Today

- Solving a second order linear homogeneous equation with constant coefficients
    - complex roots to the characteristic equation,
    - repeated roots to the characteristic equation (Reduction of Order).
- Connections to matrix algebra.
- Solving a second order linear **non**homogeneous equation.

Reminder: Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Complex roots (Section 3.3)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- When b^2 - 4ac < 0, we get complex roots:

$$\begin{aligned} r_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a} \\ &= \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i \\ &= \alpha \pm \beta i \end{aligned}$$

Complex roots (Section 3.3)

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$
\begin{aligned}
y_1(t) &= e^{(\alpha+\beta i)t} \\
&= e^{\alpha t} e^{i\beta t} \\
&= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) &= e^{(\alpha-\beta i)t} \\
&= e^{\alpha t} e^{-i\beta t} \\
&= e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t)) \\
&= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))
\end{aligned}
$$

Complex roots (Section 3.3)

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$y_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
$$y_2(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

- Instead of using these to form the general solution, let's use them to find two real valued solutions:

$$\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{\alpha t}\cos(\beta t)$$
$$\frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = e^{\alpha t}\sin(\beta t)$$

- General solution:

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)}$$

Complex roots (Section 3.3)

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1+2i$, $r_2 = 1-2i$

(B) $r_1 = -1+2i$, $r_2 = -1-2i$

(C) $r_1 = 1-2i$, $r_2 = -1+2i$

(D) $r_1 = 2+4i$, $r_2 = 2-4i$

(E) $r_1 = -2+4i$, $r_2 = -2-4i$

Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

☆ (A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

(D) $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$

Complex roots (Section 3.3)

- Example: Find the solution to the IVP

$$y'' + 2y' + 5y = 0, \;\; y(0) = 1, \;\; y'(0) = 0$$

- General solution: $\quad y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $\quad y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

(B) $\quad y(t) = e^{-t}\left(\cos(2t) - \dfrac{1}{2}\sin(2t)\right)$

(C) $\quad y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

(D) $\quad y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) + \sin(2t))$

Repeated roots (Section 3.4)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  ii.A repeated real root: $b^2 - 4ac = 0$.

  iii.Two complex roots: $b^2 - 4ac < 0$.

- For case ii ($r_1 = r_2 = r$), we need another independent solution!
- Reduction of order - a method for guessing another solution.

Reduction of order

- You have one solution $y_1(t)$ and you want to find another independent one, $y_2(t)$.
- Guess that $y_2(t) = v(t)y_1(t)$ for some as yet unknown $v(t)$. If you can find $v(t)$ this way, great. If not, gotta try something else.
- Example - $y'' + 4y' + 4y = 0.$ Only one root to the characteristic equation, r=-2, so we only get one solution that way: $y_1(t) = e^{-2t}$.
- Use Reduction of order to find a second solution.

$$y_2(t) = v(t)e^{-2t}$$

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$.

$$\Downarrow$$

$$\underline{4y_2(t) = \cancel{4v(t)e^{-2t}}}$$

$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$

$$\Downarrow$$

$$\underline{4y_2'(t) = \cancel{4v'(t)e^{-2t}} - \cancel{8v(t)e^{-2t}}}$$

$$y_2''(t) = v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$$

$$\Downarrow$$

$$\underline{y_2''(t) = v''(t)e^{-2t} - \cancel{4v'(t)e^{-2t}} + \cancel{4v(t)e^{-2t}}}$$

$$0 = y_2'' + 4y_2' + 4y_2 = v''e^{-2t}$$

$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$ (where $v(t) = C_1 t + C_2$).

$$= (C_1 t + C_2)e^{-2t}$$

$$y(t) = C_1 \underbrace{te^{-2t}}_{y_2(t)} + C_2 \underbrace{e^{-2t}}_{y_1(t)}$$

Is this the general solution? Calculate the Wronskian:

$$W(e^{-2t}, te^{-2t})(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-4t} \neq 0$$

So yes!

Summary (3.1-3.4)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

i. Two distinct real roots: $b^2 - 4ac > 0$. (r_1, r_2)

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

ii. A repeated real root: $b^2 - 4ac = 0$. (r)

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

iii. Two complex roots: $b^2 - 4ac < 0$. ($r_{1,2} = \alpha \pm i\beta$)

$$y = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

(A) $y(t) = C_1 e^{-2t} + C_2 e^{-4t}$

☆ (B) $y(t) = C_1 e^{2t} + C_2 e^{4t}$

(C) $y(t) = e^{2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(D) $y(t) = e^{-2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(E) $y(t) = C_1 e^{2t} + C_2 t e^{4t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 9y = 0$$

(A) $y(t) = C_1 e^{3t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{3t}$

(C) $y(t) = C_1 e^{3t} + C_2 e^{-3t}$

(D) $y(t) = C_1 e^{3t} + C_2 t e^{3t}$

(E) $y(t) = C_1 e^{3t} + C_2 v(t) e^{3t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 10y = 0$$

(A) $y(t) = C_1 e^{3t} + C_2 e^{t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{-t}$

(C) $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$

(D) $y(t) = e^{t}(C_1 \cos(3t) + C_2 \sin(3t))$

☆ (E) $y(t) = e^{3t}(C_1 \cos(t) + C_2 \sin(t))$

Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

- An mxn matrix is a gizmo that takes an n-vector and returns an m-vector:

$$\overline{y} = A\overline{x}$$

- It is called a linear operator because it has the following properties:

$$A(c\overline{x}) = cA\overline{x}$$

$$A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$$

- Not all operators work on vectors. Derivative operators take a function and return a new function. For example,

$$z = L[y] = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y$$

- This one is linear because

$$L[cy] = cL[y]$$

$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

Some connections to linear (matrix) algebra

- A homogeneous matrix equation has the form

$$A\overline{x} = \overline{0}$$

- A non-homogeneous matrix equation has the form

$$A\overline{x} = \overline{b}$$

- A homogeneous differential equation has the form

$$L[y] = 0$$

- A non-homogeneous differential equation has the form

$$L[y] = g(t)$$

Some connections to linear (matrix) algebra

Systems of equations written in operator notation.

