Today

- Step and ramp functions (continued)
- The Dirac Delta function and impulse force
- (Modeling with delta-function forcing)

Step function forcing

- Solve using Laplace transforms:

$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

$$y(0) = 0, \; y'(0) = 0.$$

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- So we just need h(t) and we're done.

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- Inverting H(s) to get h(t): $H(s) = \dfrac{1}{s(s^2 + 2s + 10)}$

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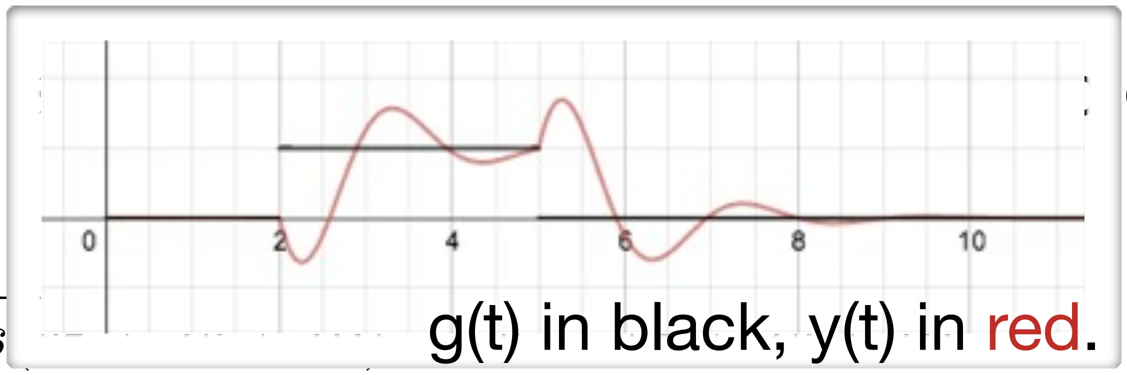

g(t) in black, y(t) in red.

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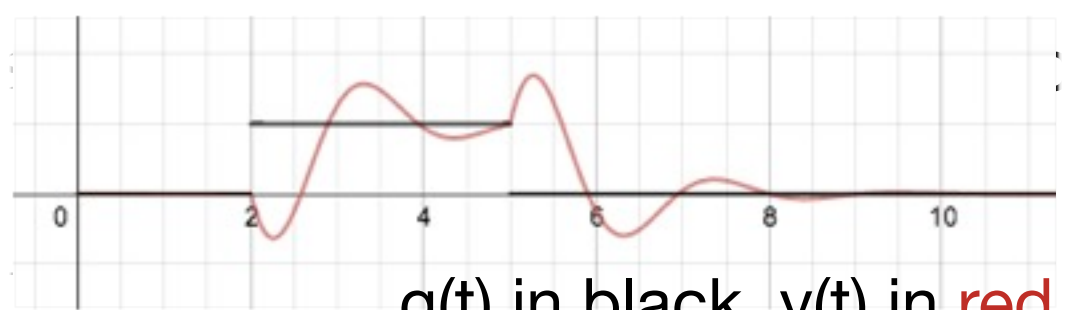

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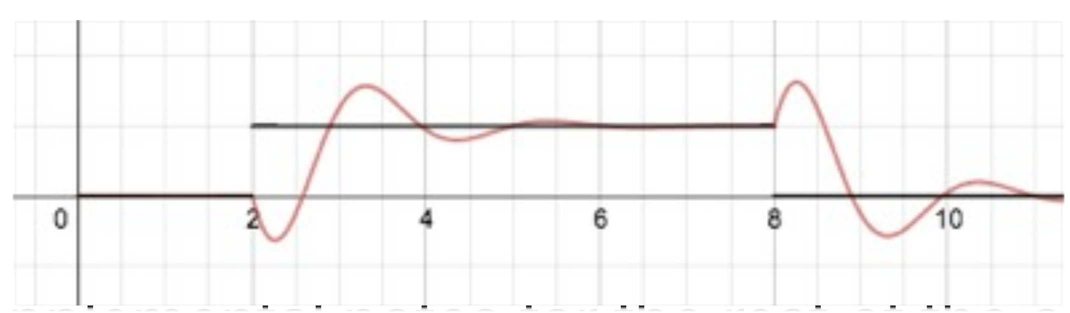

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Step function forcing

- An example with a ramped forcing function:

$$y'' + 4y = \begin{cases} 0 & \text{for } t < 5, \\ \frac{t-5}{5} & \text{for } 5 \le t < 10, \\ 1 & \text{for } t \ge 10. \end{cases}$$

