Today

- Independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Homog. eq. with constant coeff. (Section 3.1)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

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- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0,\ y'(0) = v_0$ with these two solutions?

- Solve this system for C_1, C_2...

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$$\det \begin{pmatrix} e^3 & e^{-3} \\ e^3 & e^{-3} \end{pmatrix} = 0$$

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- This quantity is called the Wronskian.

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- The Wronskian is defined for any two functions, even if they aren't solutions to an ODE.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

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- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions.

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- So for case i (distinct roots), can we form a general solution from

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So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.

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Challenge: come up with an initial condition for (iii) that has a bounded solution.

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Complex roots (Section 3.3)

- Complex number review (Euler's formula)
- Complex roots of the characteristic equation
- From complex solutions to real solutions

Complex number review

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- For α+βi, we call α the real part and β the imaginary part.

Complex number review

$$(a + bi) + (c + di) = a + c + (b + d)i$$

Complex number review

- Adding two complex numbers:

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(B) $\dfrac{c + di}{c^2 + d^2}$

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$$(c + di)\frac{c - di}{c^2 + d^2} = \frac{c^2 + d^2 - (cd - dc)i}{c^2 + d^2} = 1$$

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Complex number review

- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

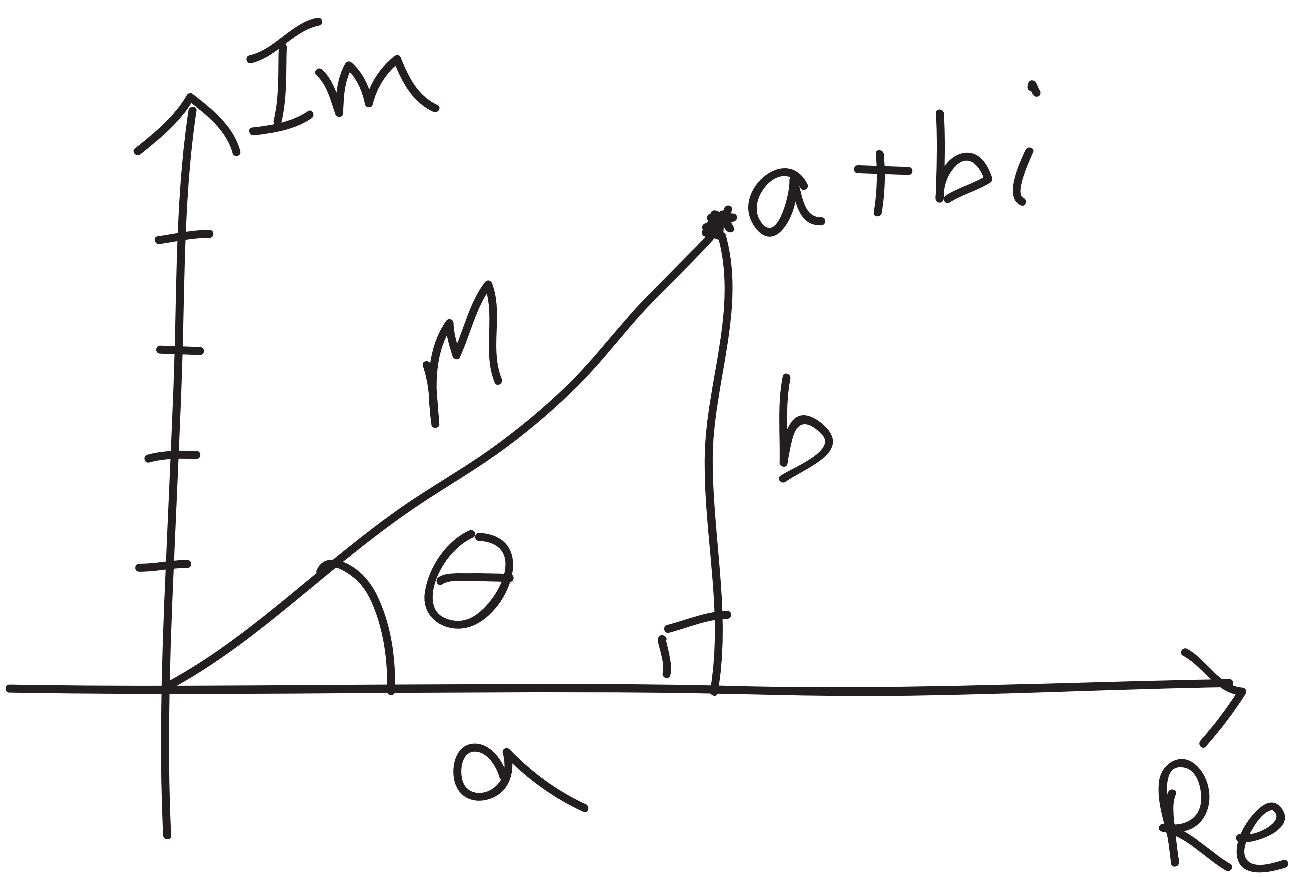
$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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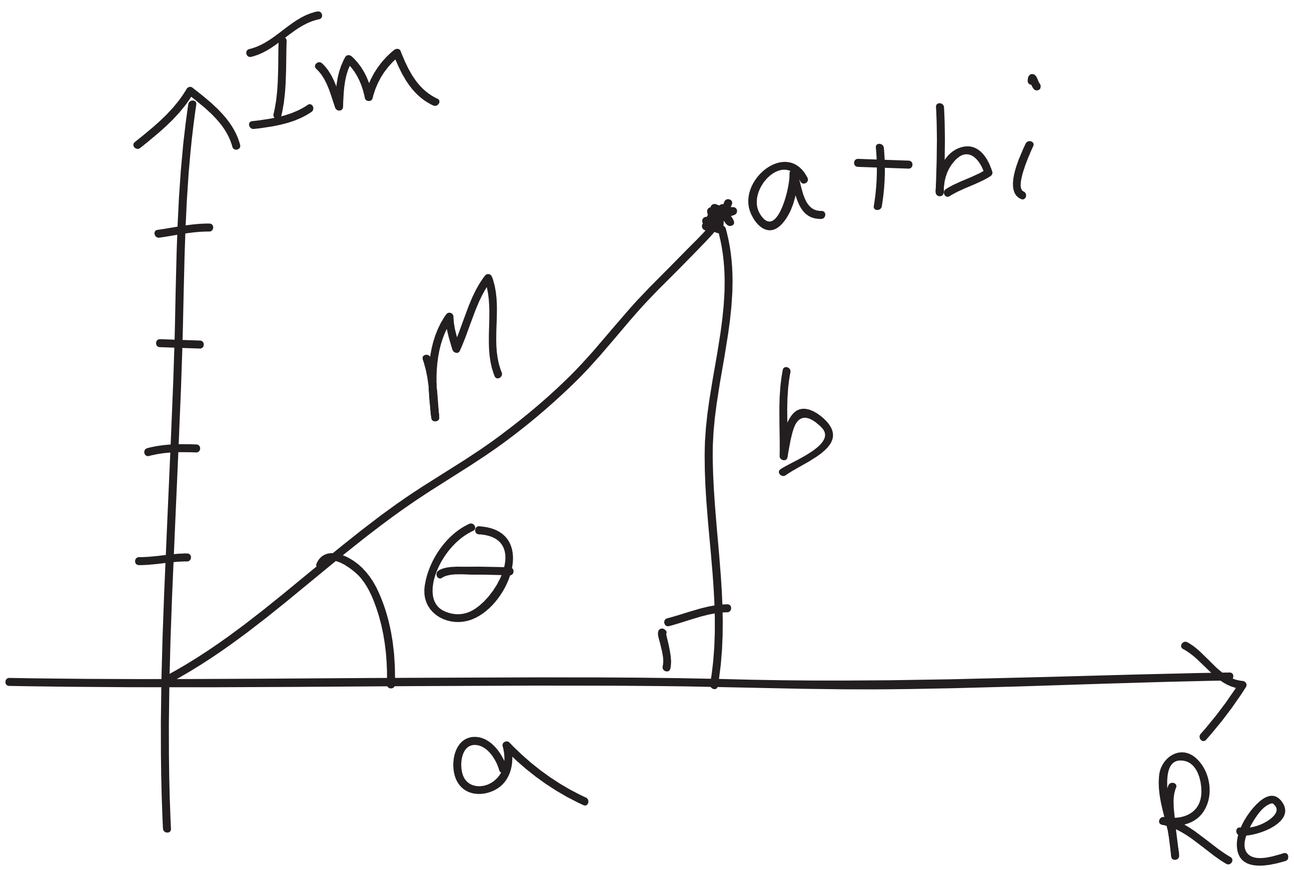
$$a = M\cos\theta$$
$$b = M\sin\theta$$



