Today

- General solution for complex eigenvalues case.
- Shapes of solutions for complex eigenvalues case.

Calculating eigenvalues - trace/det shortcut

- For the general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- find the characteristic equation and solve it to find the eigenvalues.

(A) λ^2

$$\lambda^2 - \mathrm{tr}(\mathrm{A})\lambda + \det(A) = 0$$

(B) $\lambda^2 + (b + c)\lambda + ac - bd = 0$

(C) $\lambda^2 - (a + d)\lambda + ad - bc = 0$

(D) $\lambda^2 + (a - d)\lambda + ad + bc = 0$

(E) I don’t know how to find eigenvalues.

Complex eigenvalues - example

- Find the general solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x}$.
  - The eigenvalues are
    - ★ (A) $\lambda = 1 \pm 2i$
    - (B) $\lambda = -1, 3$
    - (C) $\lambda = 2 \pm 4i$
    - (D) $\lambda = -2, 6$
    - (E) I don't know how to find eigenvalues.
  - The eigenvectors are . . .

$$A - \lambda_1 I = \begin{pmatrix} 1-(1+2i) & 1 \\ -4 & 1-(1+2i) \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \times \tfrac{1}{2} i$$

$$\sim \begin{pmatrix} -2i & 1 \\ -2i & 1 \end{pmatrix}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

Complex eigenvalues - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x(t)} = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions. Two options:
  - Convert to a second order equation as we did for real roots case.
  - Recall the sum and difference trick - it says that real and imaginary parts of a complex solution are themselves solutions.

Complex eigenvalues - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x(t)} = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$x_1'' = x_1' + x_2'$$

$$x_1'' = x_1' - 4x_1 + x_2$$

$$x_1'' = x_1' - 4x_1 + x_1' - x_1$$

$$x_1'' - 2x_1' + 5x_1 = 0$$

$$r^2 - 2r + 5 = 0$$

$$r = 1 \pm 2i$$

Complex eigenvalues - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$r = 1 \pm 2i \qquad x_1(t) = e^t \left(C_1 \cos(2t) + C_2 \sin(2t)\right)$$

$$\begin{aligned} x_1'(t) = {} & e^t\left(-2C_1 \sin(2t) + 2C_2 \cos(2t)\right) \\ & + e^t\left(C_1 \cos(2t) + C_2 \sin(2t)\right) \end{aligned}$$

$$x_2 = x_1' - x_1 = e^t\left(-2C_1 \sin(2t) + 2C_2 \cos(2t)\right)$$

Complex eigenvalues - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \quad \Rightarrow \quad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$x_1(t) = e^t (C_1 \cos(2t) + C_2 \sin(2t))$$

$$x_2(t) = e^t (-2C_1 \sin(2t) + 2C_2 \cos(2t))$$

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$

Complex eigenvalues - example

$$\frac{1}{2}\left(e^{(1+2i)t}\begin{pmatrix}1\\2i\end{pmatrix} - e^{(1-2i)t}\begin{pmatrix}1\\-2i\end{pmatrix}\right) = \boxed{}$$

$$\frac{1}{2i}\left(e^{(1+2i)t}\begin{pmatrix}1\\2i\end{pmatrix} + e^{(1-2i)t}\begin{pmatrix}1\\-2i\end{pmatrix}\right) = \boxed{}$$

- Pick one solution: $C_1 = 1,\ C_2 = 0$
- and its conjugate $C_1 = 0,\ C_2 = 1$.

$$\mathbf{x(t)} = e^{(1+2i)t}\begin{pmatrix}1\\2i\end{pmatrix}$$

$$= e^t(\cos(2t) + i\sin(2t))\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}i\right)$$

$$= \boxed{e^t\left[\begin{pmatrix}1\\0\end{pmatrix}\cos(2t) - \begin{pmatrix}0\\2\end{pmatrix}\sin(2t)\right]}$$

$$+\boxed{e^t\left[\begin{pmatrix}1\\0\end{pmatrix}\sin(2t) + \begin{pmatrix}0\\2\end{pmatrix}\cos(2t))\right]}\,i$$

Complex eigenvalues - general case

- Find e-values, $\lambda = \alpha \pm \beta i$, and e-vectors, $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$= \mathbf{a} + i\mathbf{b}$$

- Write down solution (or use method on previous slide for formula-free):

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(\beta t) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin(\beta t) \right) \right.$$

$$\left. + C_2 \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin(\beta t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos(\beta t) \right) \right]$$

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 (\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)) \right.$$

$$\left. + C_2 (\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)) \right]$$

Complex eigenvalues - example

- Suppose you find eigenvalue $\lambda = 2\pi i$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Which of the following is a solution to the original equation?

