Today

- Comment on pre-lecture problems
- Finish up with integrating factors
- The structure of solutions
- Separable equations

Pre-lecture assignment comments

Consider the initial value problem

$$\frac{dy}{dt} - 5y = 4e^{2t}, \quad y(0) = A.$$

Solve the differential equation $\frac{dy}{dx} = \frac{x}{9y}$.

a. Find the equation of the solution through the point $(x, y) = (-3, 1)$.

help (equations)

b. Find the equation of the solution through the point $(x, y) = (0, -3)$.

help (equations)

$y \to 0$ if $A \in$

$y \to -\infty$ if $A \in$

Method of integrating factors (Section 2.1)

$$\frac{d}{dt}\left(t^2 y(t)\right) =$$

(A) $2t\dfrac{dy}{dt}$

(B) $t^2\dfrac{dy}{dt}$

(C) $2ty$

(D) $t^2\dfrac{dy}{dt} + 2ty$

(E) Don’t know.

Method of integrating factors (Section 2.1)

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(E) Don’t know.

Method of integrating factors (Section 2.1)

- Given that $$\frac{d}{dt}\left(t^2 y(t)\right) = t^2 \frac{dy}{dt} + 2ty$$
- if you're given the equation $$t^2 \frac{dy}{dt} + 2ty = 0$$
- you can rewrite is as $$\frac{d}{dt}\left(t^2 y(t)\right) = 0$$
- so the solution is $t^2 y(t) = C$ or equivalently $y(t) = \dfrac{C}{t^2}$.

arbitrary constant that appeared at an integration step

Method of integrating factors (Section 2.1)

- Solve the equation $t^2 \frac{dy}{dt} + 2ty(t) = \sin(t)$ (not brute force checking).

(A) $y(t) = -\cos(t) + C$

(B) $y(t) = \frac{C - \cos(t)}{t^2}$

(C) $y(t) = \sin(t) + C$

(D) $y(t) = -\frac{1}{t^2} \cos(t)$

(E) Don’t know.

Method of integrating factors (Section 2.1)

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(A) $y(t) = -\cos(t) + C$

(B) $y(t) = \frac{C - \cos(t)}{t^2}$ ← general solution (although that's not obvious)

(C) $y(t) = \sin(t) + C$

(D) $y(t) = -\frac{1}{t^2}\cos(t)$ ← a particular solution

(E) Don't know.

Initial conditions (IC) and initial value problems (IVP)

- An initial condition is an added constraint on a solution.
- e.g. Solve $t^2\frac{dy}{dt} + 2ty(t) = \sin(t)$ subject to the IC $y(\pi) = 0$.

(A) $y(t) = -\frac{C + \cos(\pi)}{\pi^2}$

(B) $y(t) = -\frac{1 - \cos(t)}{t^2}$

(C) $y(t) = \frac{1 + \cos(t)}{t^2}$

(D) $y(t) = -\frac{1 + \cos(t)}{t^2}$

(E) Don't know.

- An Initial Value Problem (IVP) is a ODE together with an IC.

Method of integrating factors (Section 2.1)

- What's the integrating factor?

$$t\frac{dy}{dt} + 2y(t) = 1 \qquad \rightarrow f(t) = t$$

$$t^2\frac{dy}{dt} + 4ty(t) = \frac{1}{\ldots} \qquad \rightarrow f(t) = t^2$$

In class, I referred to these f(t) functions as the integrating factors. This is incorrect. They are functions that you can multiply the ORIGINAL equation by to get a perfect "product rule" form. But an integrating factor is defined as the function that you would multiply the "normalized" equation by to get the "product rule" form. By "normalized", I mean after dividing through by the coefficient on dy/dt. So for the first example above, the integrating factor would be $I(t)=t^2$ and the equation you multiply it through is $dy/dt + 2/t\, y = 1/t$.

$$\frac{dy}{dt} + g'(t)y(t) = 0 \qquad \rightarrow f(t) = e^{g(t)}$$

Method of integrating factors (Section 2.1)

- General case - all first order linear ODEs can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

- The appropriate integrating factor is $e^{\int p(t)dt}$.

- The equation can be rewritten $\frac{d}{dt}\left(e^{\int p(t)dt}y\right) = e^{\int p(t)dt}q(t)$ which is solvable provided you can find the antiderivative of the right hand side.

$$e^{\int p(t)dt}y(t) = \int e^{\int p(t)dt}q(t)dt + C$$

$$y(t) = e^{-\int p(t)dt}\int e^{\int p(t)dt}q(t)dt + Ce^{-\int p(t)dt}$$

The structure of solutions

- When the equation is of the form (called homogeneous)

$$\frac{dy}{dt} + p(t)y = 0$$

- the solution is

$$y(t) = C\mu(t)^{-1}$$

- where

$$\mu(t) = \exp\left(\int p(t)dt\right)$$

- is the integrating factor.