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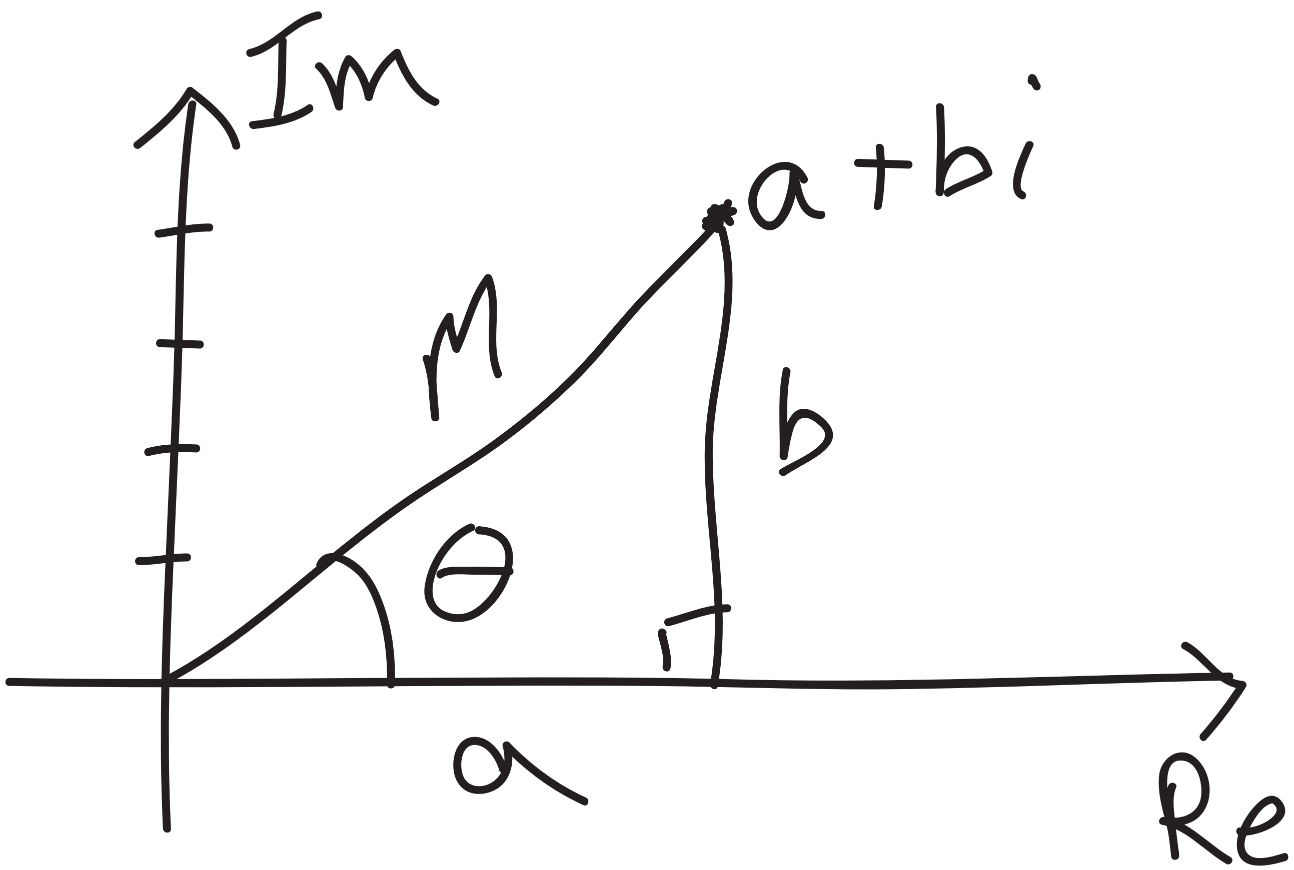

