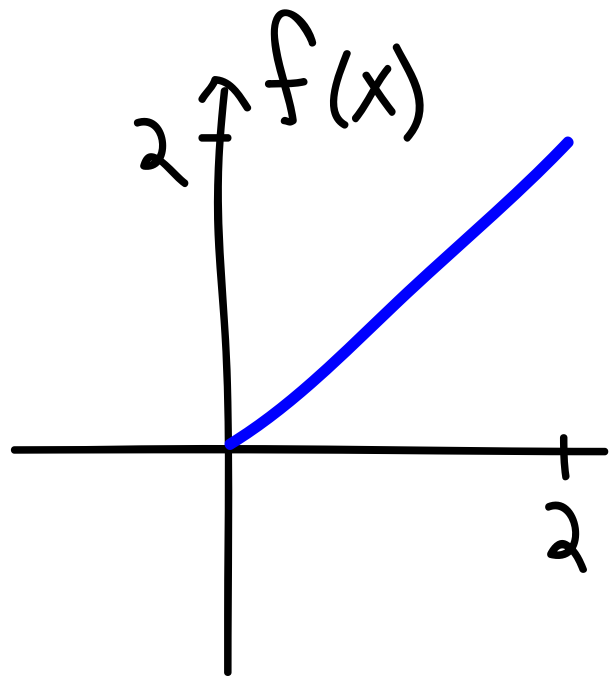
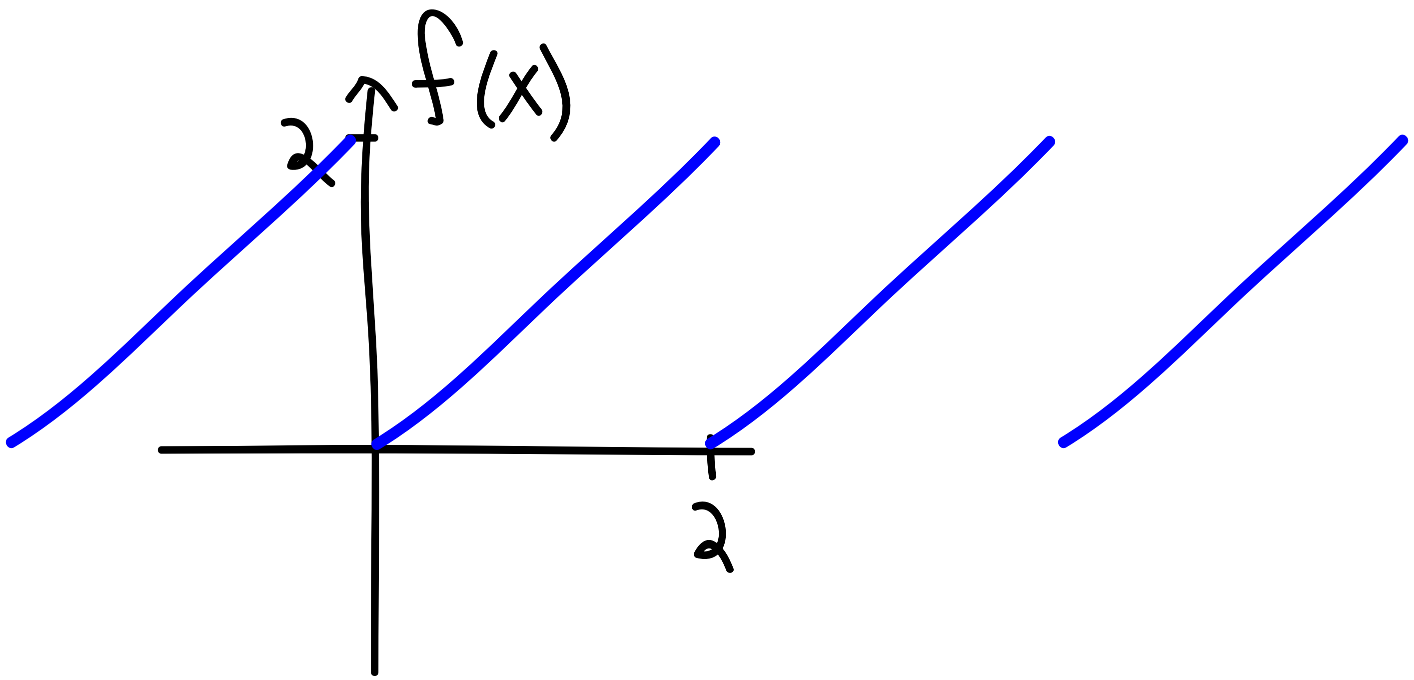
Today

- Fourier Series examples - even and odd extensions, other symmetries
- Using Fourier Series to solve the Diffusion Equation

Examples - calculate the Fourier Series



Examples - calculate the Fourier Series



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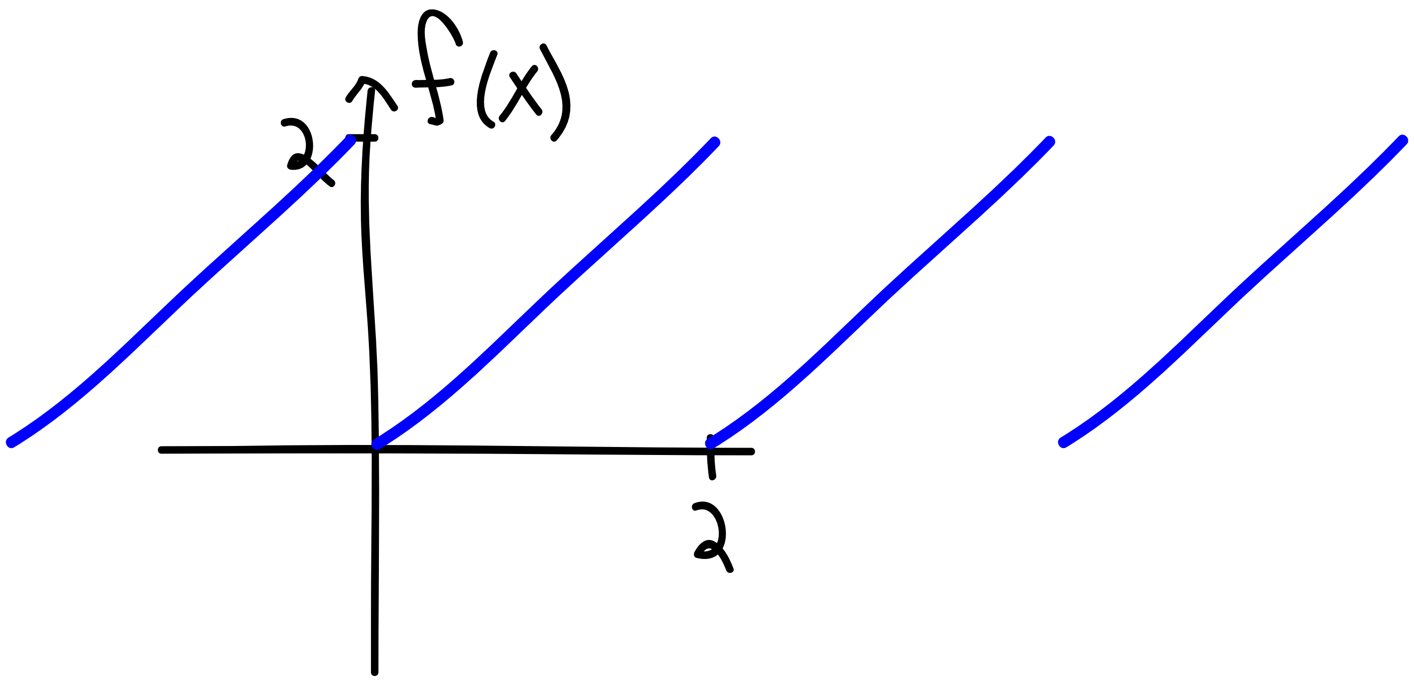
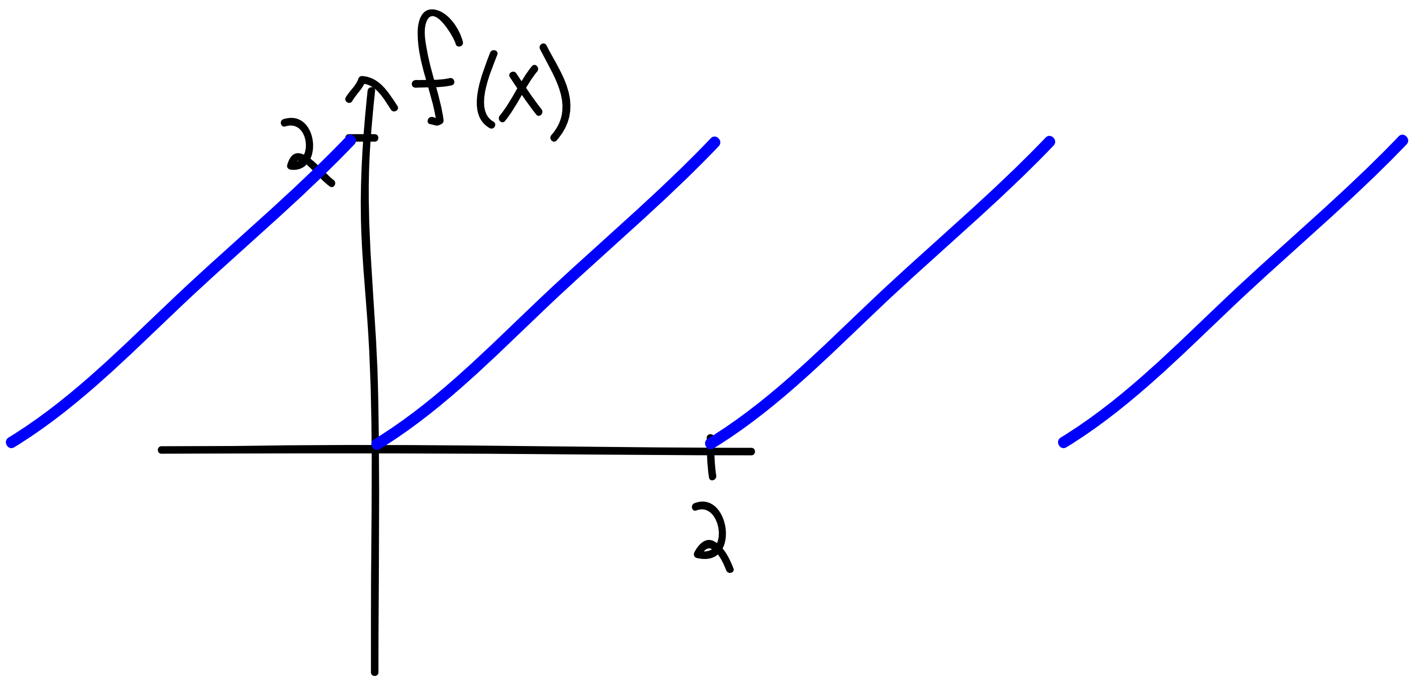


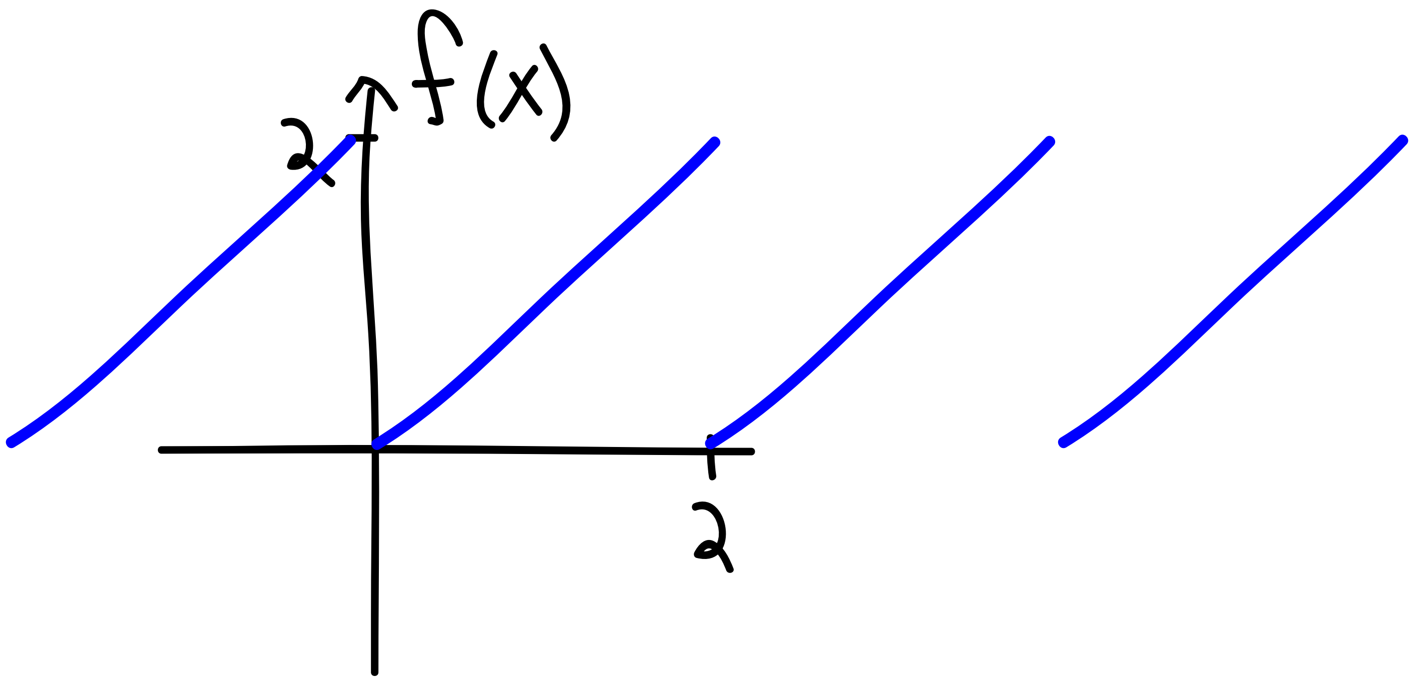
What is L?

