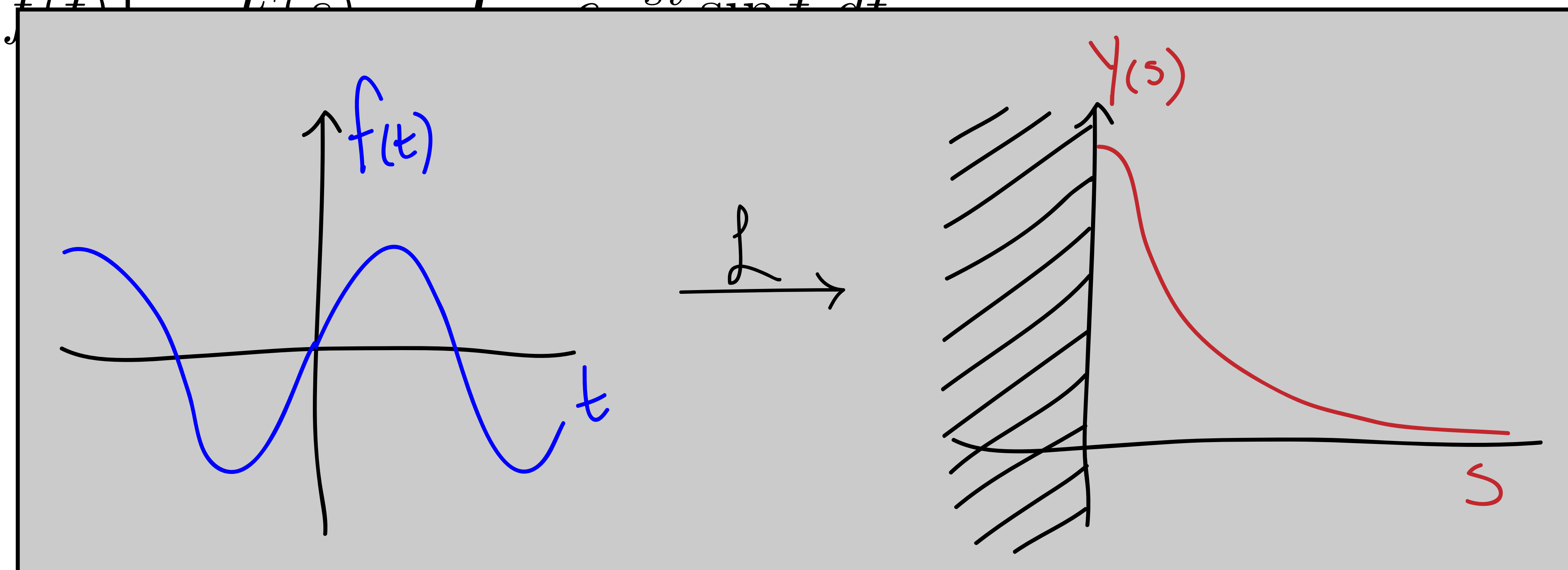
(D) $Y(s) = \dfrac{1}{6-s} \qquad s > 0$



Laplace transforms - examples (6.1)

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\sin t\ dt$$



$$= 1 - s\left(e^{-st}\sin t\Big|_0^\infty + s\int_0^\infty e^{-st}\sin t\ dt\right)$$

$$= 1 - s^2F(s) \qquad s > 0$$

$$(1+s^2)F(s) = 1$$

$$F(s) = \frac{1}{1+s^2} \qquad s > 0$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $h(t) = \sin(\omega t)$? $\quad (\omega > 0)$

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st} \sin(\omega t)\ dt$$

$$u = \omega t$$
$$du = \omega dt$$

☆ (A) $H(s) = \dfrac{\omega}{\omega^2 + s^2}$

(B) $H(s) = \dfrac{1}{1 + \left(\frac{s}{w}\right)^2}$

(C) $H(s) = \dfrac{1}{\omega} \dfrac{1}{1 + s^2}$

(D) $H(s) = \dfrac{1}{1 + s^2}$

(E) Huh?

$$H(s) = \int_0^\infty e^{-s\frac{u}{\omega}} \sin u \ \frac{du}{\omega}$$

$$= \frac{1}{\omega} \int_0^\infty e^{-\frac{s}{\omega}u} \sin u \ du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

$$= \frac{1}{\omega}\ \frac{1}{1 + \left(\frac{s}{\omega}\right)^2} \qquad s > 0$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $g(t) = \cos t$?