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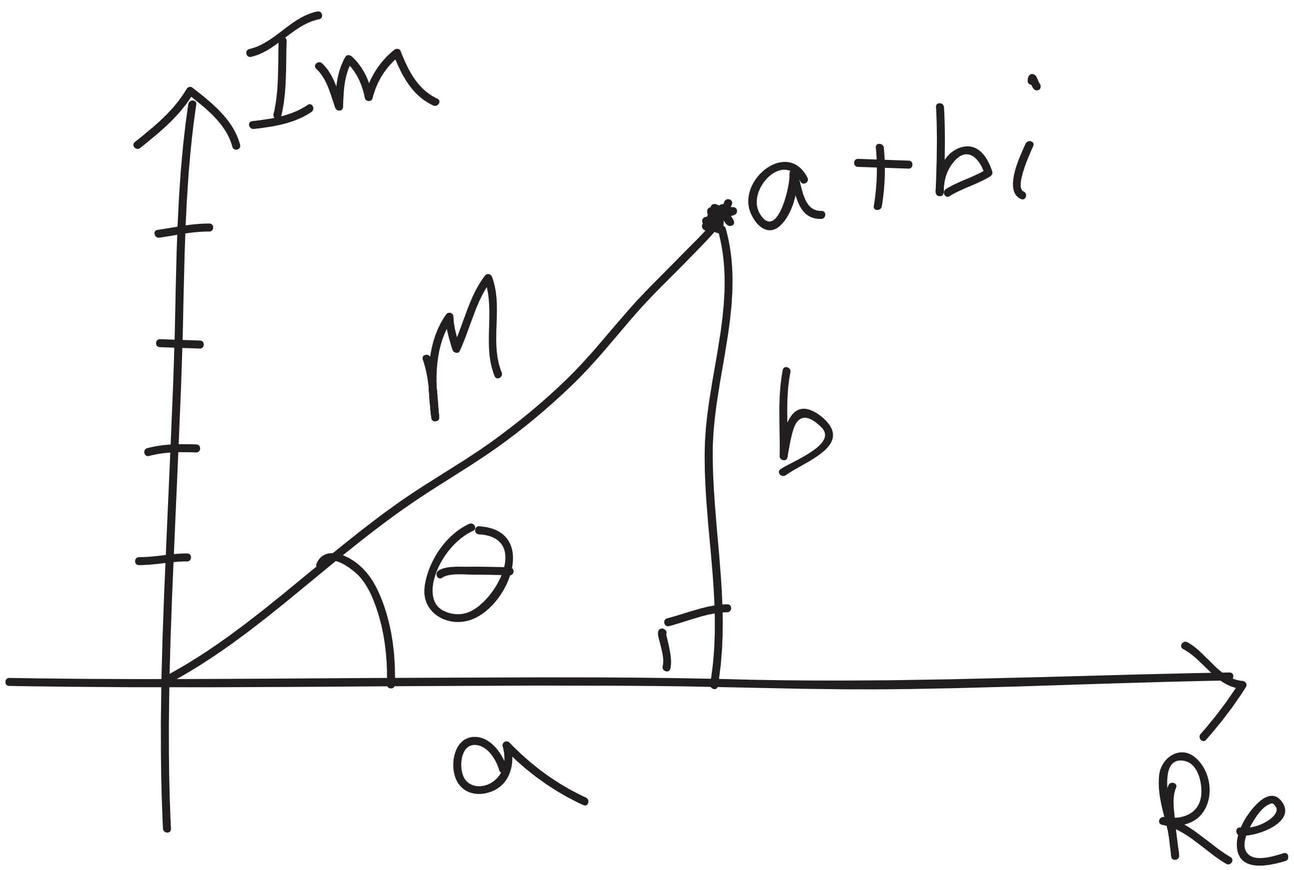