Examples - calculate the Fourier Series



What is L? $L=1$

Examples - calculate the Fourier Series



$a_0 = \frac{1}{2}$

What is L? $L=1$

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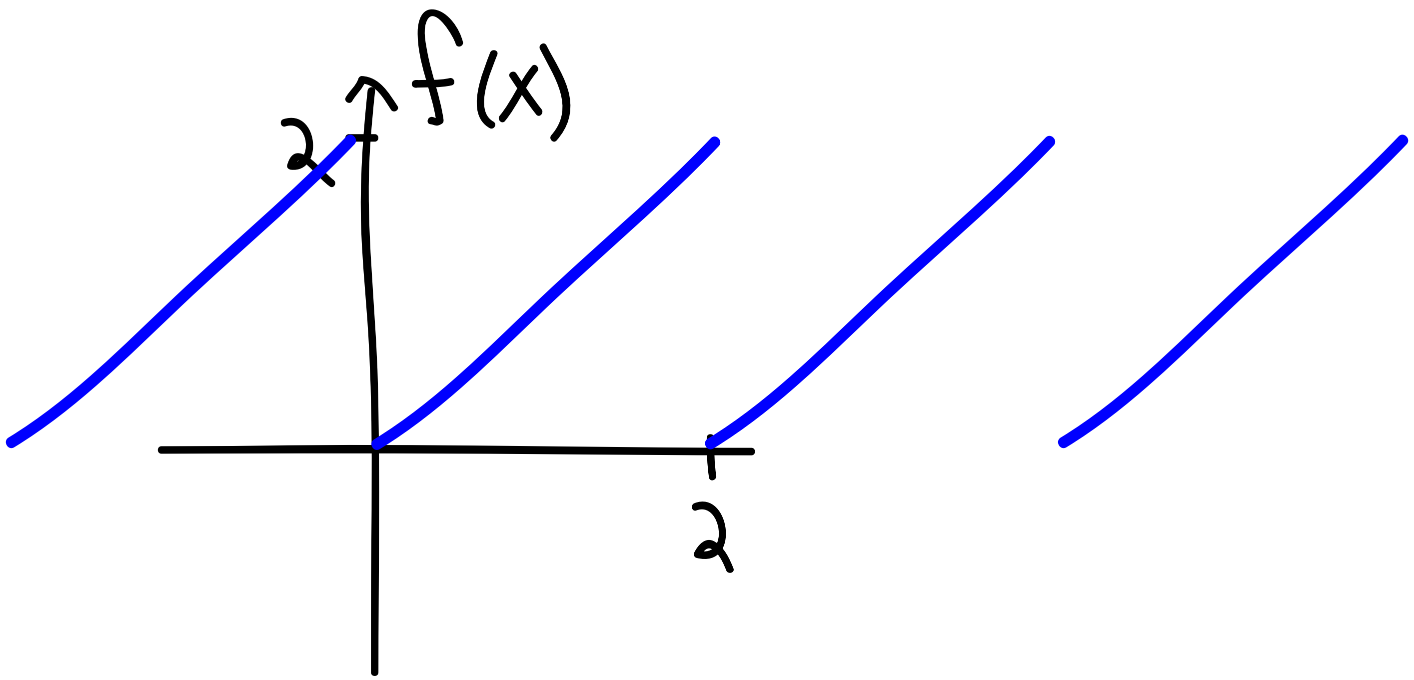

$a_0 = \frac{1}{2}$

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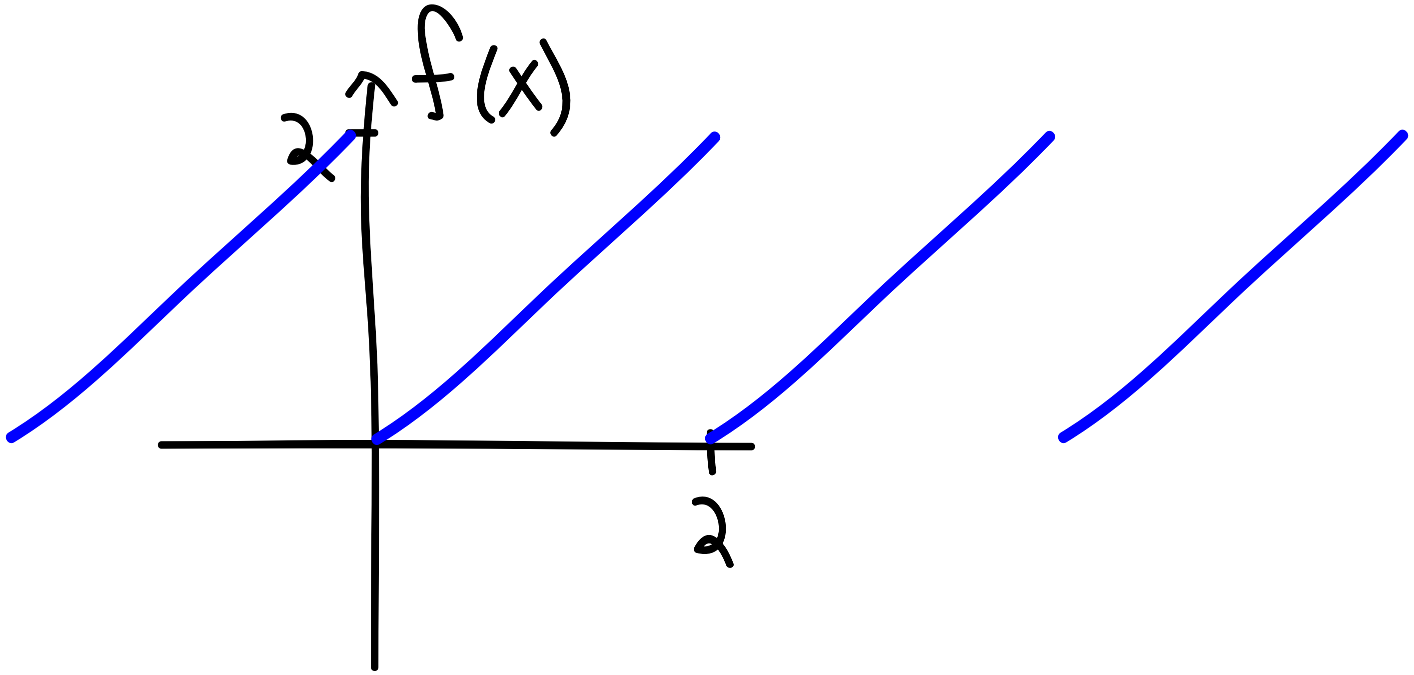

$$a_0 = \frac{1}{2}$$

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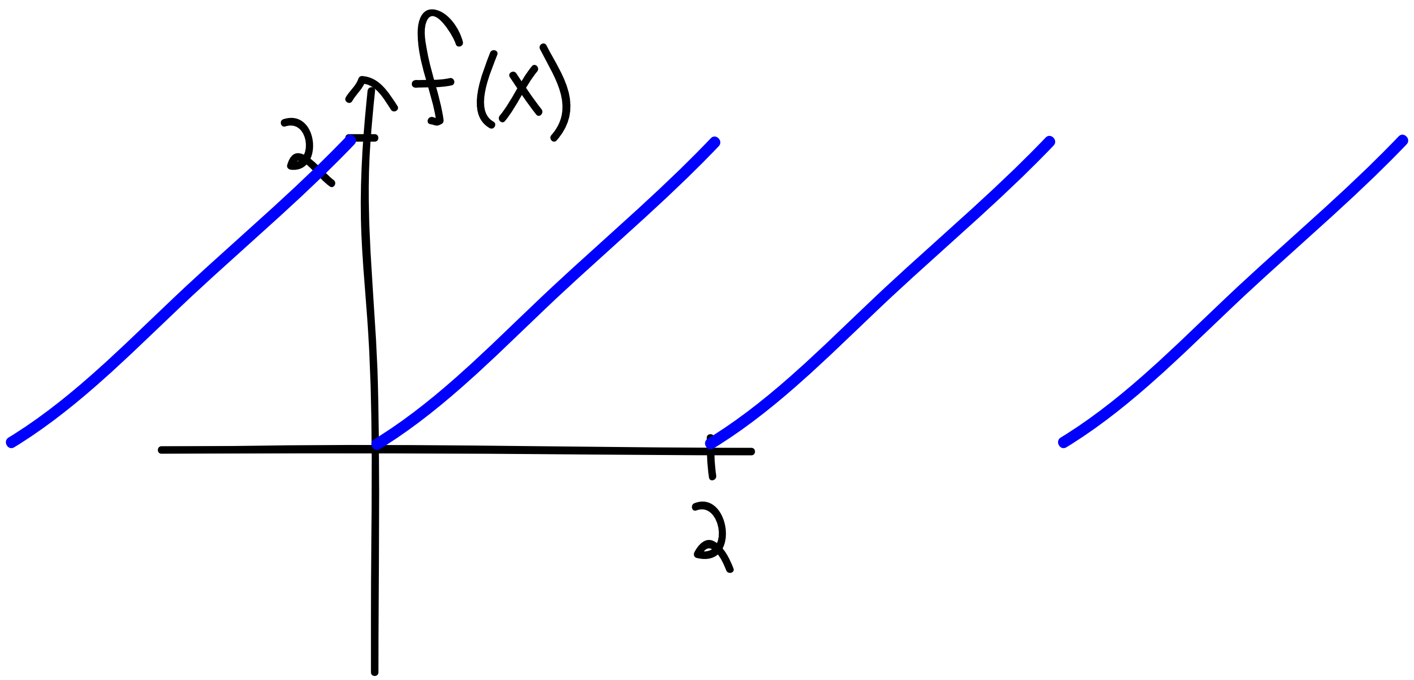

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What is L? $L = 1$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \cos n\pi x$$

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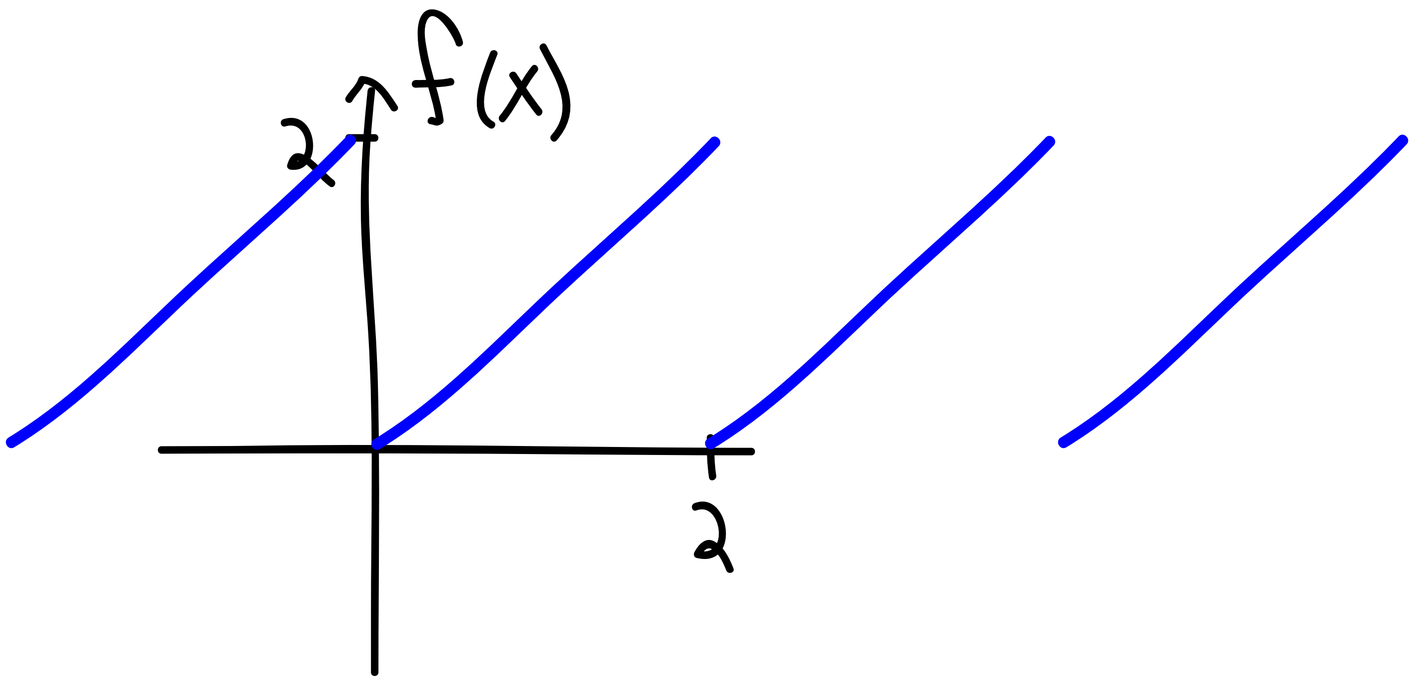

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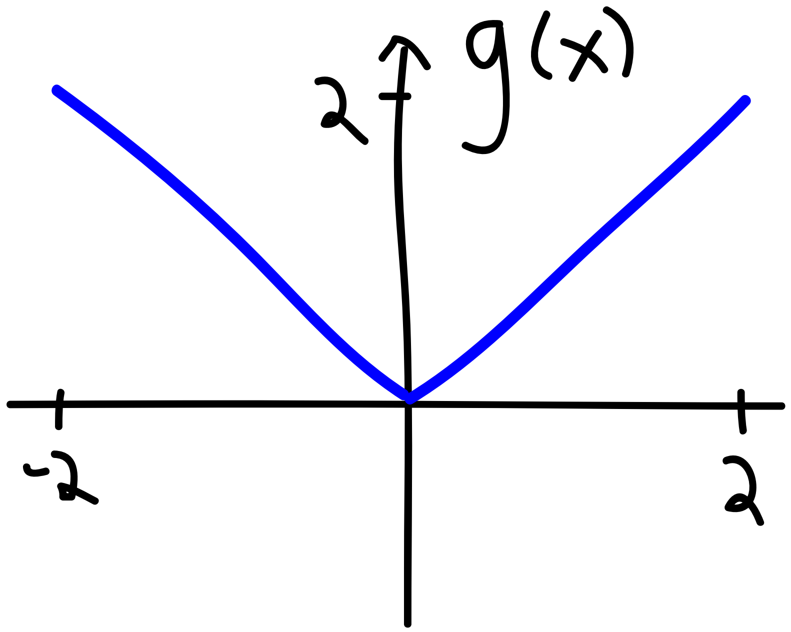

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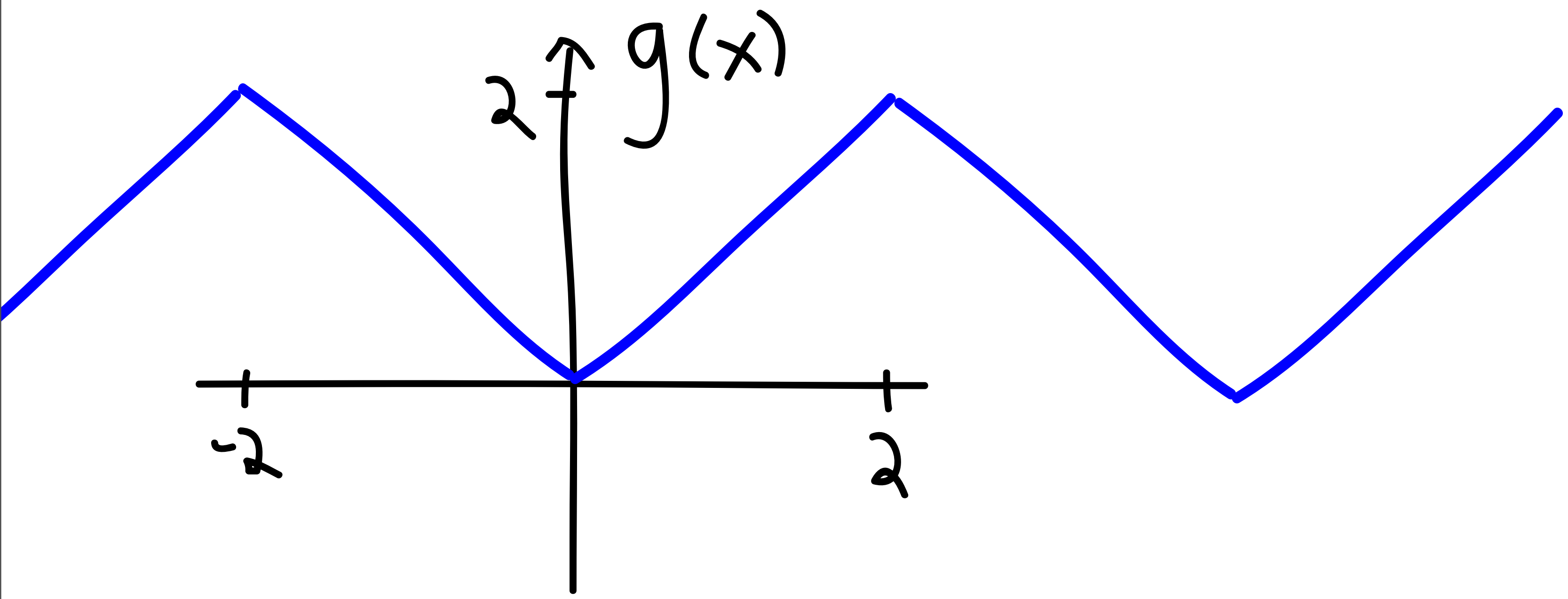