- Could calculate directly but note that g(t) = f’(t) where f(t)=sin t.

$$F(s) = \mathcal{L}\{$$

$$\boxed{G(s) = \mathcal{L}\{\cos t\} = \frac{s}{1+s^2}}$$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0 e \quad f'(t)\ dt$$

Add sketch.

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

$$= -f(0) + sF(s) \qquad s > 0$$

$$= -0 + s\frac{1}{1+s^2} = \frac{s}{1+s^2}$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

- Recall two examples back:

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st}\ f(\omega t)\ \ dt$$

$$u = \omega t$$
$$du = \omega dt$$

$$= \int_0^\infty e^{-s\frac{u}{\omega}}\ f(u)\ \frac{du}{\omega}$$

$$= \frac{1}{\omega}\int_0^\infty e^{-\frac{s}{\omega}u}\ f(u)\,du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

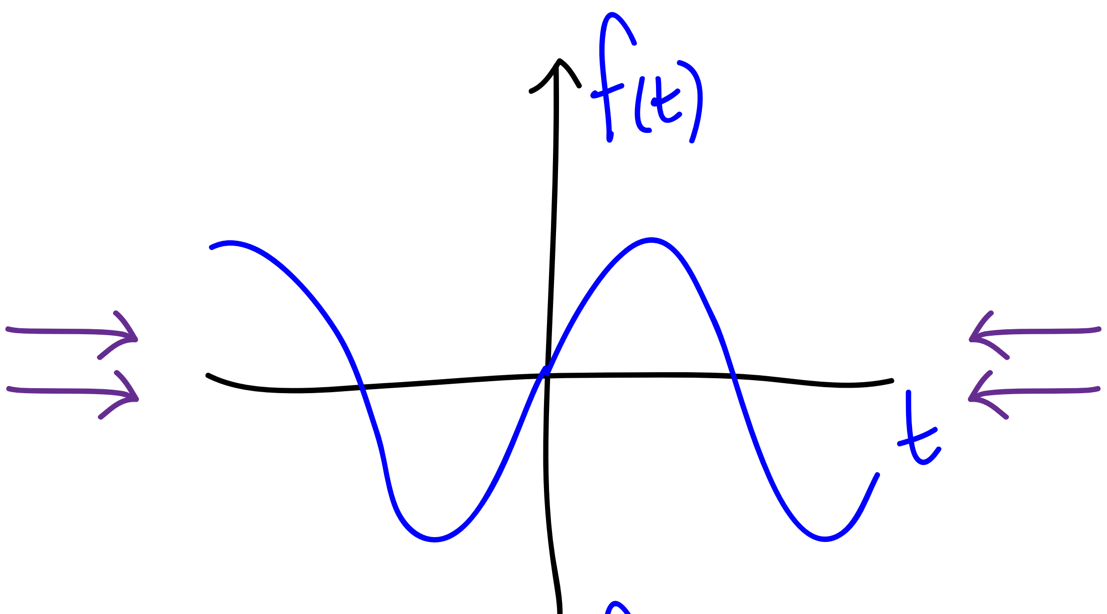
$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

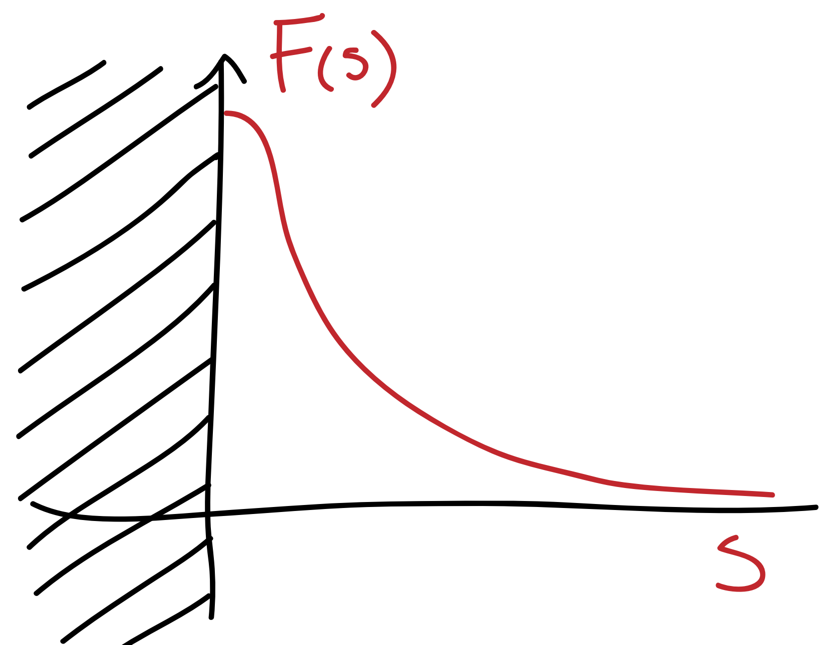
$$\mathcal{L}\{\cos(\omega t)\}$$
$$= \frac{1}{\omega}\ \frac{\frac{s}{\omega}}{1+\left(\frac{s}{\omega}\right)^2}$$
$$= \frac{s}{\omega^2+s^2}$$