☆ (A) $\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$

(B) $\mathbf{x(t)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t)$

☆ (C) $\mathbf{x(t)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t)$

(D) $\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$

Complex eigenvalues - example

- Suppose you find eigenvalue $\lambda = 2\pi i$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Which of the following is a solution to the original equation?

$$\overline{\mathbf{x}}(\mathbf{t}) = e^{2\pi i t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= (\cos(2\pi t) + i \sin(2\pi t)) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\pi t) + i \sin(2\pi t)) \\ -\sin(2\pi t) + i \cos(2\pi t)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$$+ i \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) \right]$$

- Sum and difference trick lets us take the Real and Imaginary parts as two indep. solutions

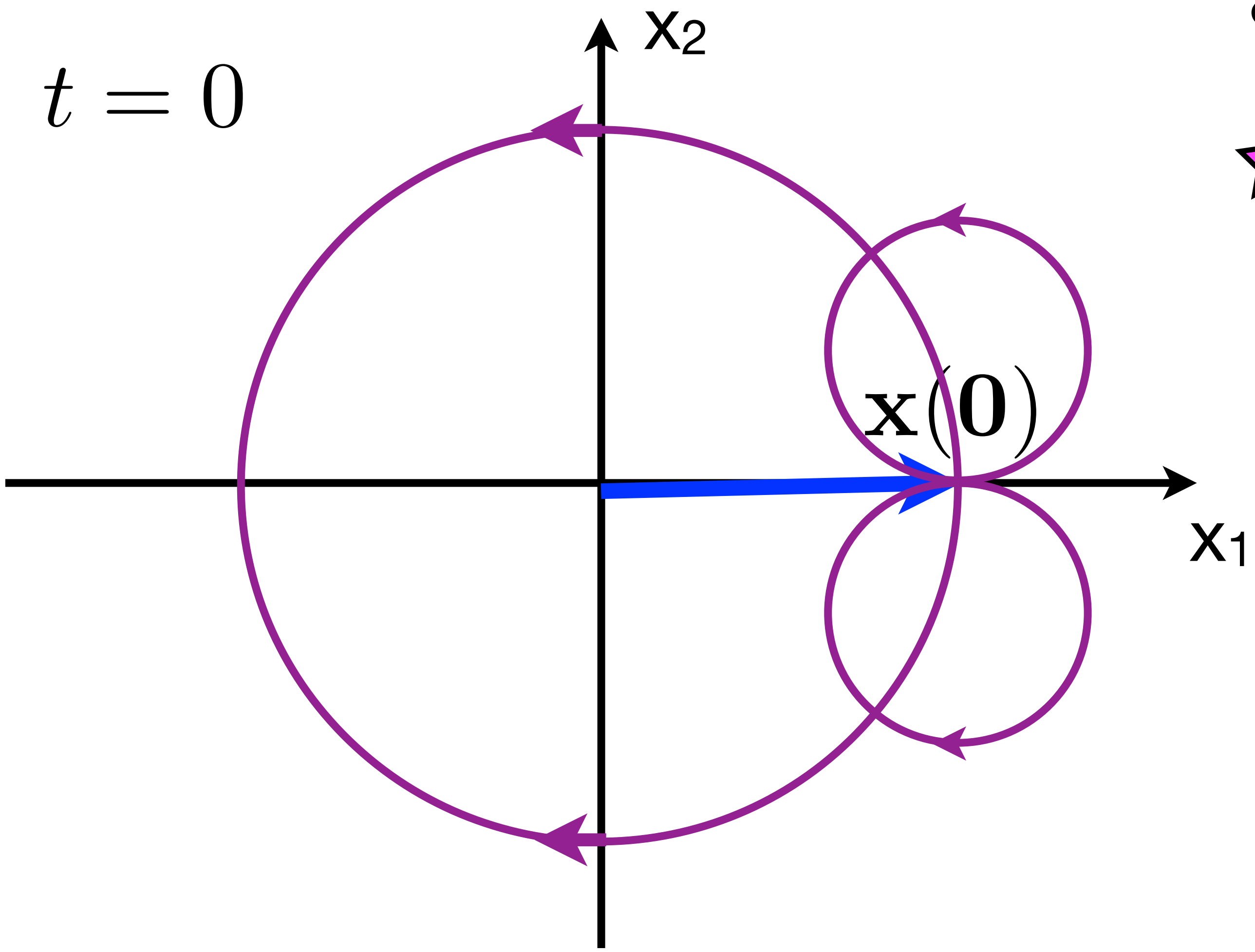
Complex eigenvalues - example

- But what about $\lambda_2 = -2\pi i$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$?