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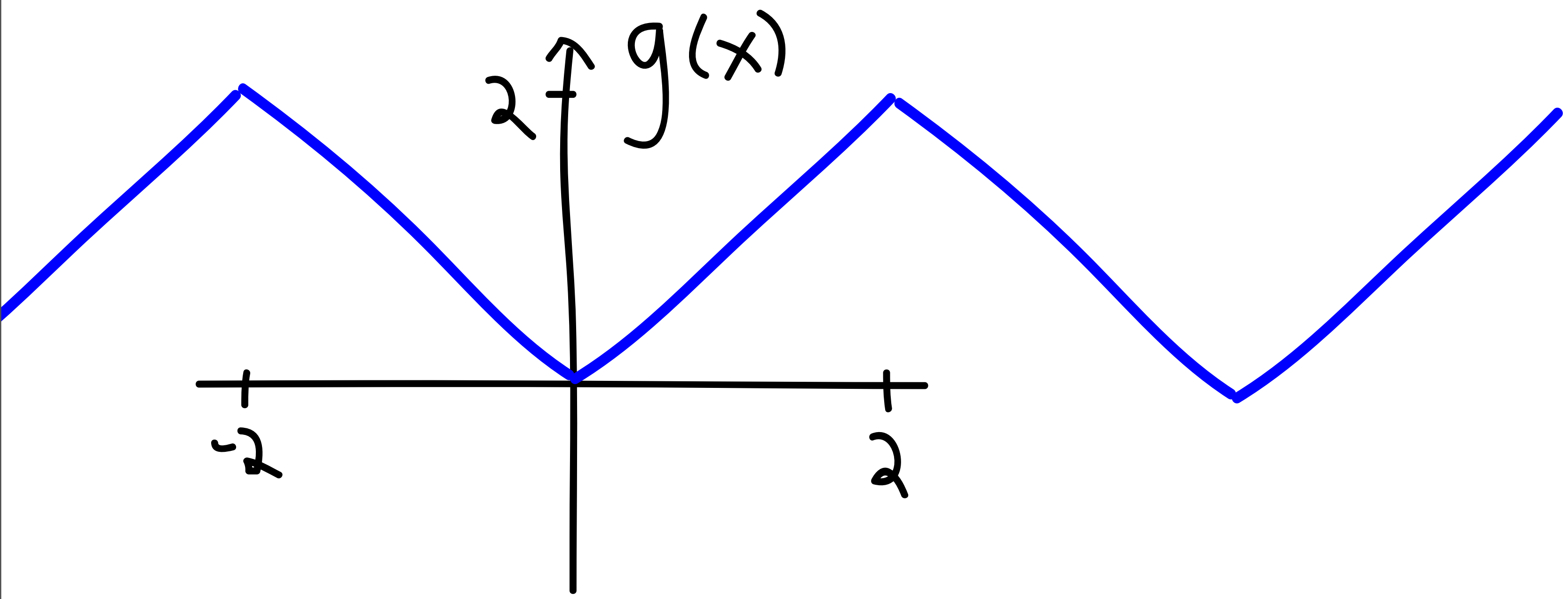

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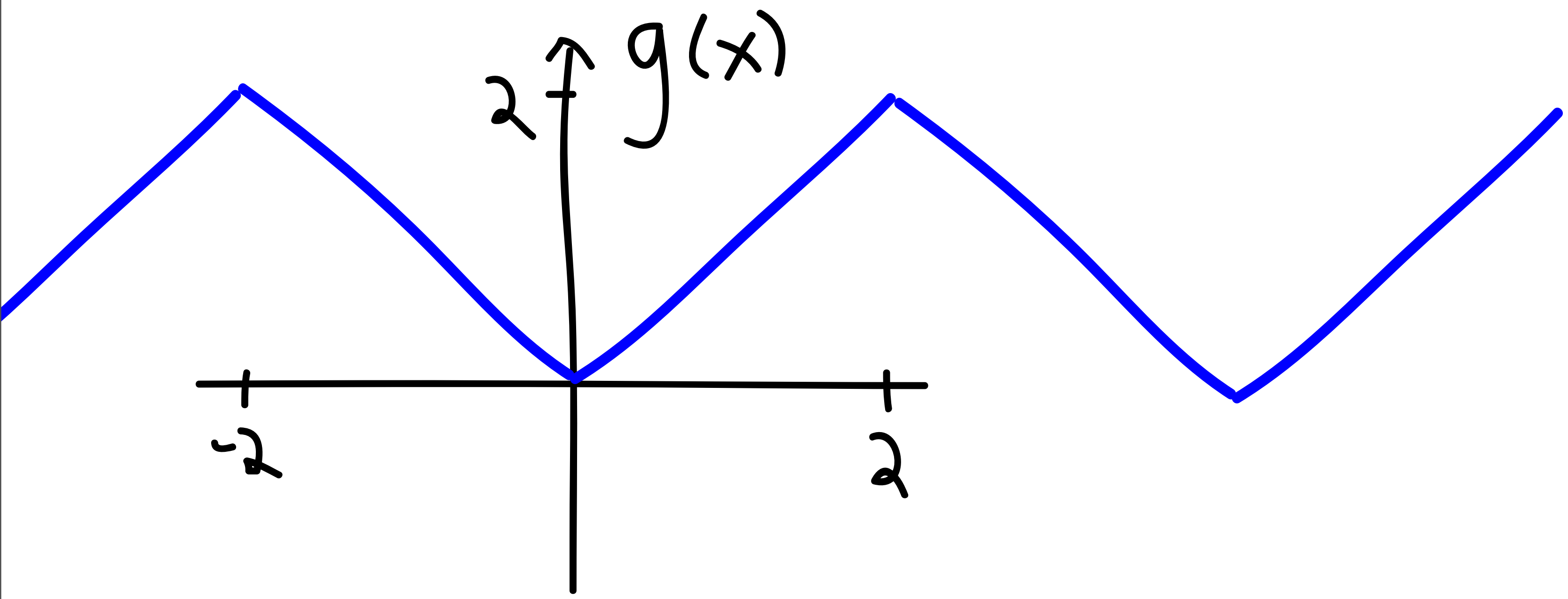

What is L? $L=2$

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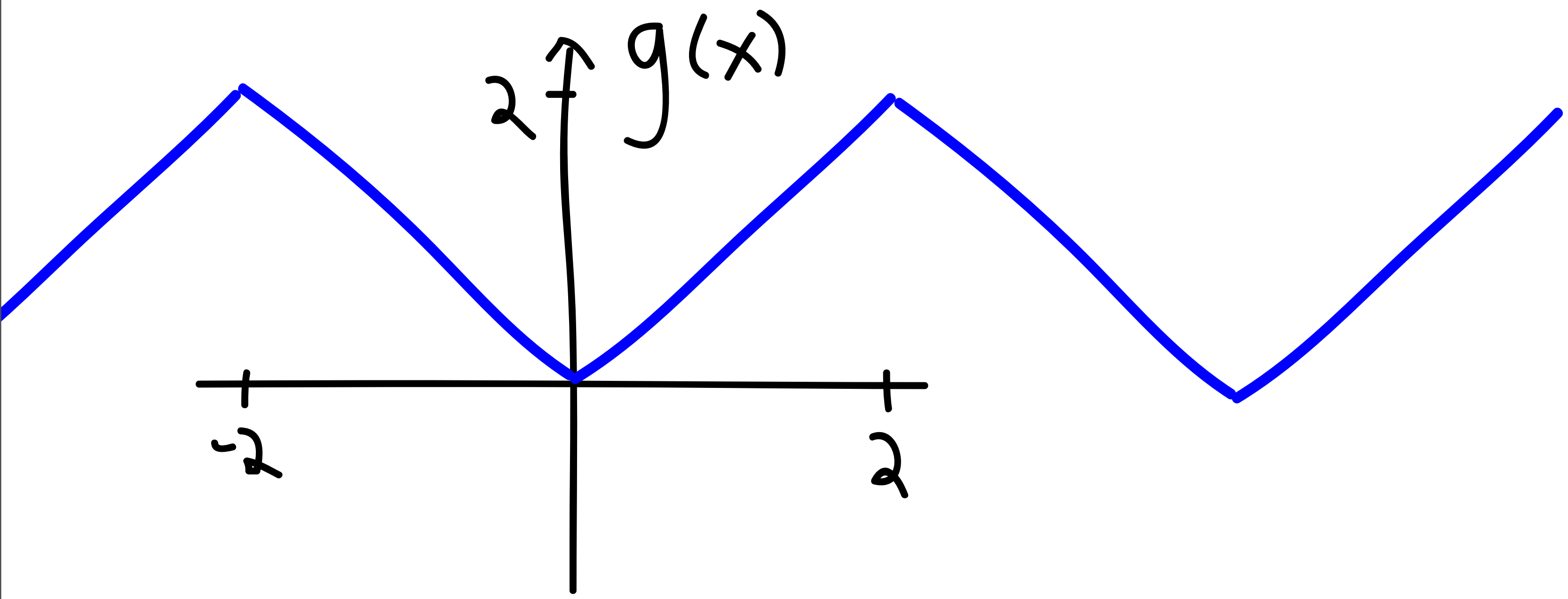

$$a_n = \begin{cases} 0 & n \text{ even} \\ -\frac{8}{n^2\pi^2} & n \text{ odd} \end{cases}$$

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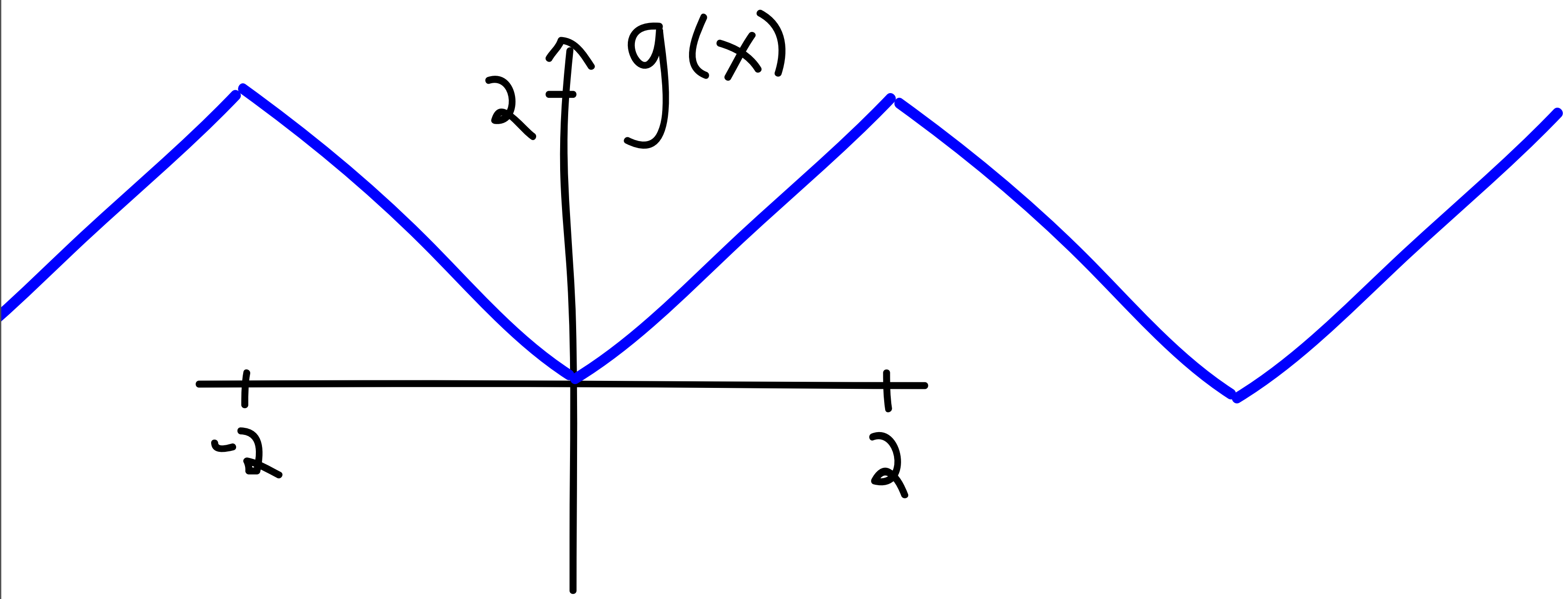

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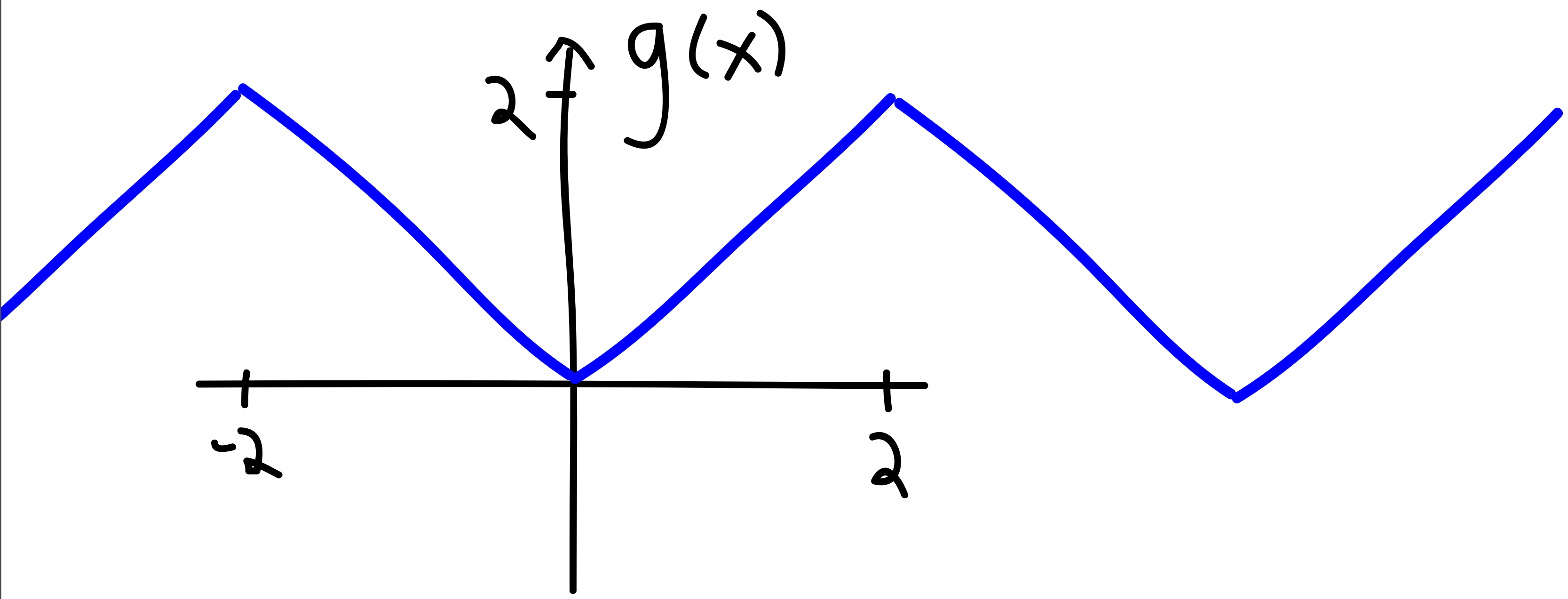