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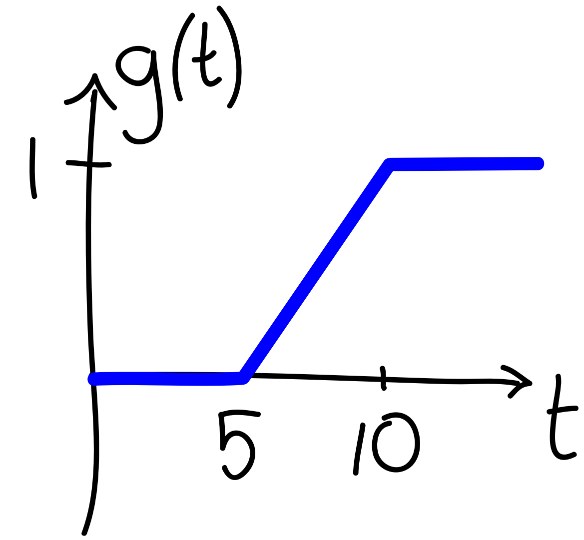

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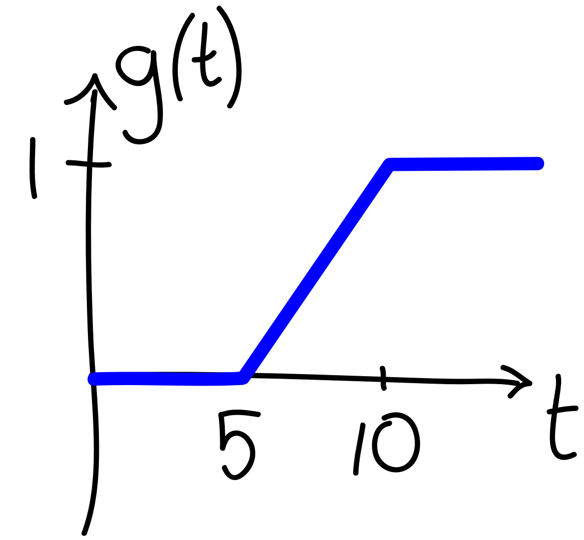

- Write g(t) is terms of $u_c(t)$:

(A) $g(t) = u_5(t) - u_{10}(t)$

(B) $g(t) = u_5(t)(t-5) - u_{10}(t)(t-5)$

(C) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/5$

(D) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/10$

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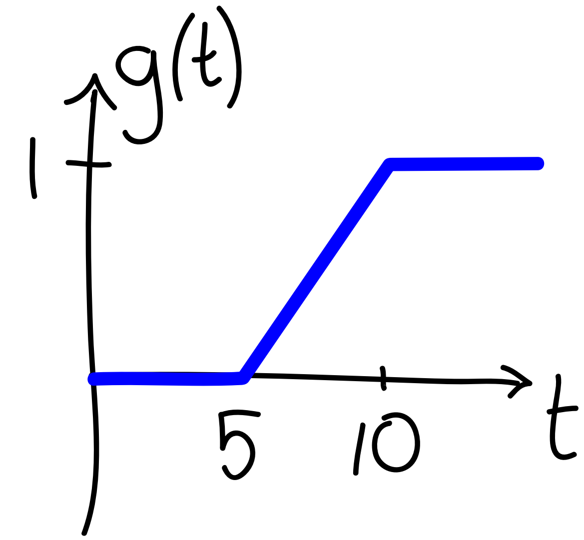

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Two methods:

1. Build from left to right, adding/subtracting what you need to make the next section:

$$g(t) = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

2. Build each section independently:

$$g(t) = \left(u_5(t) - u_{10}(t)\right)\frac{1}{5}(t-5) + u_{10}(t)\cdot 1$$

- W

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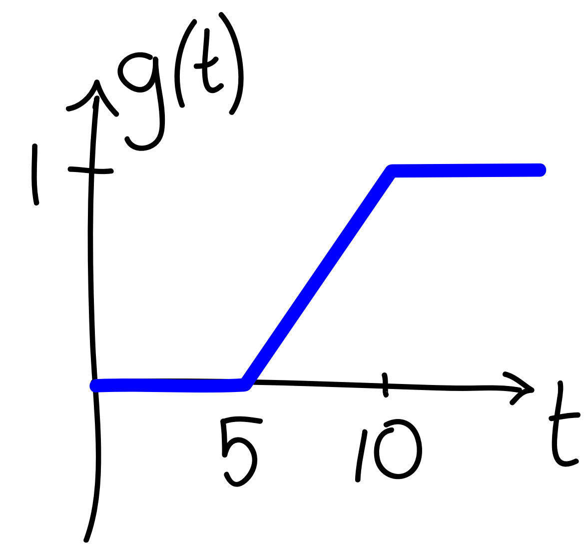