Laplace transforms - examples (6.1)

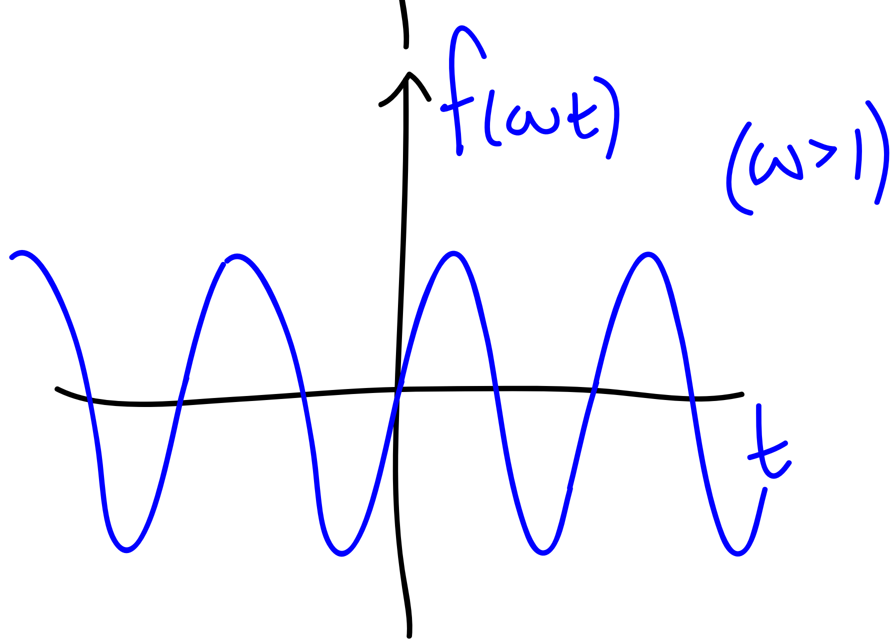
- What is the Laplace transform of $f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

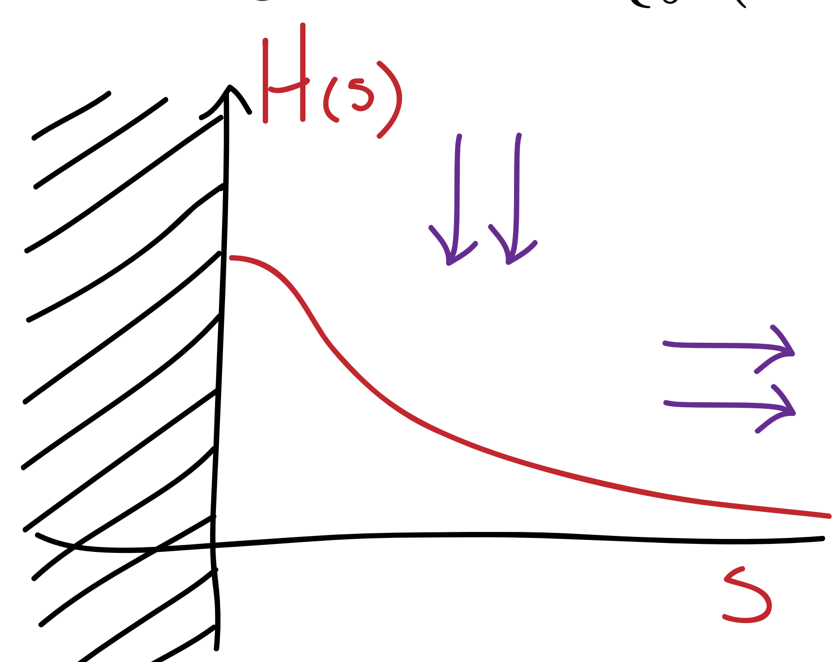
$$\mathcal{L}\{f(\omega t)\} = \frac{1}{\omega}\, F\left(\frac{s}{\omega}\right)$$





Sketch graph of $\mathcal{L}\{f(\omega t)\}$.