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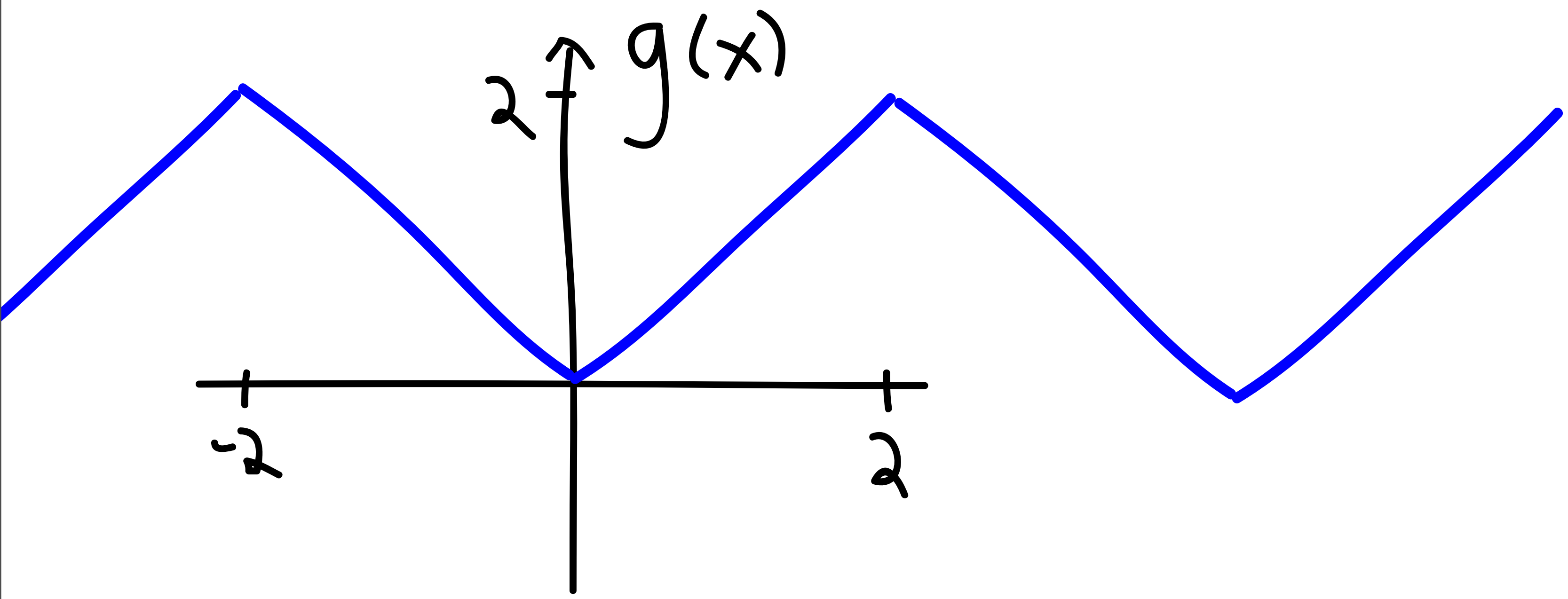

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What is L? L = 2

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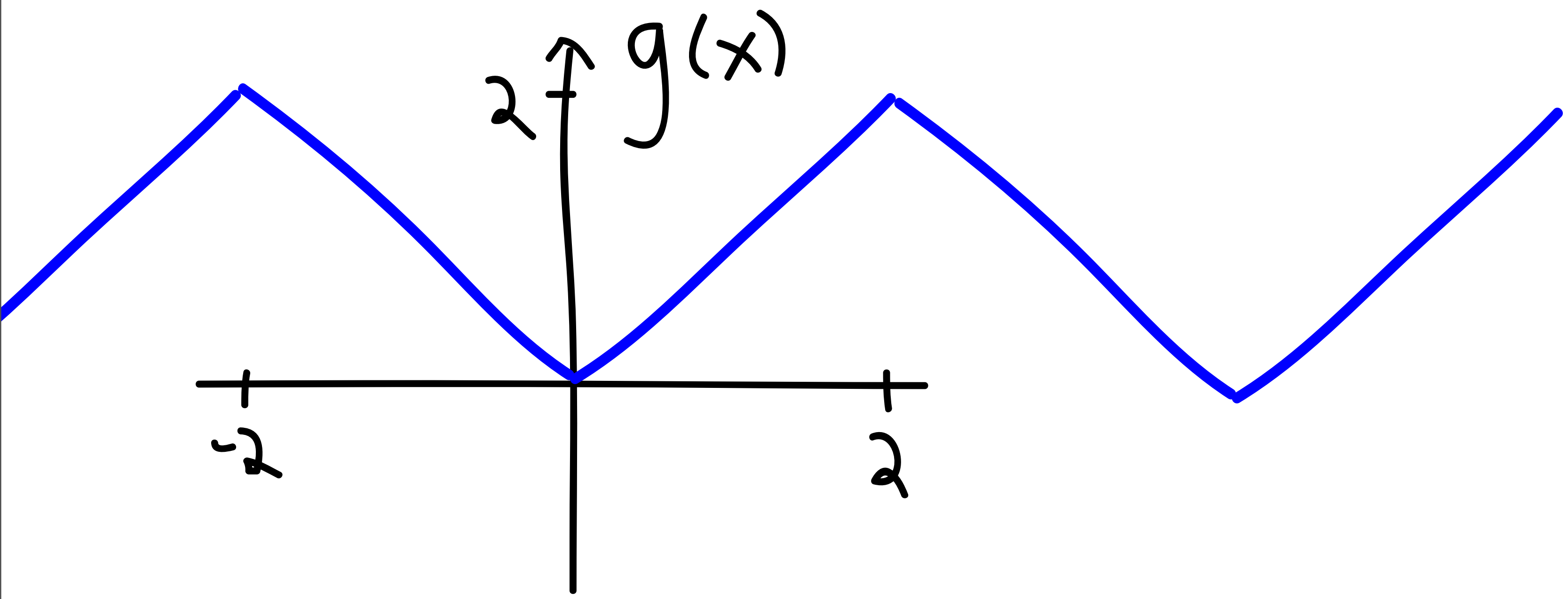
Examples - calculate the Fourier Series



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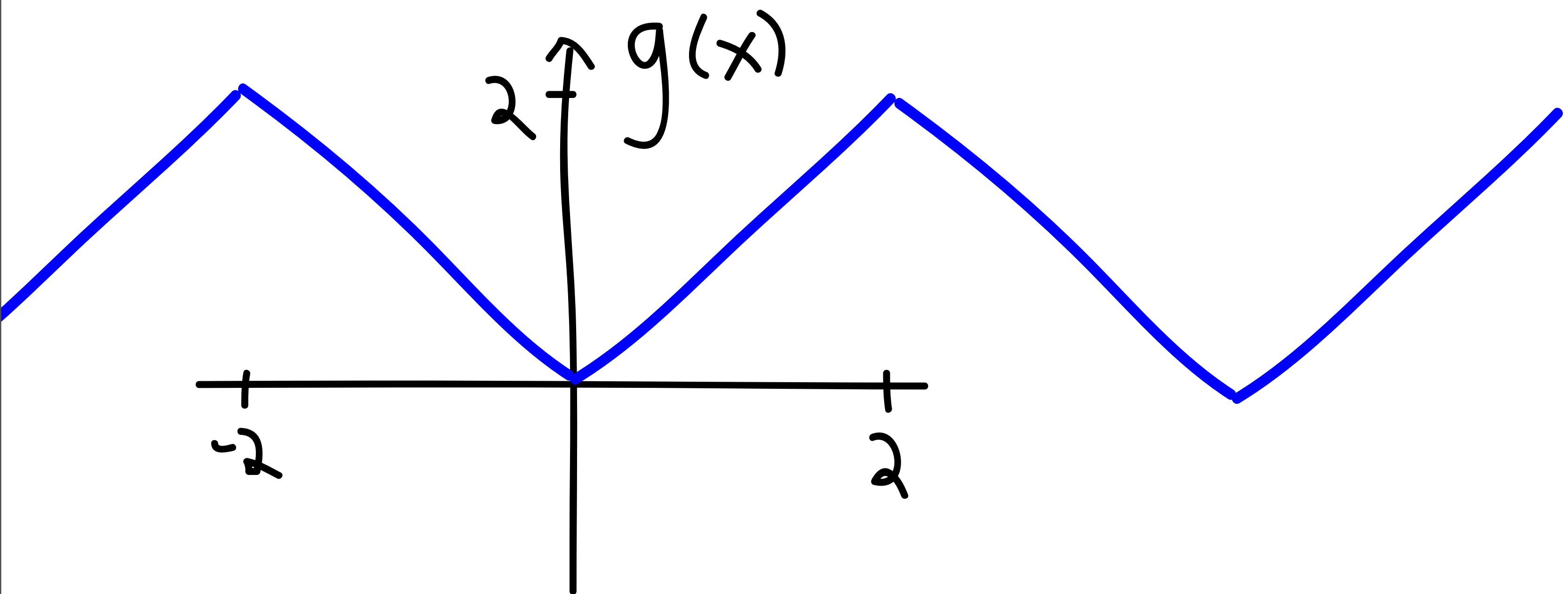
Examples - calculate the Fourier Series



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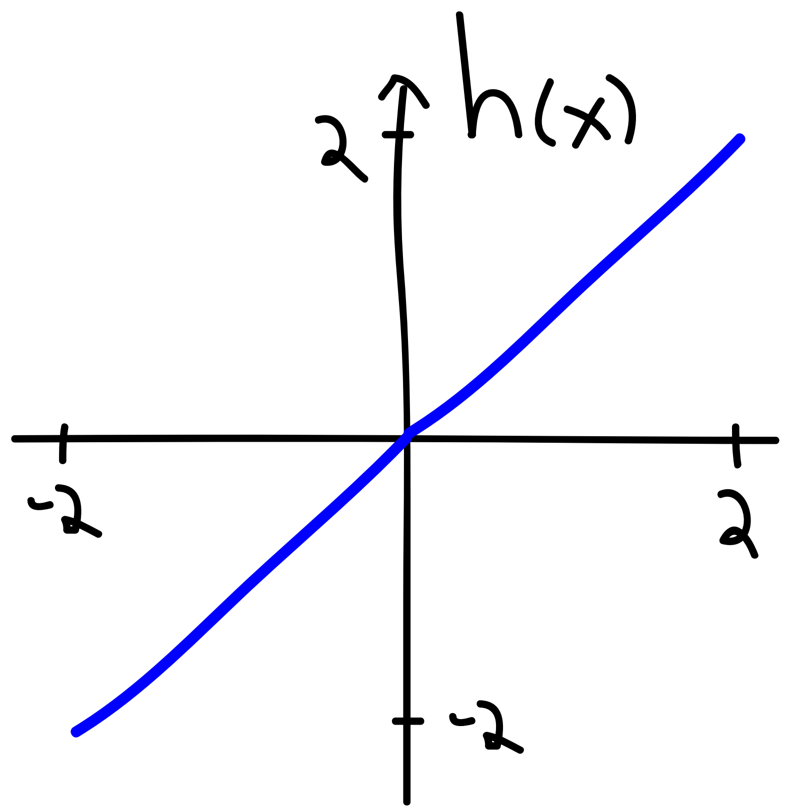
Examples - calculate the Fourier Series



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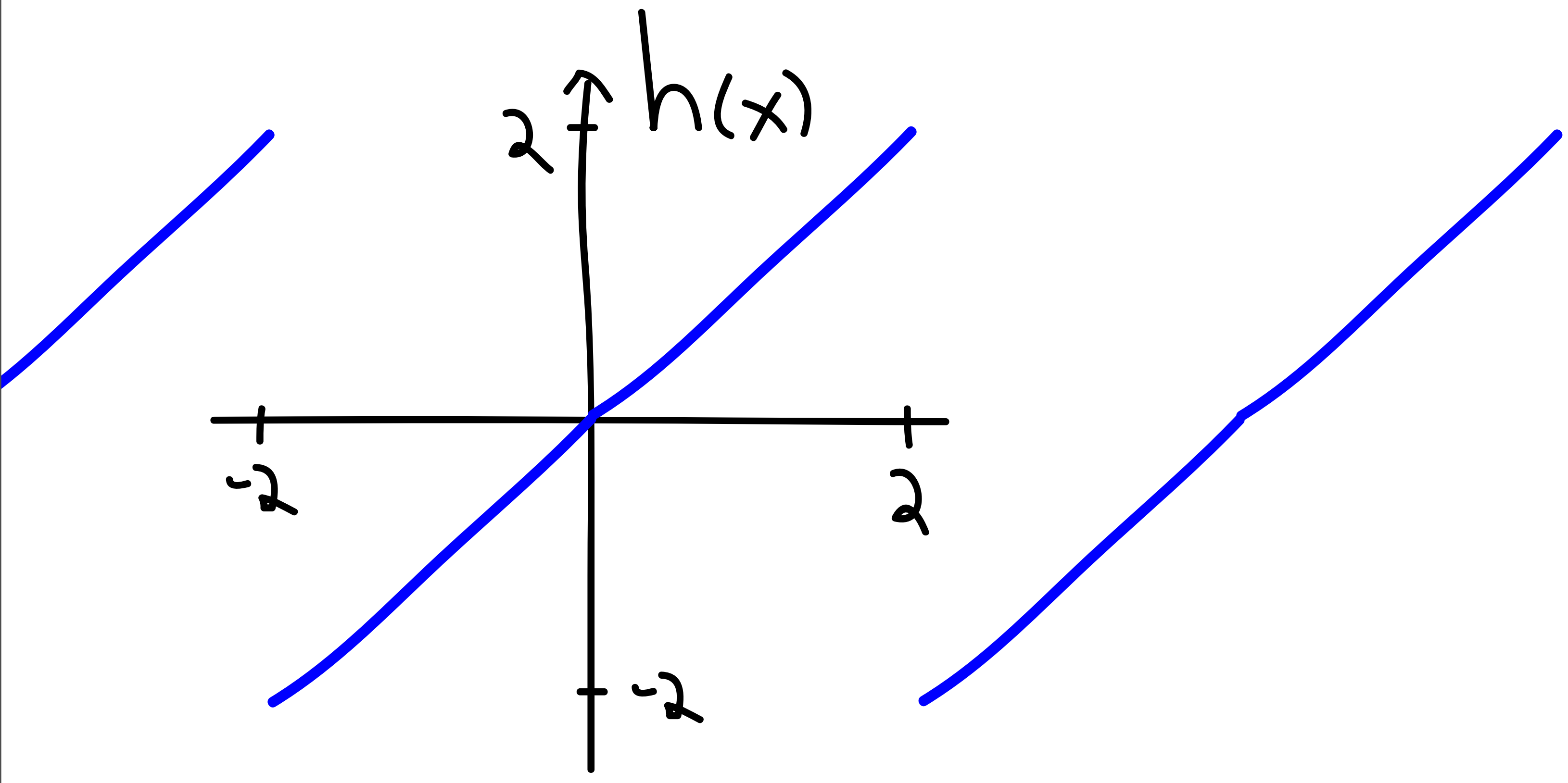