$$\begin{aligned}
\overline{\mathbf{x}}(\mathbf{t}) &= e^{-2\pi i t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= (\cos(-2\pi t) + i\sin(-2\pi t)) \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= \begin{pmatrix} \cos(2\pi t) - i\sin(2\pi t)) \\ -\sin(2\pi t) - i\cos(2\pi t)) \end{pmatrix} \\
&= \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)} \\
&\quad - i \boxed{\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) \right]}
\end{aligned}$$

Complex eigenvalues - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



- What happens as t increases?

★ (A) The vector rotates clockwise.

(B) The vector rotates counter-clockwise.

(C) The tip of the vector maps out a circle in the first quadrant.

(D) The tip of the vector maps out a circle in the fourth quadrant.

(E) Explain please.

Complex eigenvalues - example

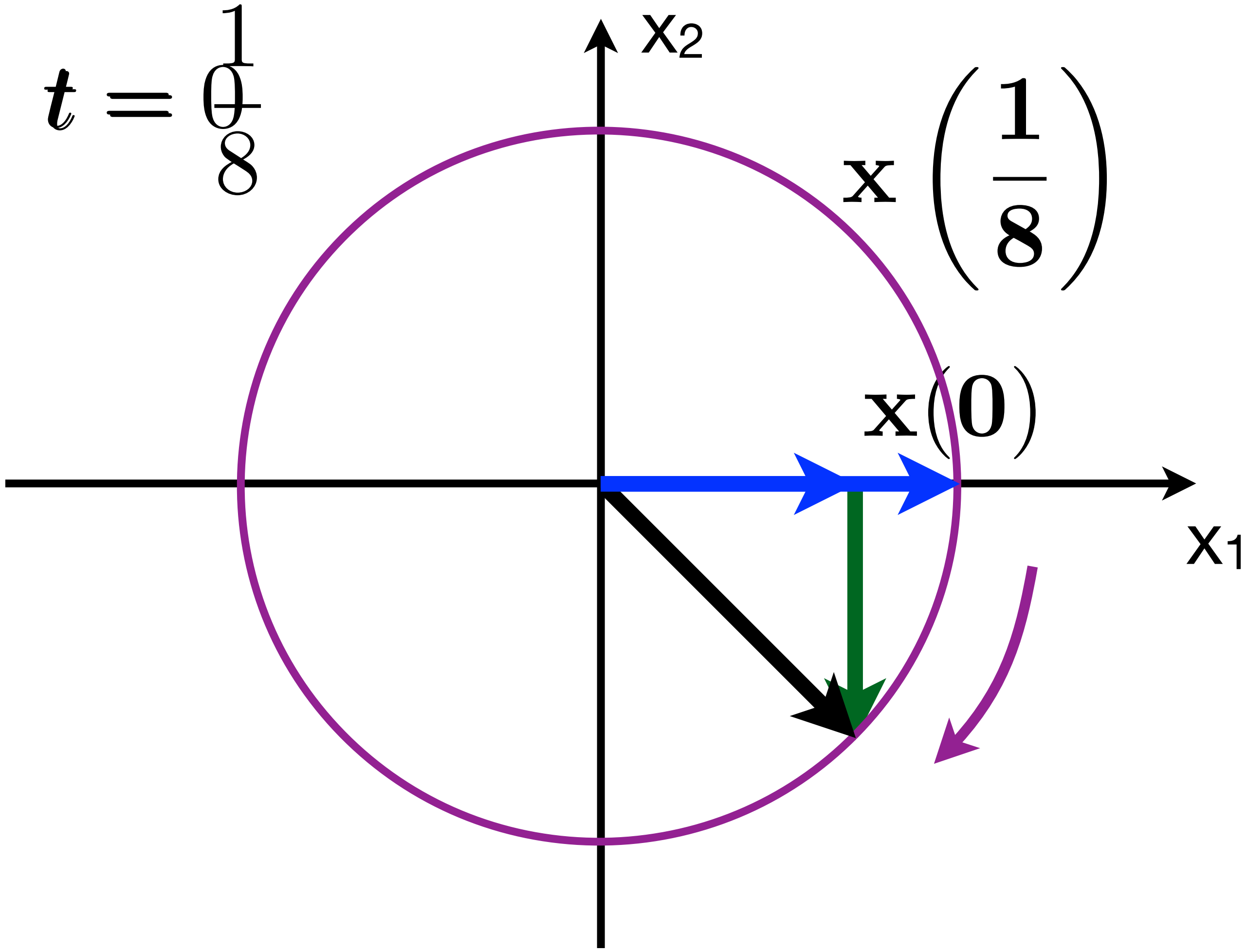
$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos\left(\frac{\pi}{4}\right) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{4}\right)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\boldsymbol{t} = 0\ \frac{1}{8}$