Laplace transforms - examples (6.1)

- What is the Laplace transform of $k(t) = e^{at} f(t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

$$\mathcal{L}\{k(t)\} = K(s) = \int_0^\infty e^{-st} e^{at} f(t)\ dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\ dt$$

$$= F(s-a)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{e^{-3t}\cos t\} =$$

(A) $\dfrac{s}{1+(s+3)^2}$

(B) $\dfrac{1}{1+(s+3)^2}$

★(C) $\dfrac{s+3}{s^2+6s+10}$

(D) $\dfrac{1}{s^2+6s+10}$

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $ay'' + by' + cy = 0$ using Laplace transforms.
- Recall that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.
- Applying this to f”, we find that

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(sF(s) - f(0)) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

- Transforming both sides of the equation,

$$\mathcal{L}\{ay'' + by' + cy\} = 0$$

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

$$a\underline{\mathcal{L}\{y''\}} + b\underline{\mathcal{L}\{y'\}} + c\underline{\mathcal{L}\{y\}} = 0$$

$$a(\underline{s^2Y(s) - sy(0) - y'(0)}) + b(\underline{sY(s) - y(0)}) + c\underline{Y(s)} = 0$$

$$(as^2 + bs + c)Y(s) = asy(0) + ay'(0) + by(0)$$

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 4y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c}$$
$$= \frac{s}{s^2 + 4}$$

- To find y(t), we have to invert the transform. What y(t) would have Y(s) as its transform?

- Recall that $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{\omega^2 + s^2}$. So $y(t) = \cos(2t)$.

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2} \qquad \longrightarrow \qquad Y(s) = \frac{s+6}{s^2+6s+13}$$

$$\lambda = \frac{-6 \pm i\sqrt{52-36}}{2} = -3 \pm 2i$$

- To find y(t), we hav … would have Y(s) as its transform?

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{e^{-3t}\cos t\} = \frac{s+3}{1+(s+3)^2}$$

$$Y(s) = \frac{s+3+3}{s^2+6s+9+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{2}\frac{2}{(s+3)^2+4}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

Solving IVPs using Laplace transforms (6.2)

- What is the transformed equation for the IVP

$$y' + 6y = e^{2t}$$
$$y(0) = 2$$

(A) $Y'(s) + 6Y(s) = \dfrac{1}{s+2}$

(B) $Y'(s) + 6Y(s) = \dfrac{1}{s-2}$

(C) $sY(s) - 2 + 6Y(s) = \dfrac{1}{s+2}$

☆(D) $sY(s) - 2 + 6Y(s) = \dfrac{1}{s-2}$

(E) Explain, please.

$$\mathcal{L}\{y'(t)\} = sY(s) - 2$$
$$\mathcal{L}\{6y(t)\} = 6Y(s)$$
$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$$

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{(2-s)t}\, dt$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Solving IVPs using Laplace transforms (6.2)

- Find the solution to $y' + 6y = e^{2t}$, subject to IC $y(0) = 2$.

$$sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$$

$$Y(s) = \left(2 + \frac{1}{s-2}\right)/(s+6)$$

$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

$$\downarrow$$

$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

$$y(t) = 2e^{-6t} + \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s+6}\right)$$

$$y(t) = 2e^{-6t} + \frac{1}{8}e^{2t} - \frac{1}{8}e^{-6t}$$

$$\frac{1}{(s-2)(s+6)} = \frac{A}{s-2} + \frac{B}{s+6}$$

$$1 = A(s+6) + B(s-2)$$

$$(s=2) \qquad 1 = 8A$$

$$(s=-6) \qquad 1 = -8B$$

$$y(t) = \frac{15}{8}e^{-6t} + \frac{1}{8}e^{2t}$$

$$y_h(t) = Ce^{-6t}$$

$$C = \frac{15}{8} \qquad\qquad y_p(t) = \frac{1}{8}e^{2t}$$