System of equations	Operator definition	Equation in operator notation
$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 4x_2 &= 7 \end{aligned}$	$A\overline{x} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	$A\overline{x} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

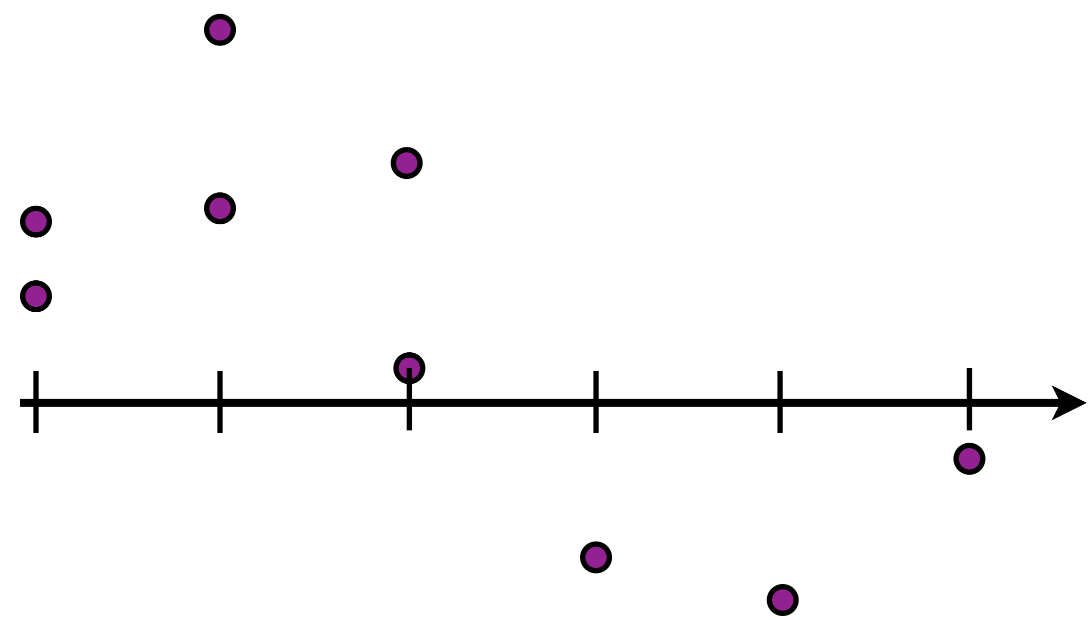
Some differential equations we've seen, written in operator notation.

Differential equation	Operator definition	Equation in operator notation
$t\dfrac{dy}{dt} + 2y = 4t^2$	$L[y] = t\dfrac{dy}{dt} + 2y$	$L[y] = 4t^2$
$y'' + 4y' + 4y = 0$	$L[y] = y'' + 4y' + 4y$	$L[y] = 0$

Some connections to linear (matrix) algebra

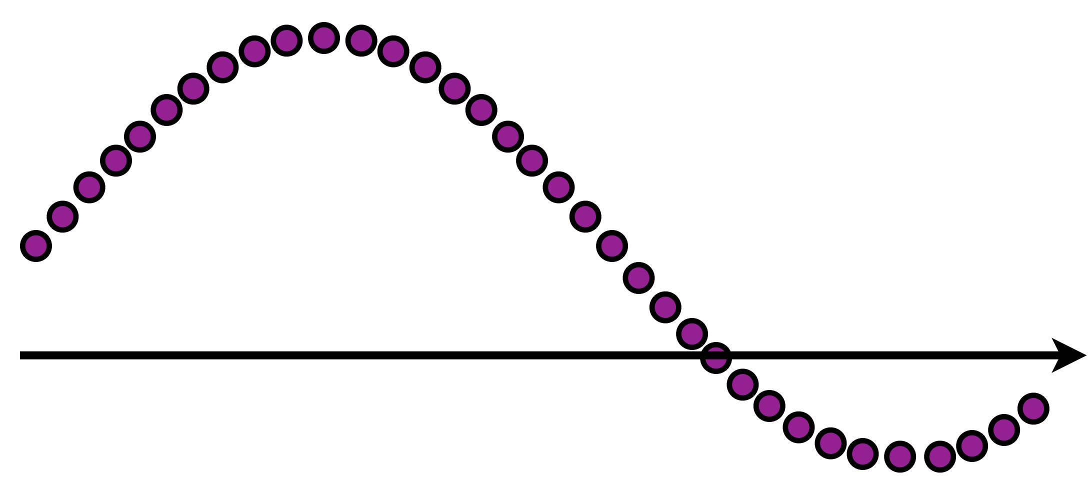
- A more detailed connection between matrix equations and DEs:
  - A vector as a function

$$\bar{b} = \begin{pmatrix} \sin(1) \\ \sin(2) \\ \sin(3) \\ \sin(4) \\ \sin(5) \\ \sin(6) \end{pmatrix}$$



  - A function is just a vector with an infinite number of entries.

$$y(t) = \sin(t)$$



  - A differential operator is just a really big matrix.