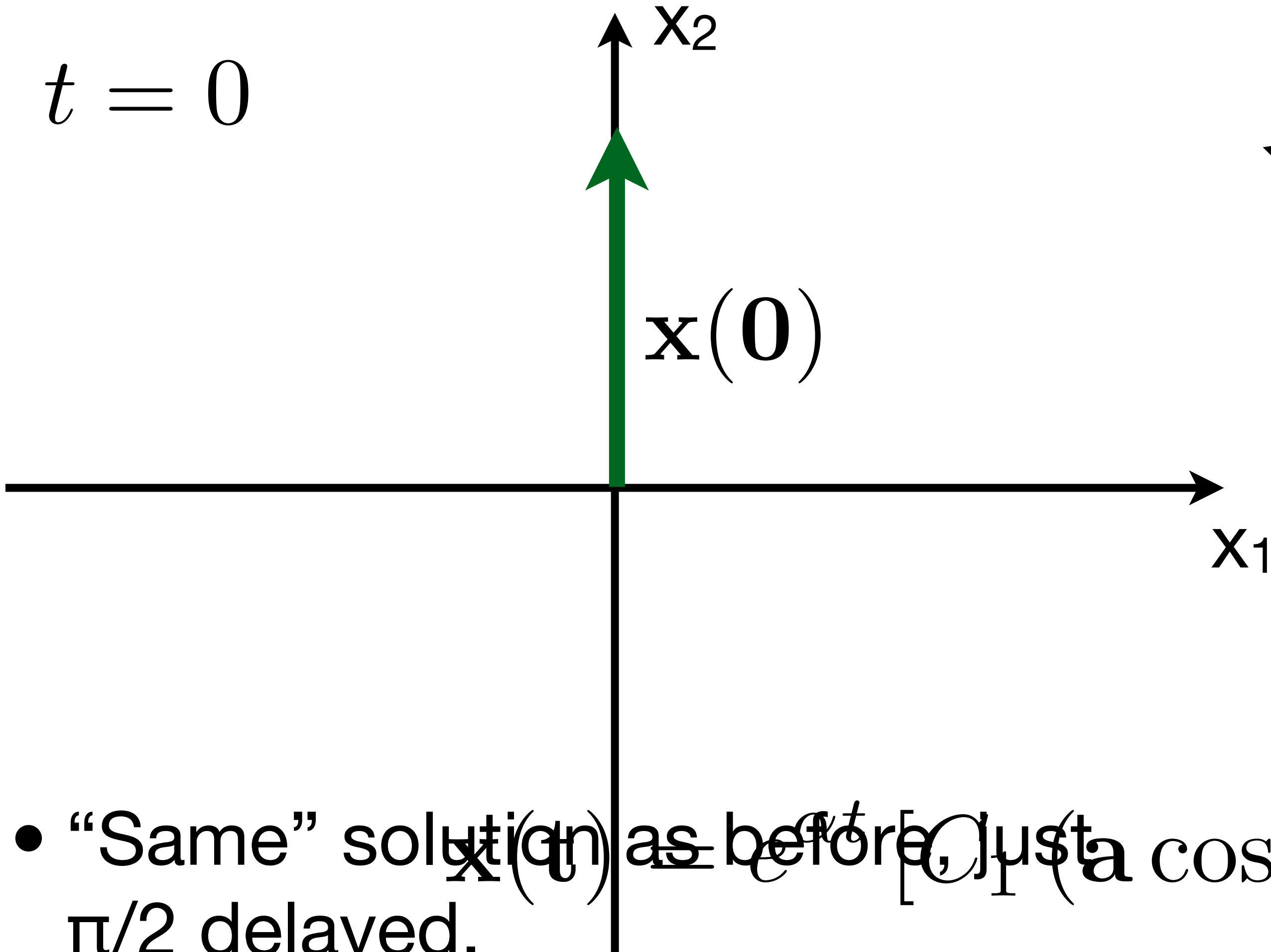


Complex eigenvalues - example

- Same equation, initial condition chosen so that $C_1=0$ and $C_2=1$.

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix}1\\0\end{pmatrix}\sin(2\pi t) + \begin{pmatrix}0\\1\end{pmatrix}\cos(2\pi t)$$

$t = 0$



- What happens as t increases?

★ (A) The vector rotates clockwise.

(B) The vector rotates counter-clockwise.

(C) The tip of the vector maps out a circle in the first quadrant.

(D) The tip of the vector maps out a circle in the second quadrant.

(E) Explain please.

- “Same” solution as before, just π/2 delayed.

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}[C_1(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)) + C_2(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t))]$$

Complex eigenvalues - general case