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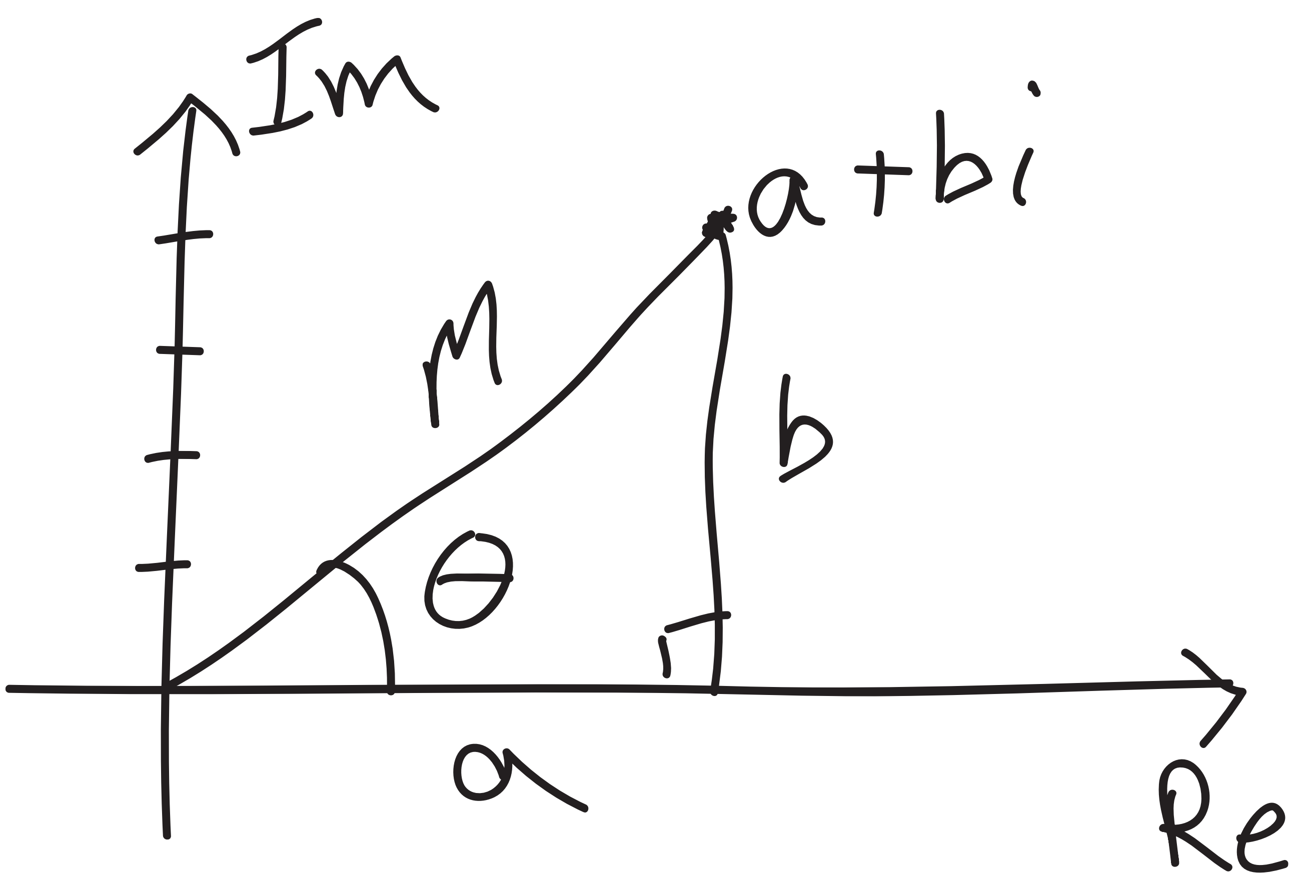

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θ is sometimes called the argument or phase of $a + bi$.