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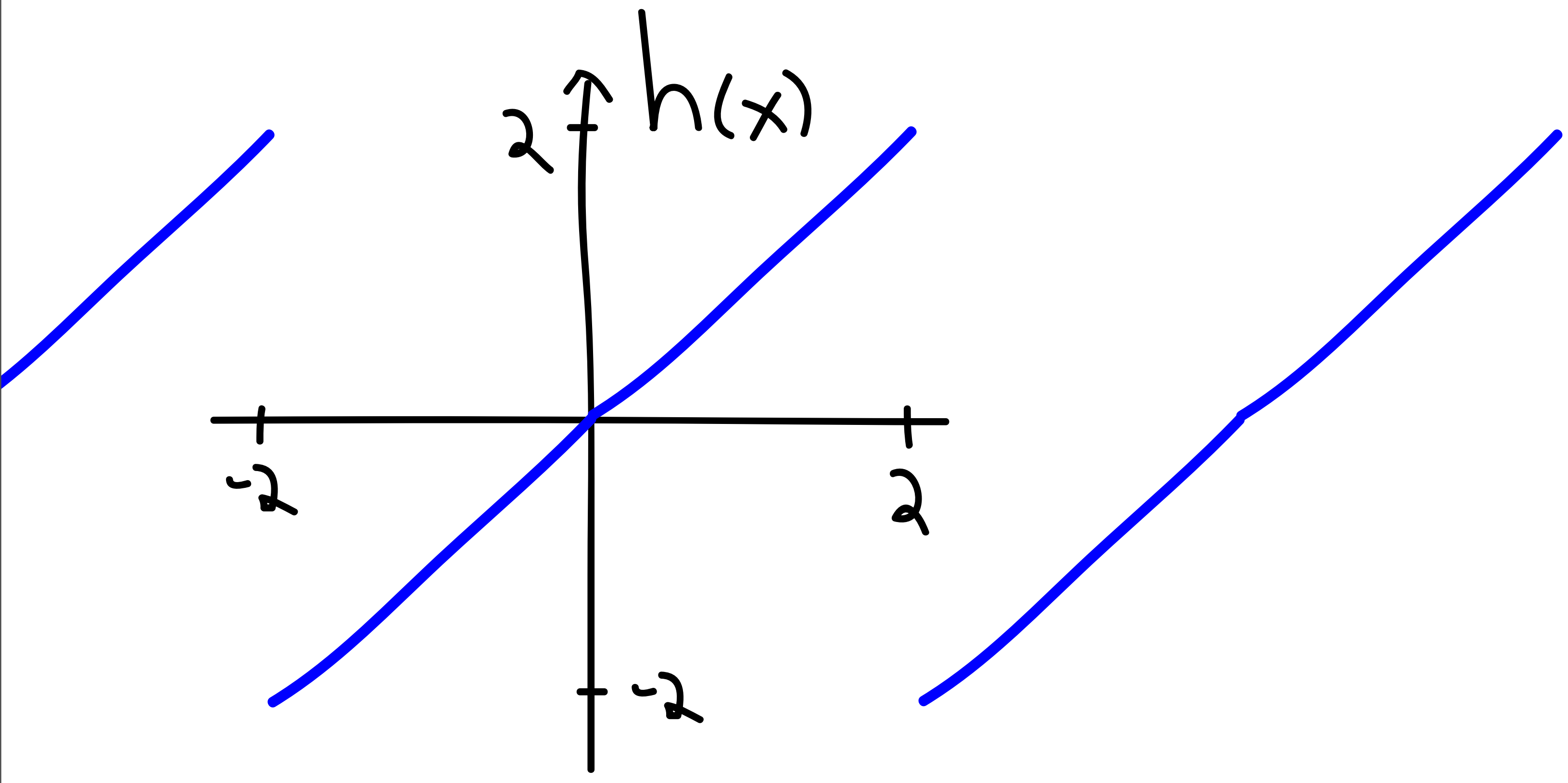

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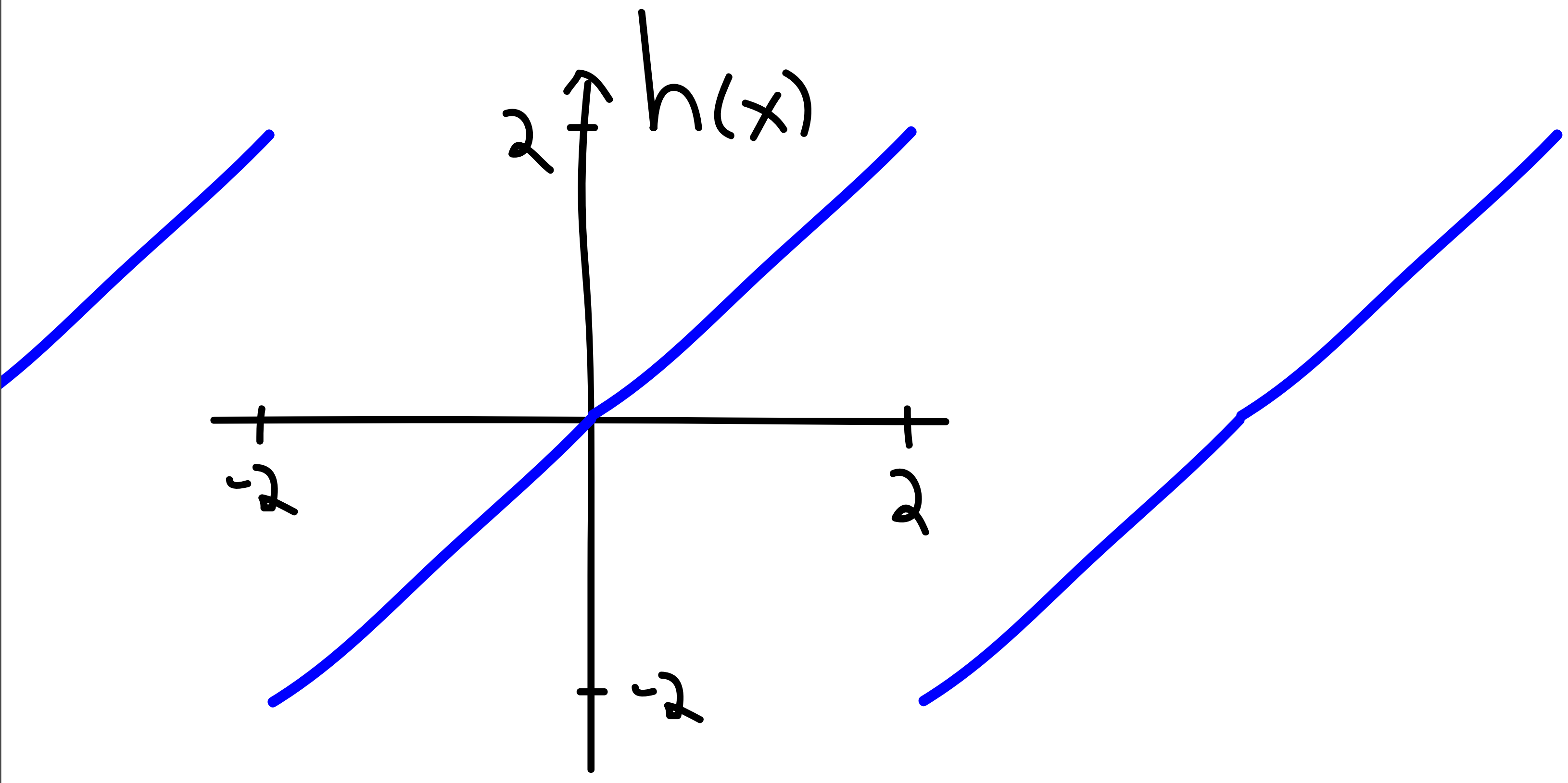
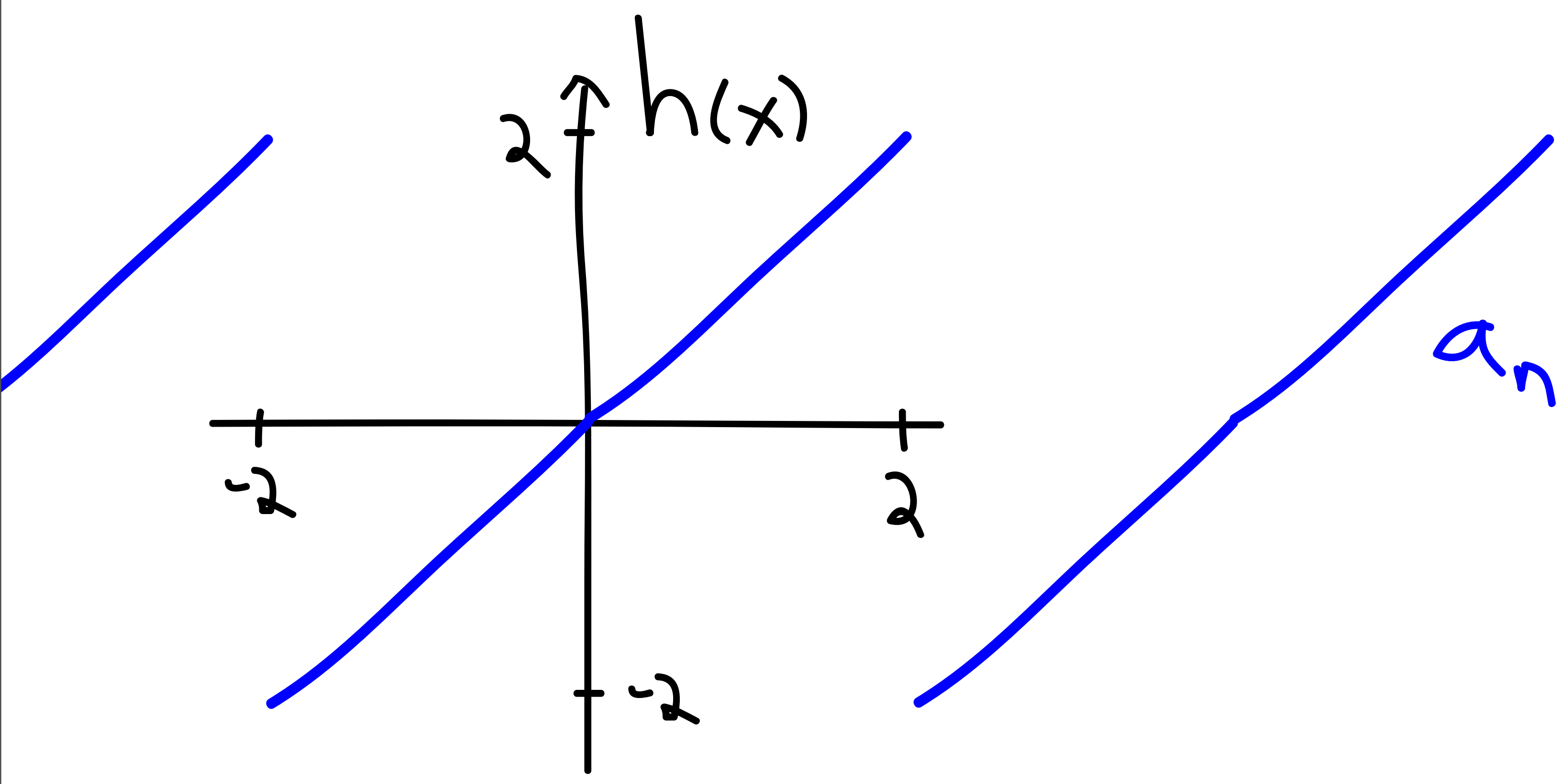
Examples - calculate the Fourier Series



What is L? L=2

Examples - calculate the Fourier Series



$a_n = 0$

What is L? L=2

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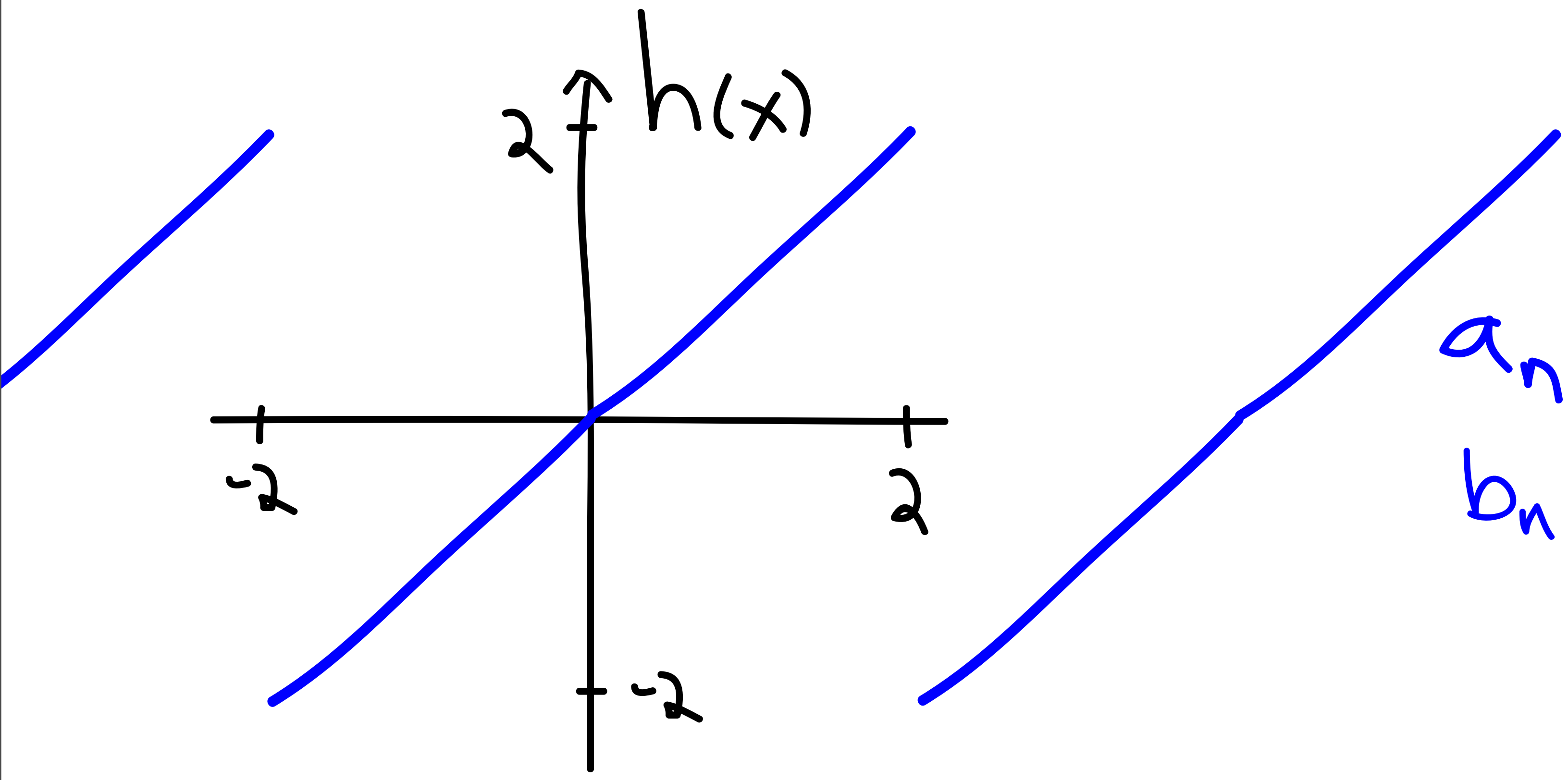
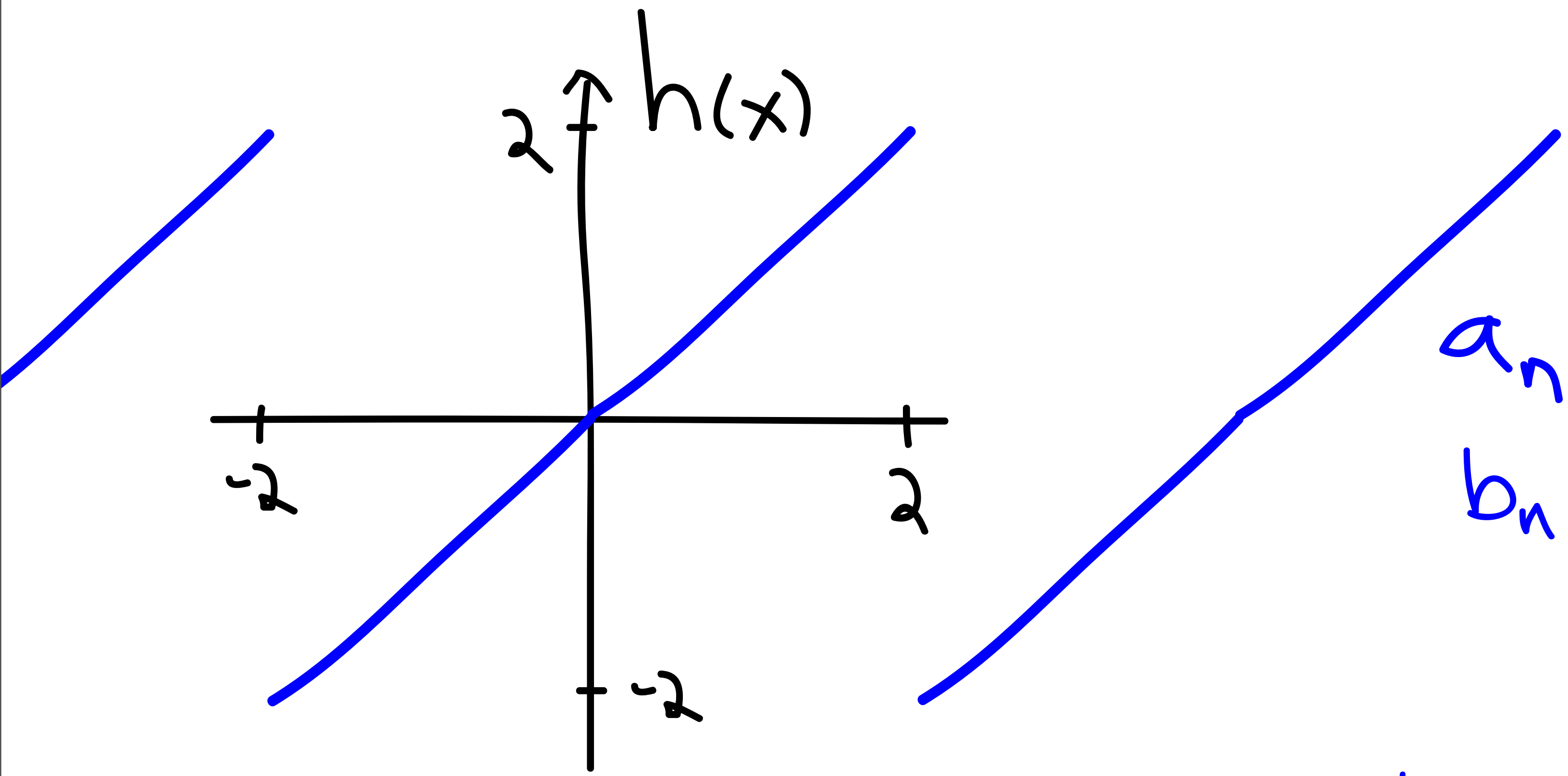