- Looking at the general solution again...

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right)\right.$$
$$\left. + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\right]$$

- Both parts rotate in the exact same way but the C_2 part is delayed by a quarter phase.

- If an initial condition lies neither parallel to vector **a** nor to vector **b**, C_1 and C2 allow for intermediate phases to be achieved.

- **x(t)** can be rewritten (using trig identities) as

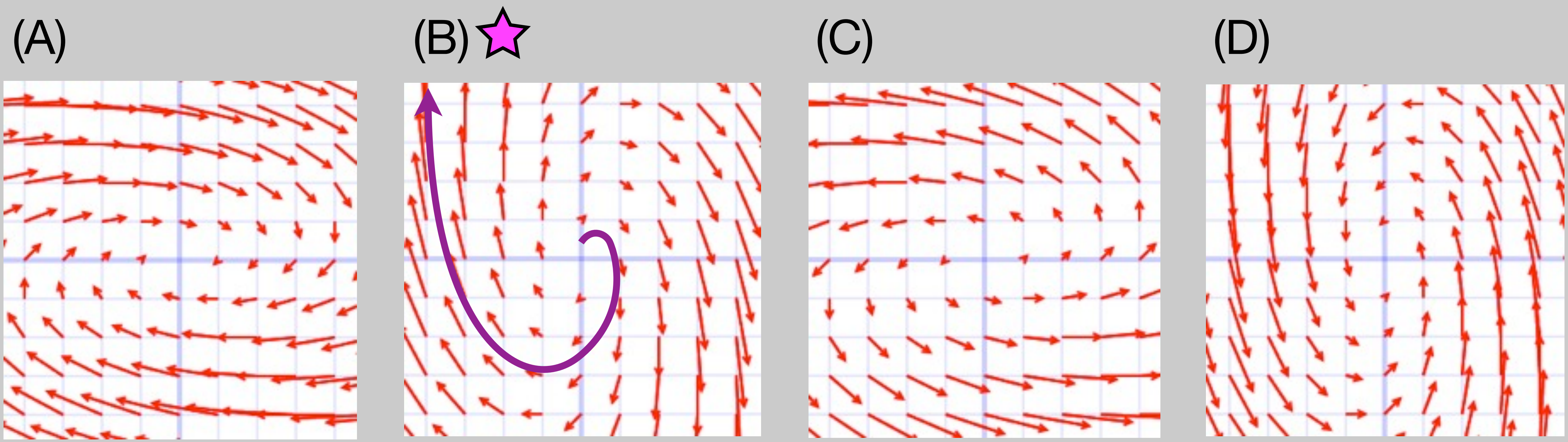
$$\mathbf{x}(\mathbf{t}) = Me^{\alpha t}\left(\mathbf{a}\cos(\beta t - \phi) - \mathbf{b}\sin(\beta t - \phi)\right)$$

where M and ϕ are constants to replace C_1 and C_2.

Complex eigenvalues - example

- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$



(E) Explain, please.