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  - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

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$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta$$

$$= e^{i\theta}$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

Euler's formula:

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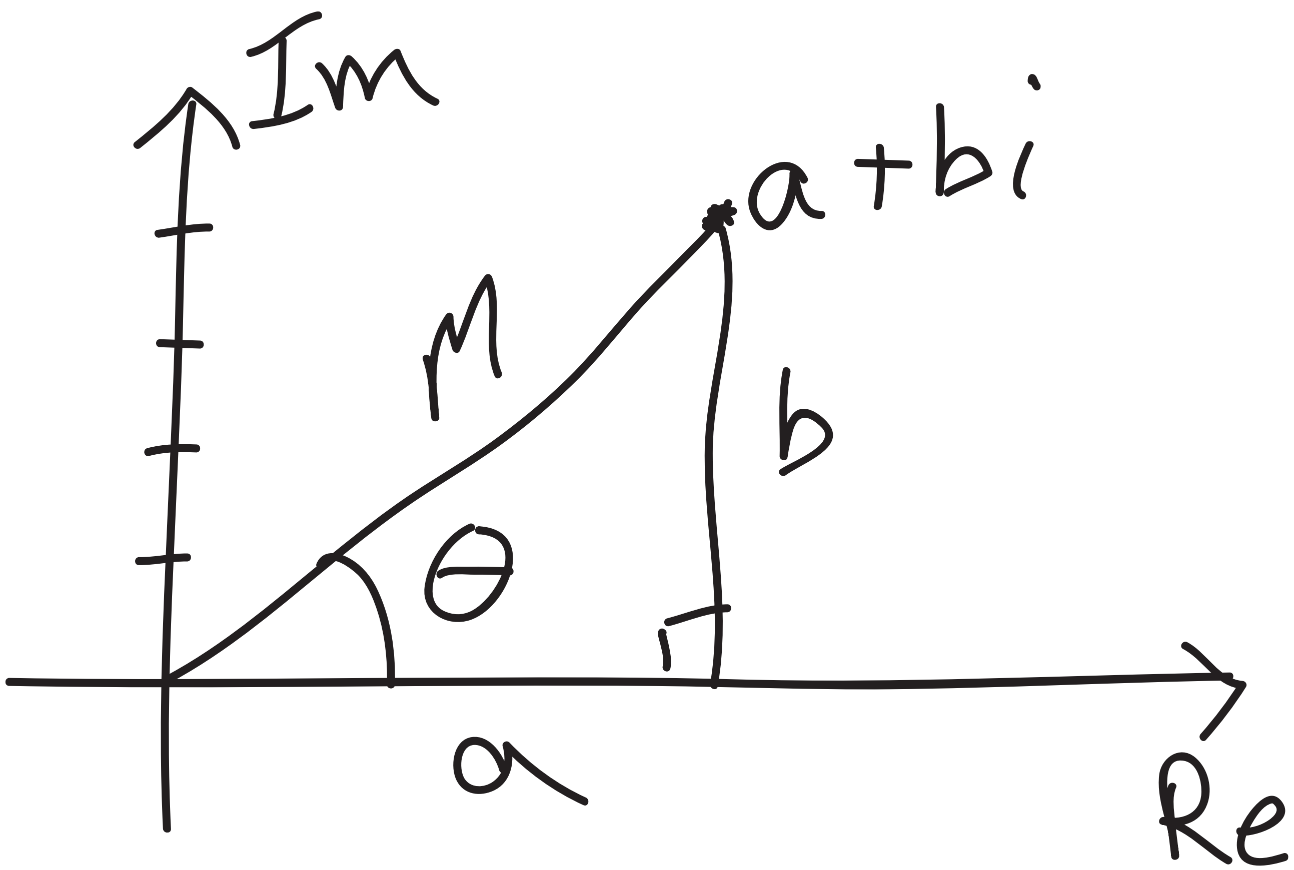
Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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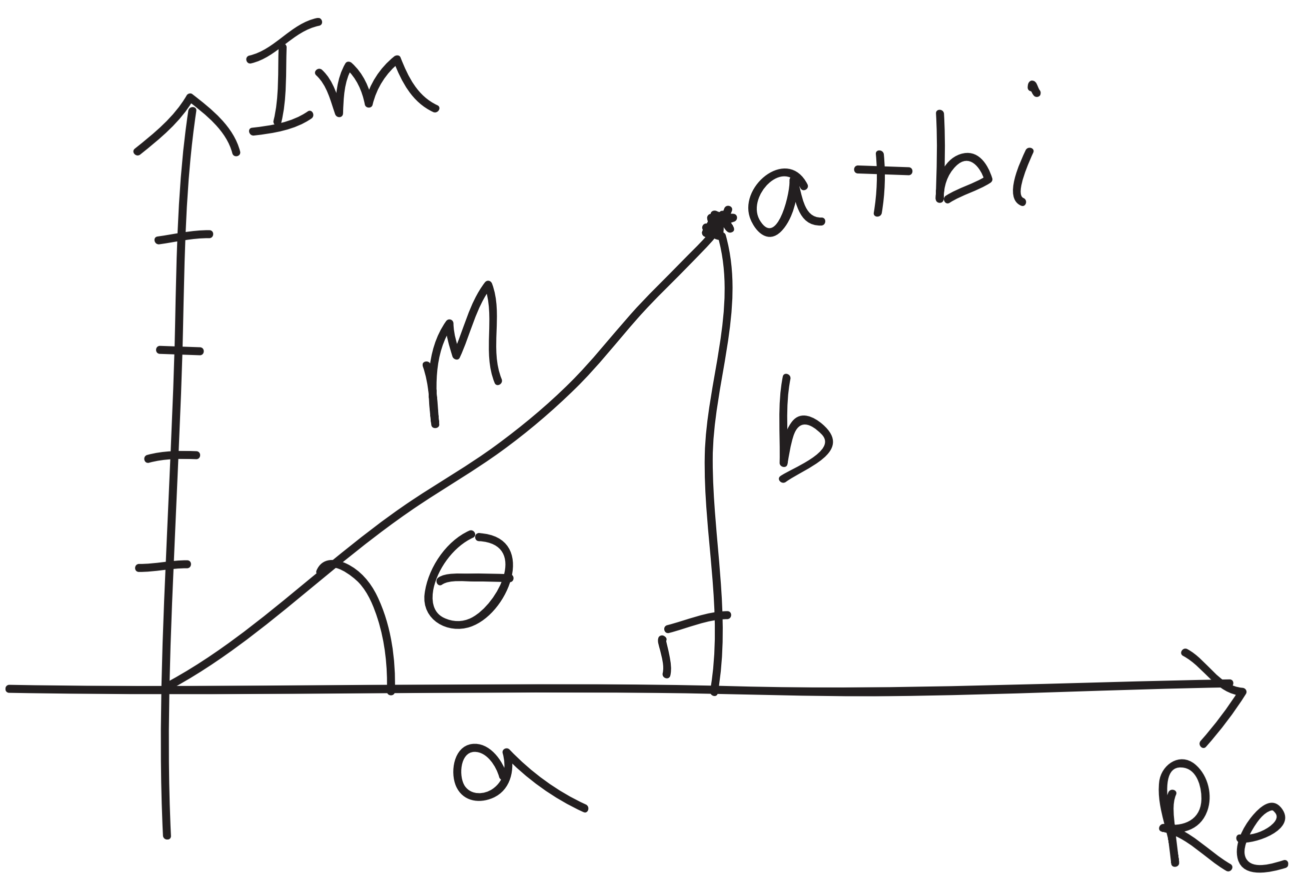