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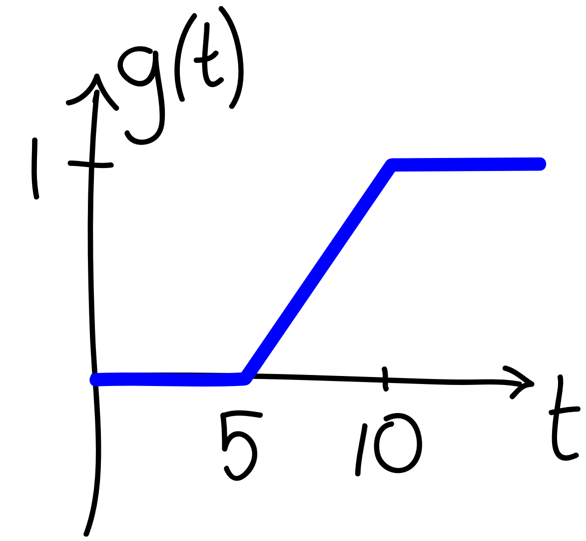

$$s^2Y + 4Y =$$

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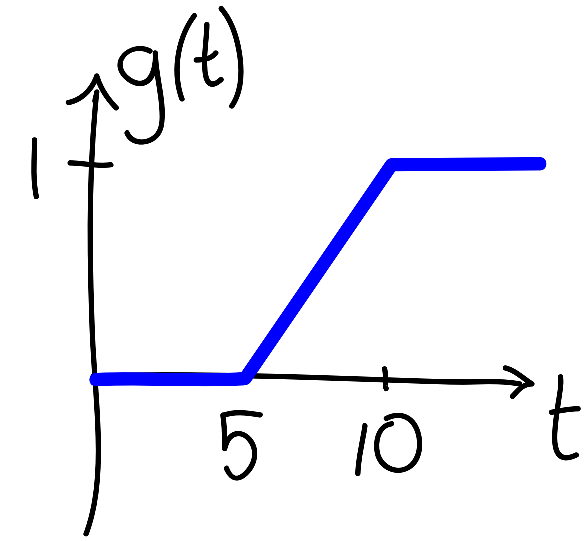

$$s^2Y + 4Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$Y(s) = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)} \quad = \frac{1}{5}(e^{-5s} - e^{-10s})H(s)$$

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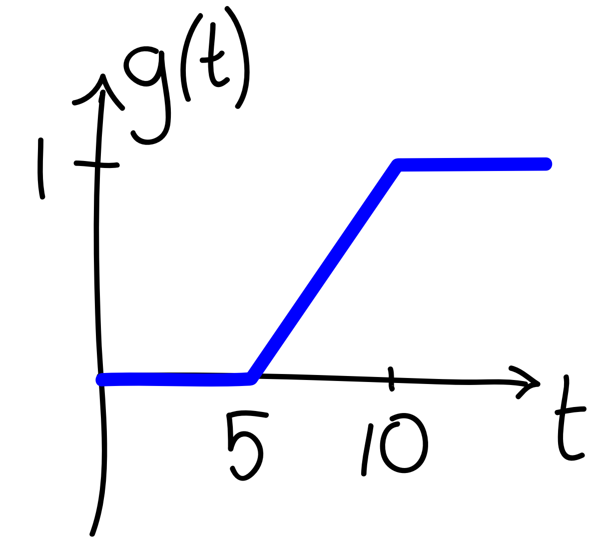

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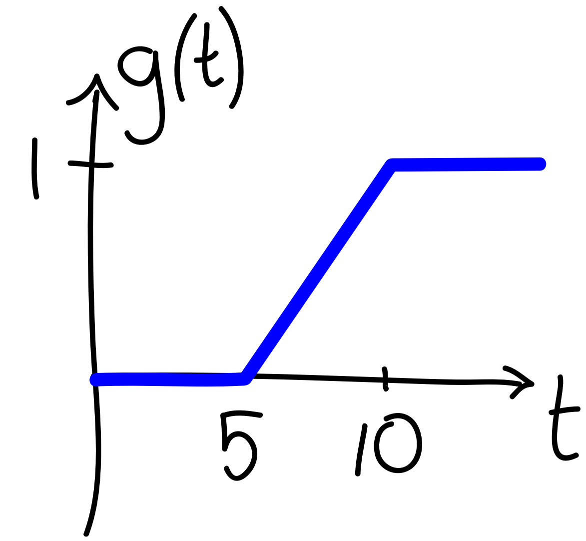

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$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$