$a_n = 0$

$b_n = \frac{(-1)^{n+1} 4}{n\pi}$

What is L? L=2

Examples - calculate the Fourier Series



What is L? L=2

$$a_n = 0$$

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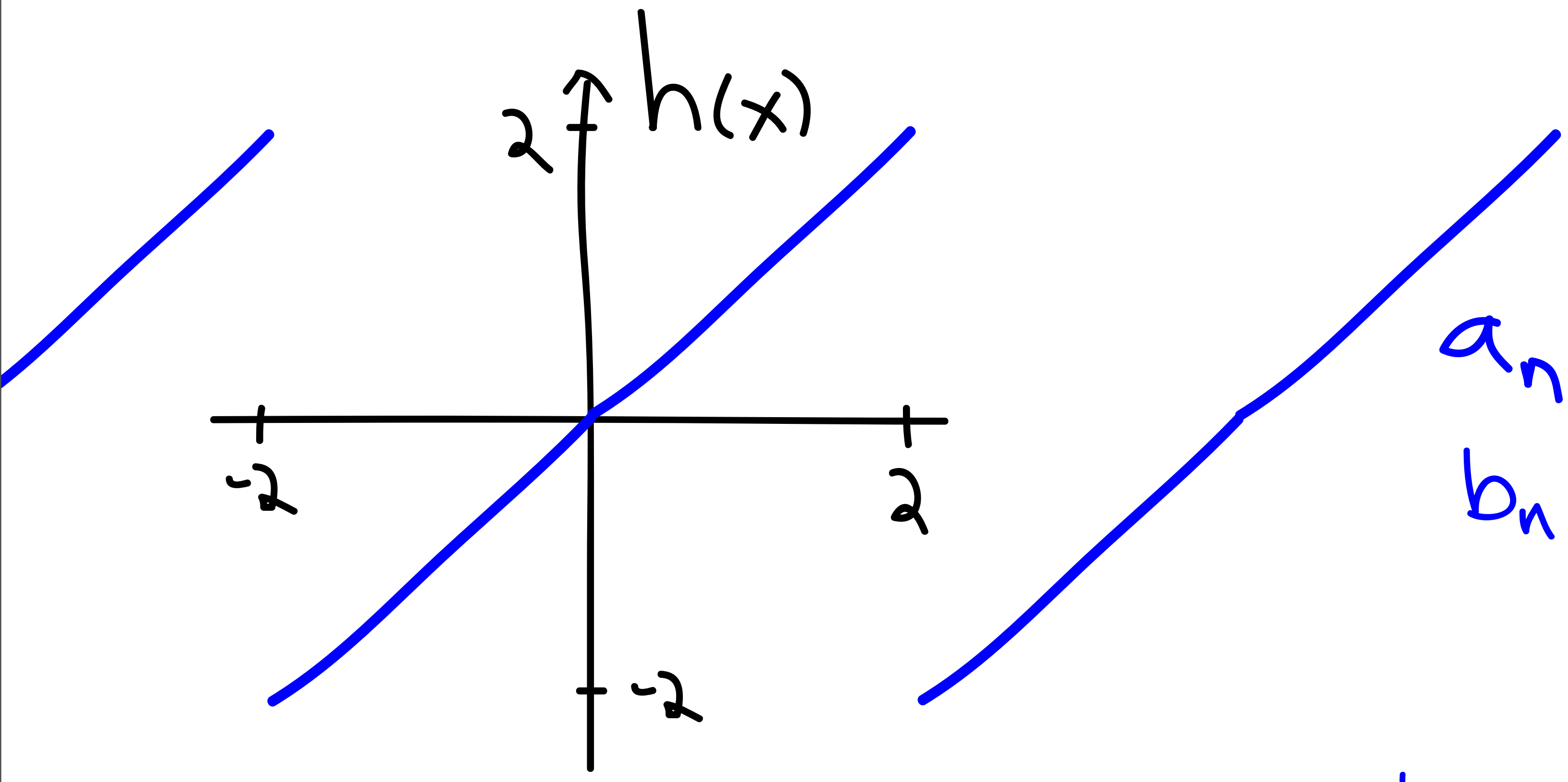

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for $x \neq -2, 2$.

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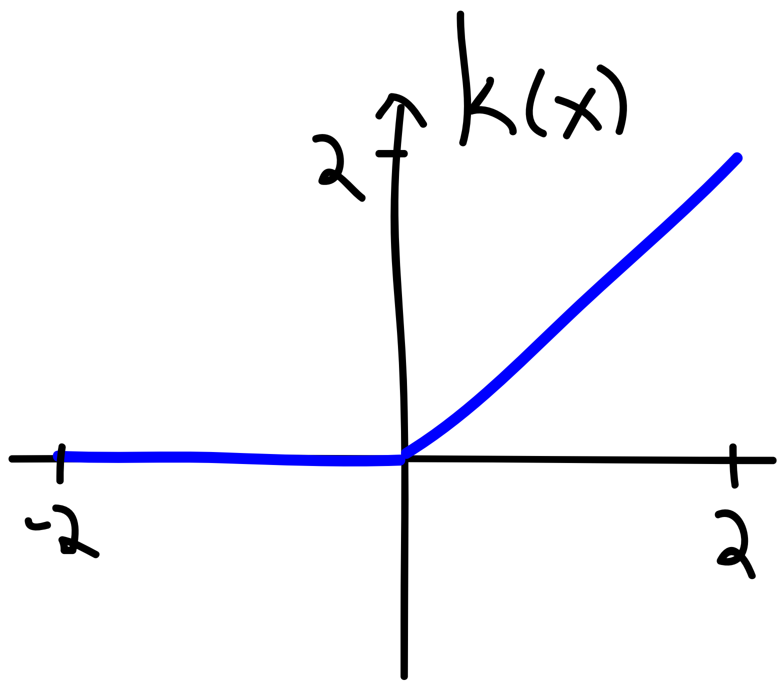

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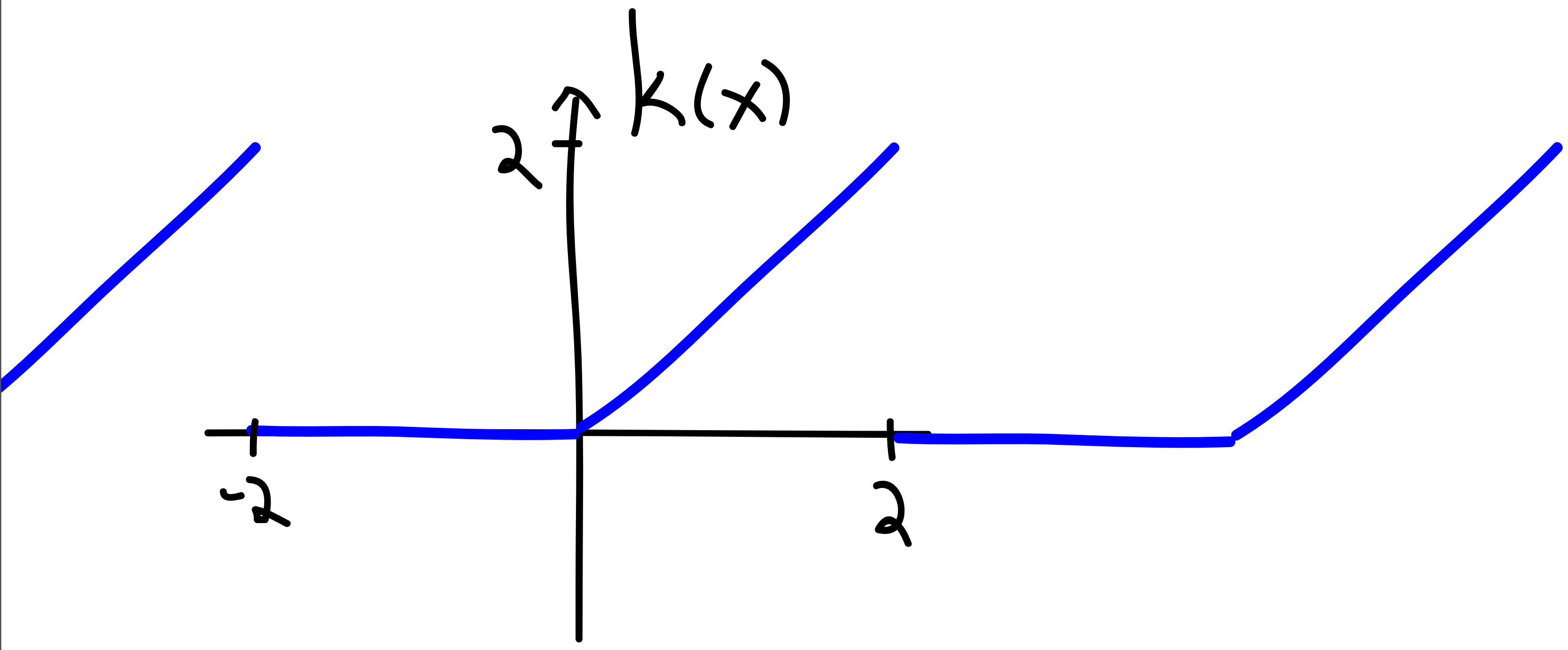

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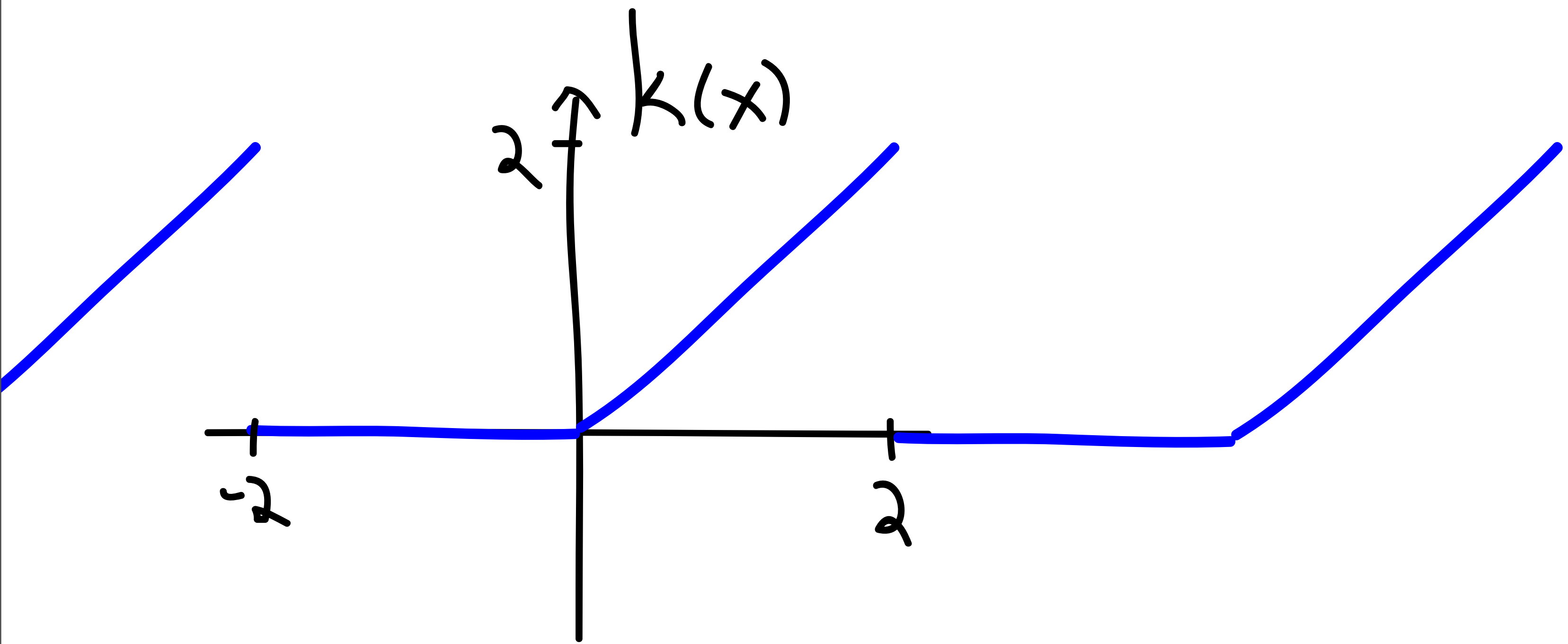