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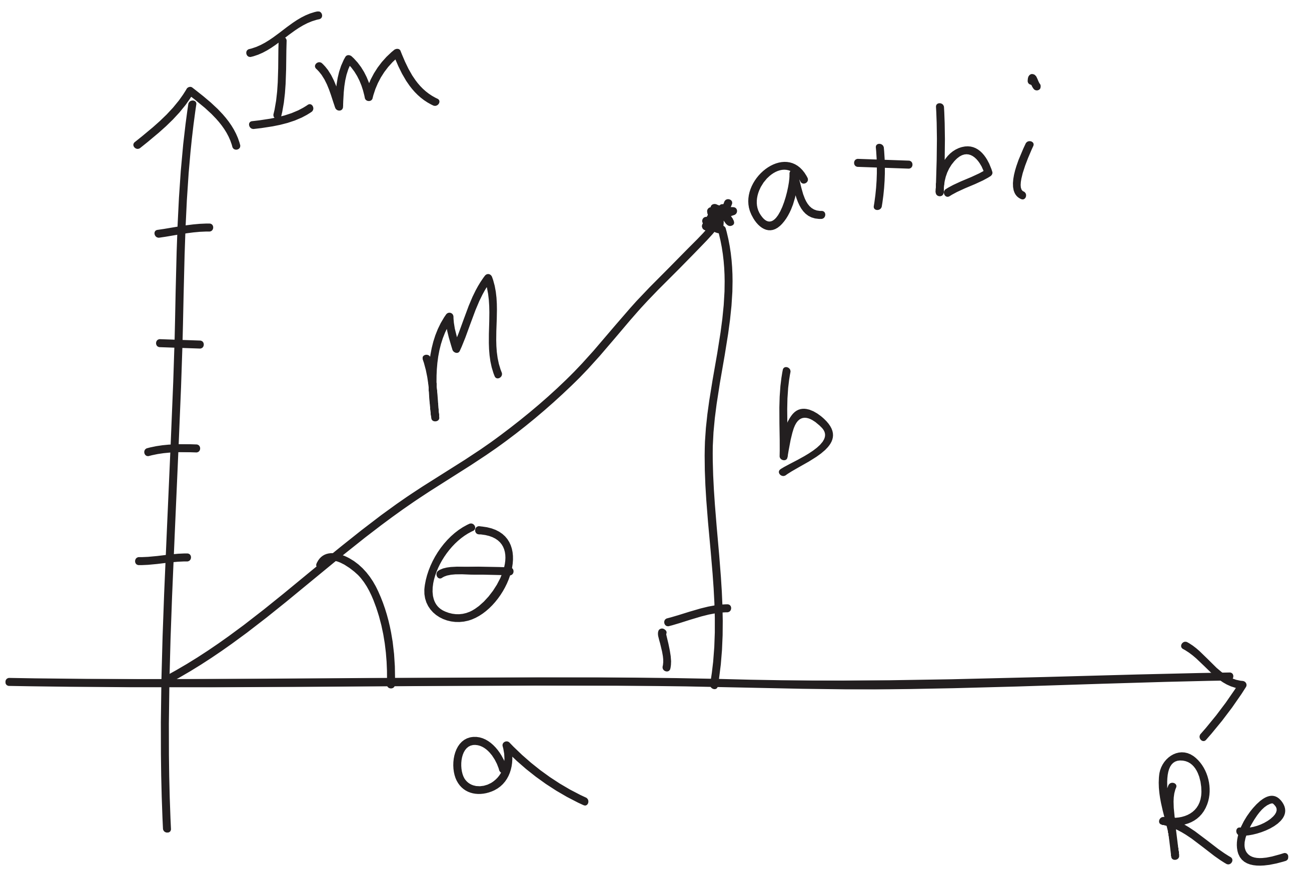

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(Polar form makes multiplication much cleaner)