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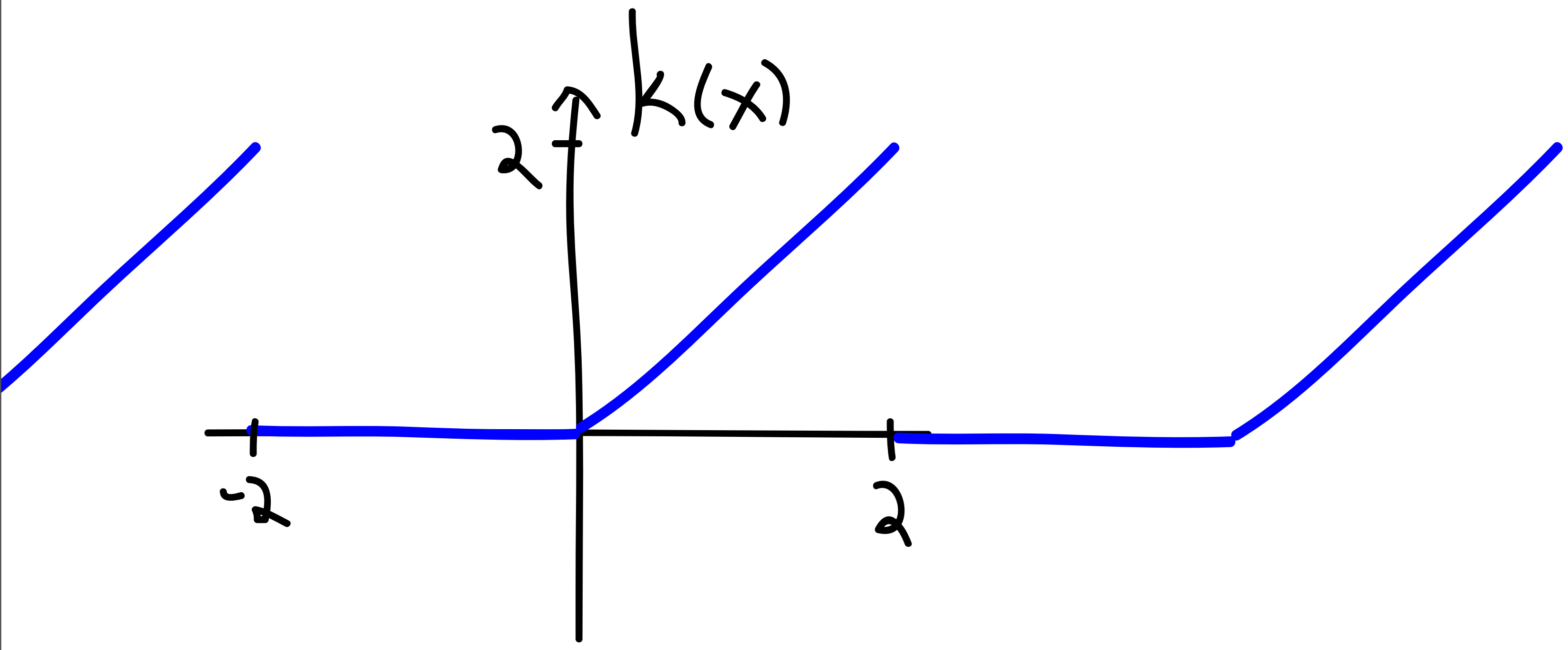

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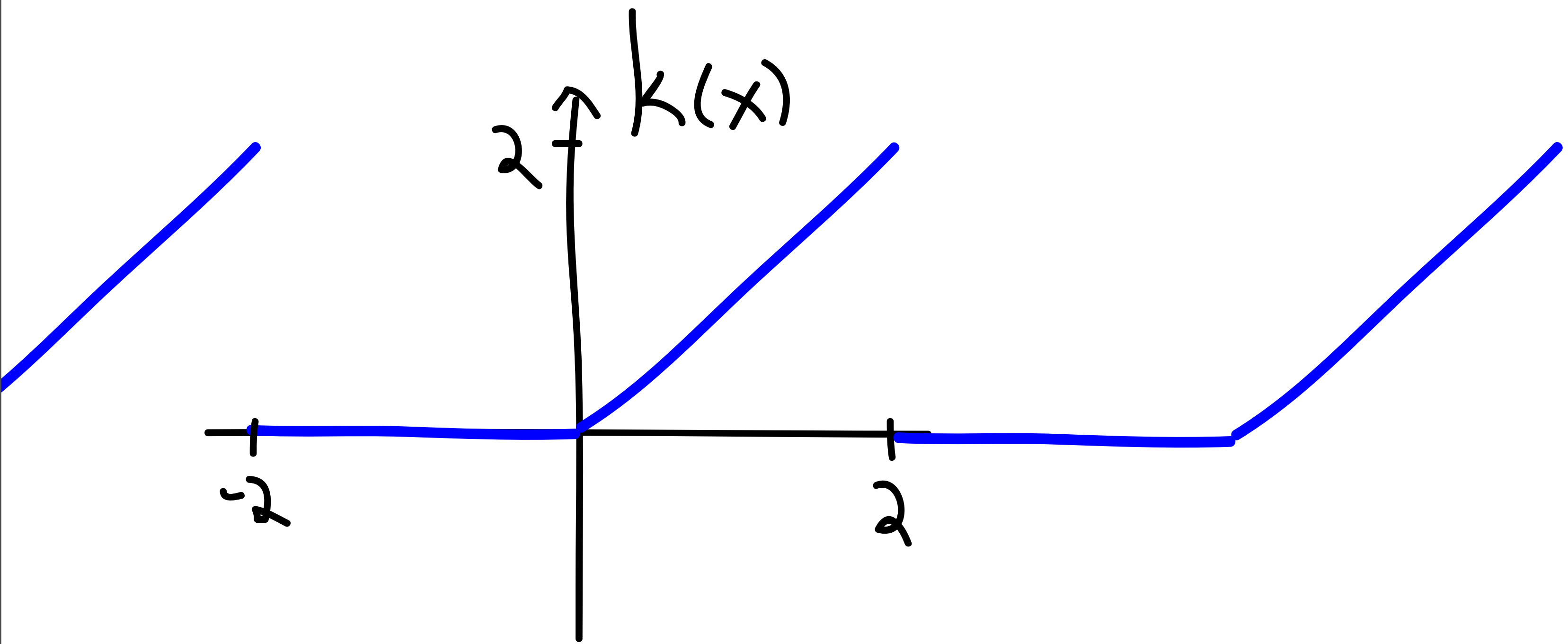

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What is L? $L=2$

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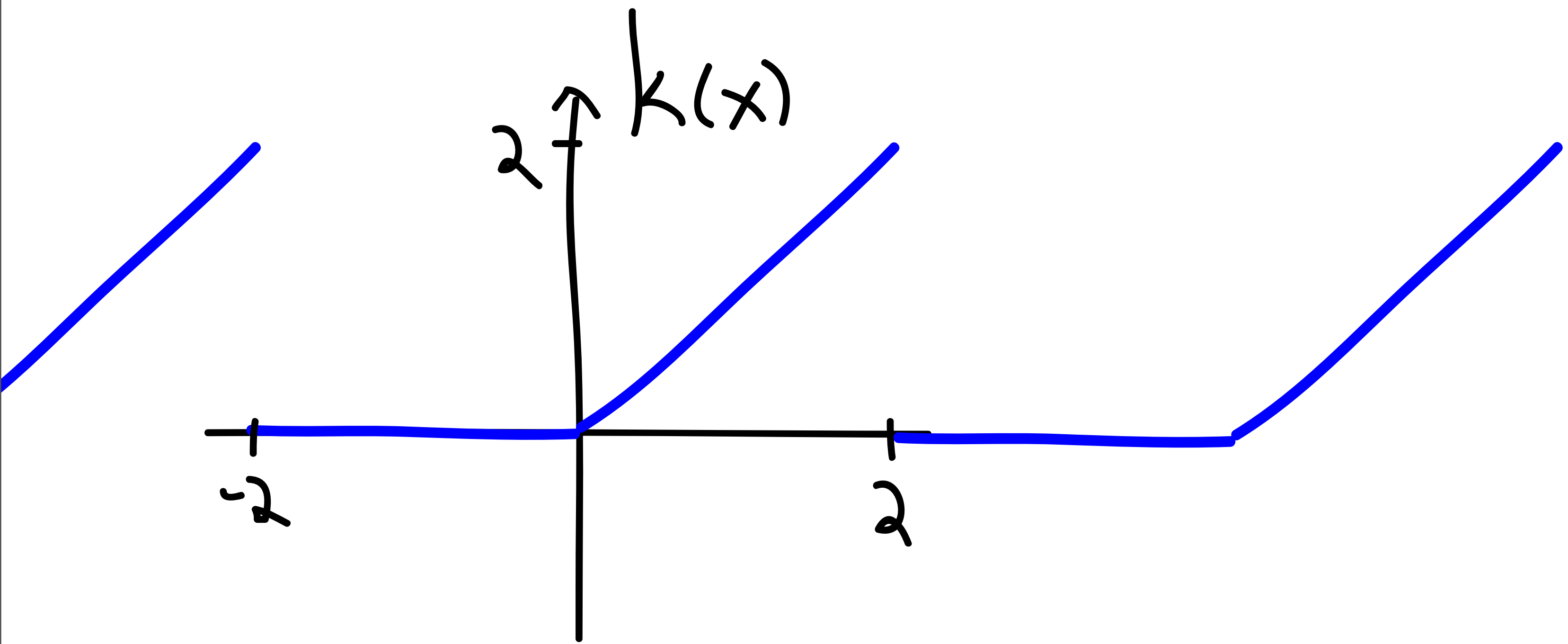

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What is L? L=2

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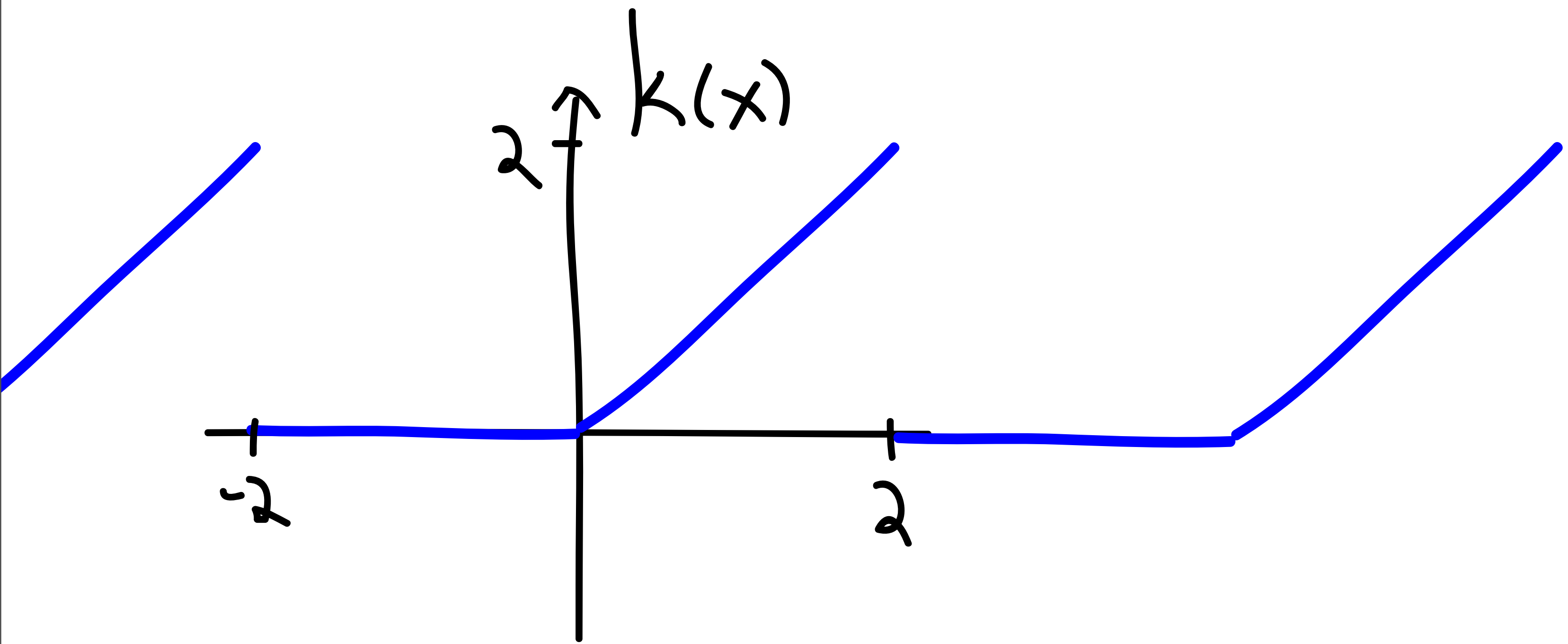

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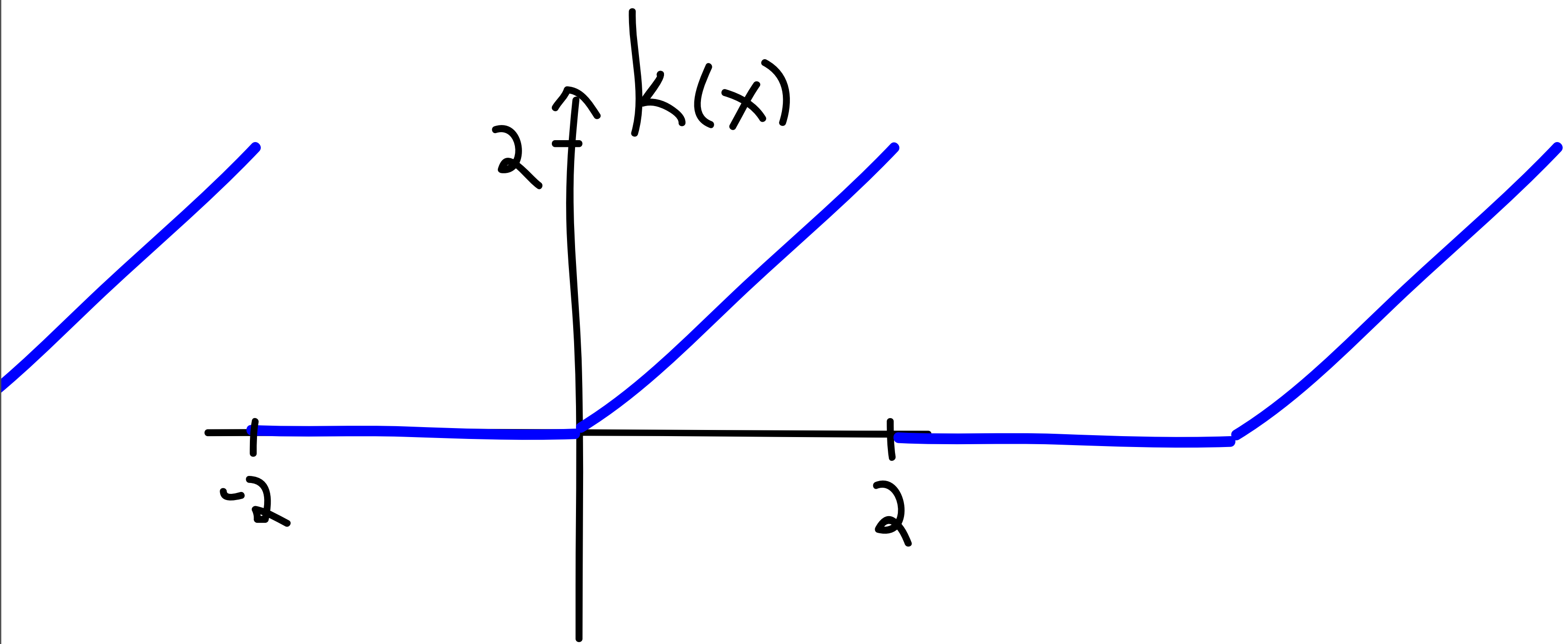

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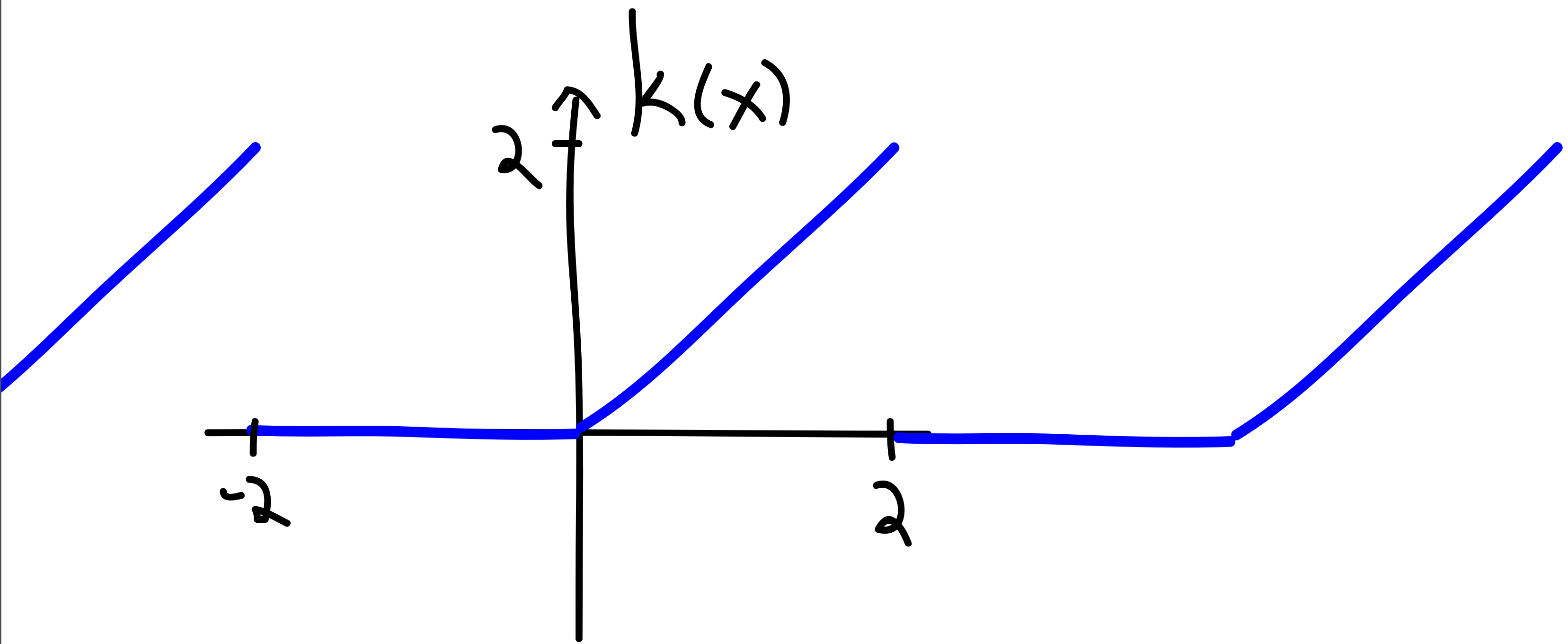

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Even and odd extensions

- For a function *f(x)* defined on [0,L], the even extension of *f(x)* is the function

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L, \\ f(-x) & \text{for } -L \le x < 0. \end{cases}$$

- For a function *f(x)* defined on [0,L], the odd extension of *f(x)* is the function

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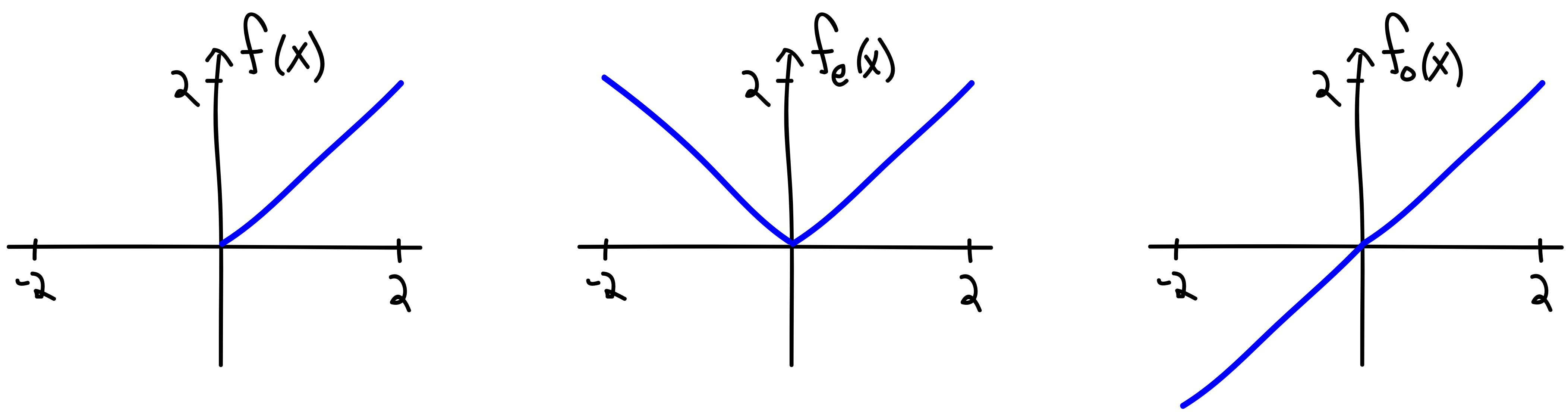

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- Because these functions are even/odd, their Fourier Series have a couple simplifying features:

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

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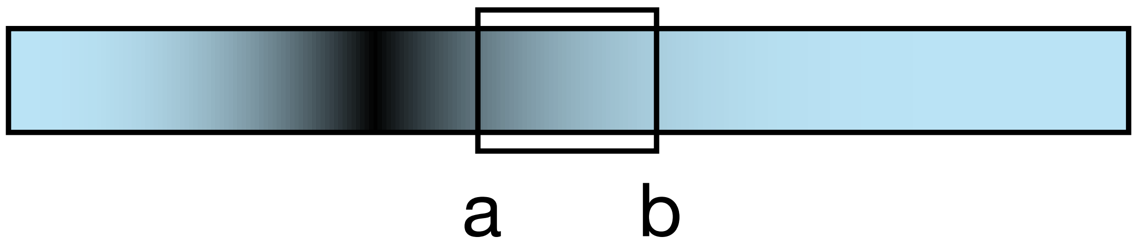
The Diffusion Equation

c(x,t) is linear mass density of ink in a long narrow tube.

$$Q_{ab}(t) = \int_a^b c(x,t)\ dx$$



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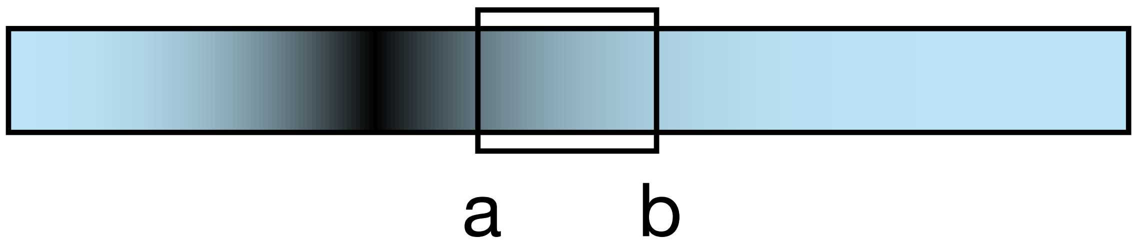

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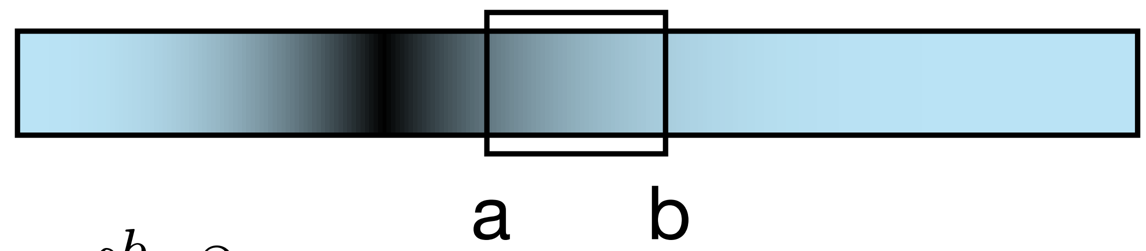

$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt}\int_a^b c(x,t)\ dx\ = \int_a^b \frac{\partial}{\partial t}c(x,t)\ dx$$

Define the flux J_a to be the amount of mass crossing the line x=a (+ -->) .

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Need a model for flux, here, chemical diffusion: $J_a = -D\ \left.\frac{\partial c}{\partial x}\right|_{x=a}$

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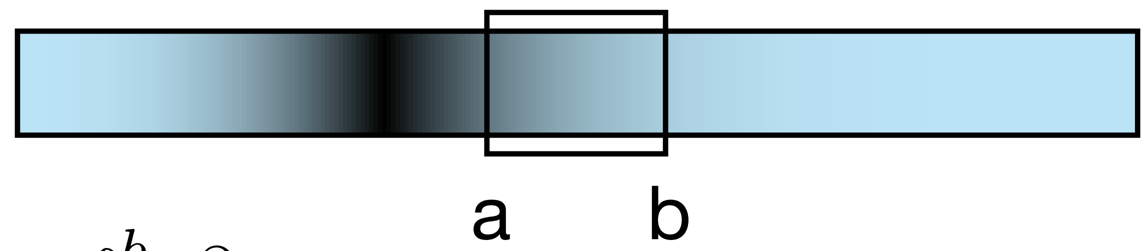

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- Two space derivatives require two boundary conditions c(0,t) and c(L,t).

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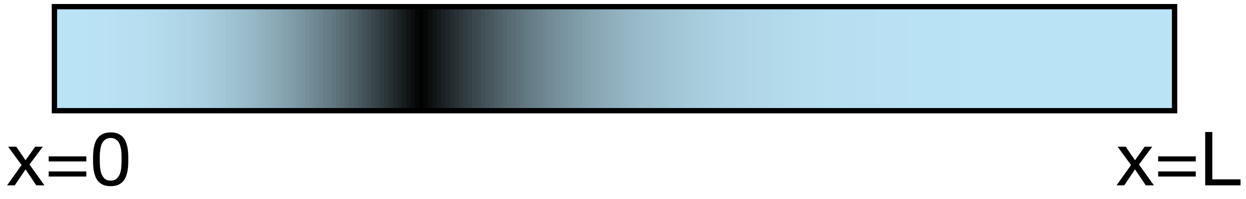

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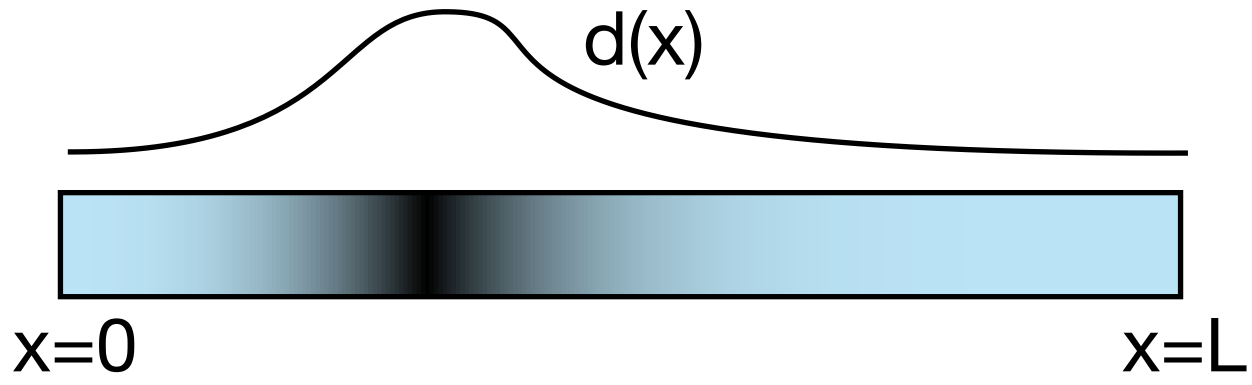

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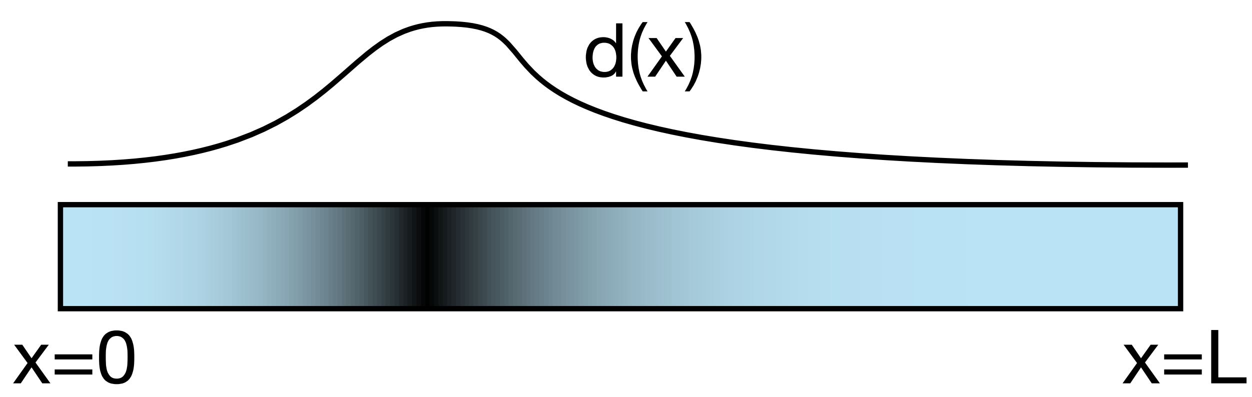

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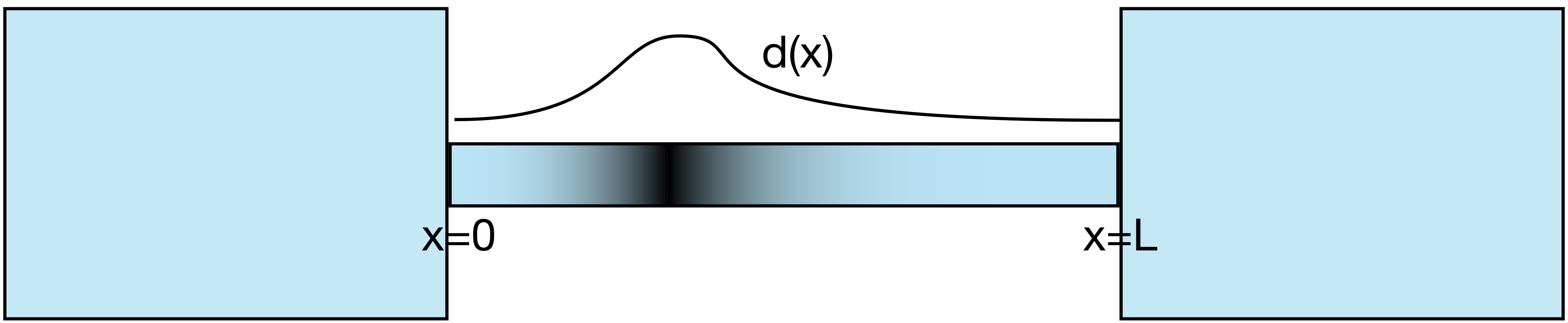

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