The structure of solutions

- When the equation is of the form (called nonhomogeneous)

$$\frac{dy}{dt} + p(t)y = q(t)$$

- the solution is $y(t) = k(t) + C\mu(t)^{-1}$
- where k(t) involves no arbitrary constants.
- Think about this expression as $y(t) = y_p(t) + y_h(t)$
- Directly analogous to solving the vector equations $A\overline{x} = 0$ and $A\overline{x} = \overline{b}$.

Examples

- Find the general solution to

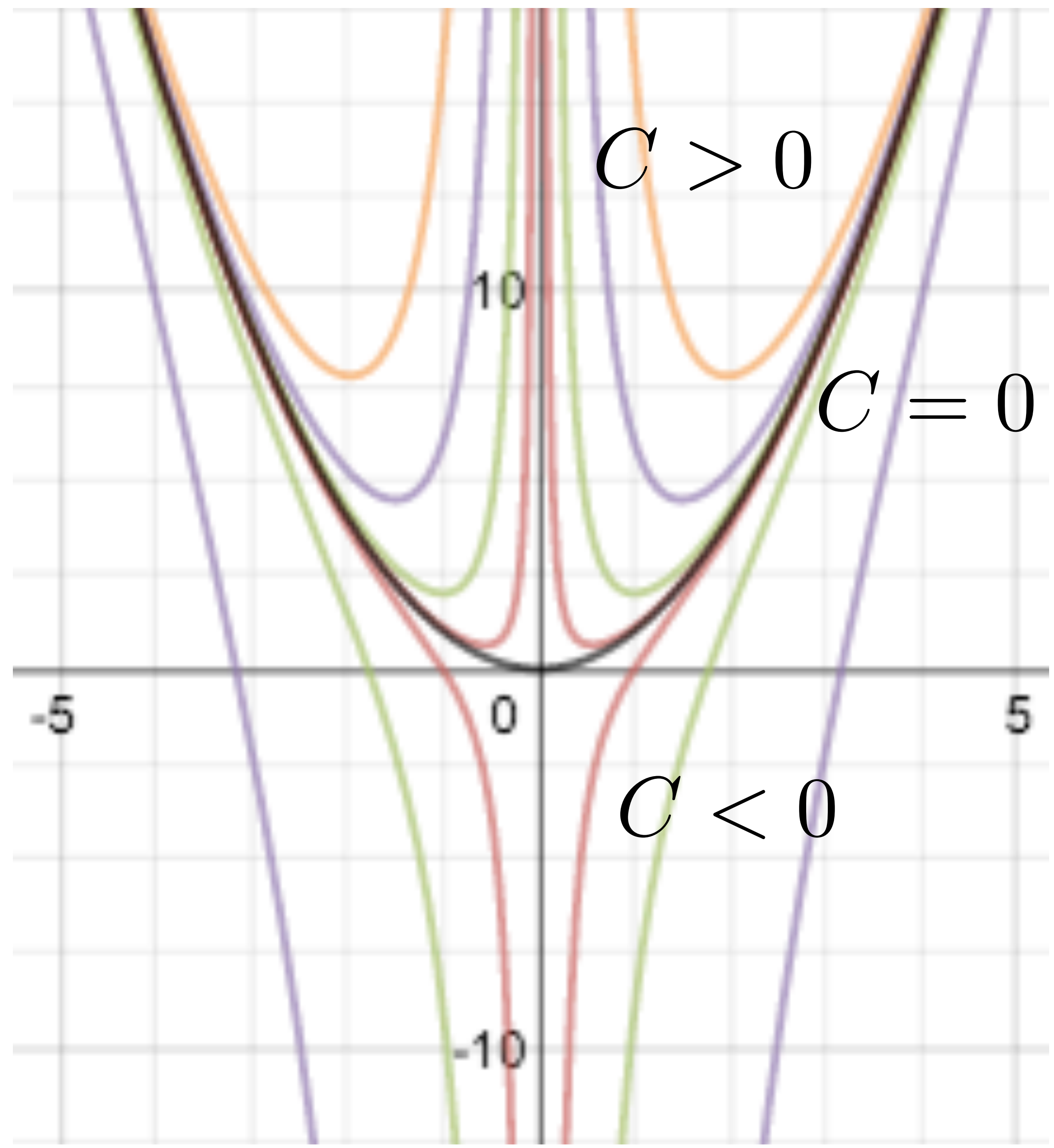
$$t\frac{dy}{dt} + 2y = 4t^2$$

- and plot a few of the integral curves.

- Integral curve - the graph of a solution to an ODE.

$$y(t) = t^2 + C\frac{1}{t^2}$$

- Steps: divide through by t, calculate I(t), take antiderivatives, solve for y. Or shortcut.



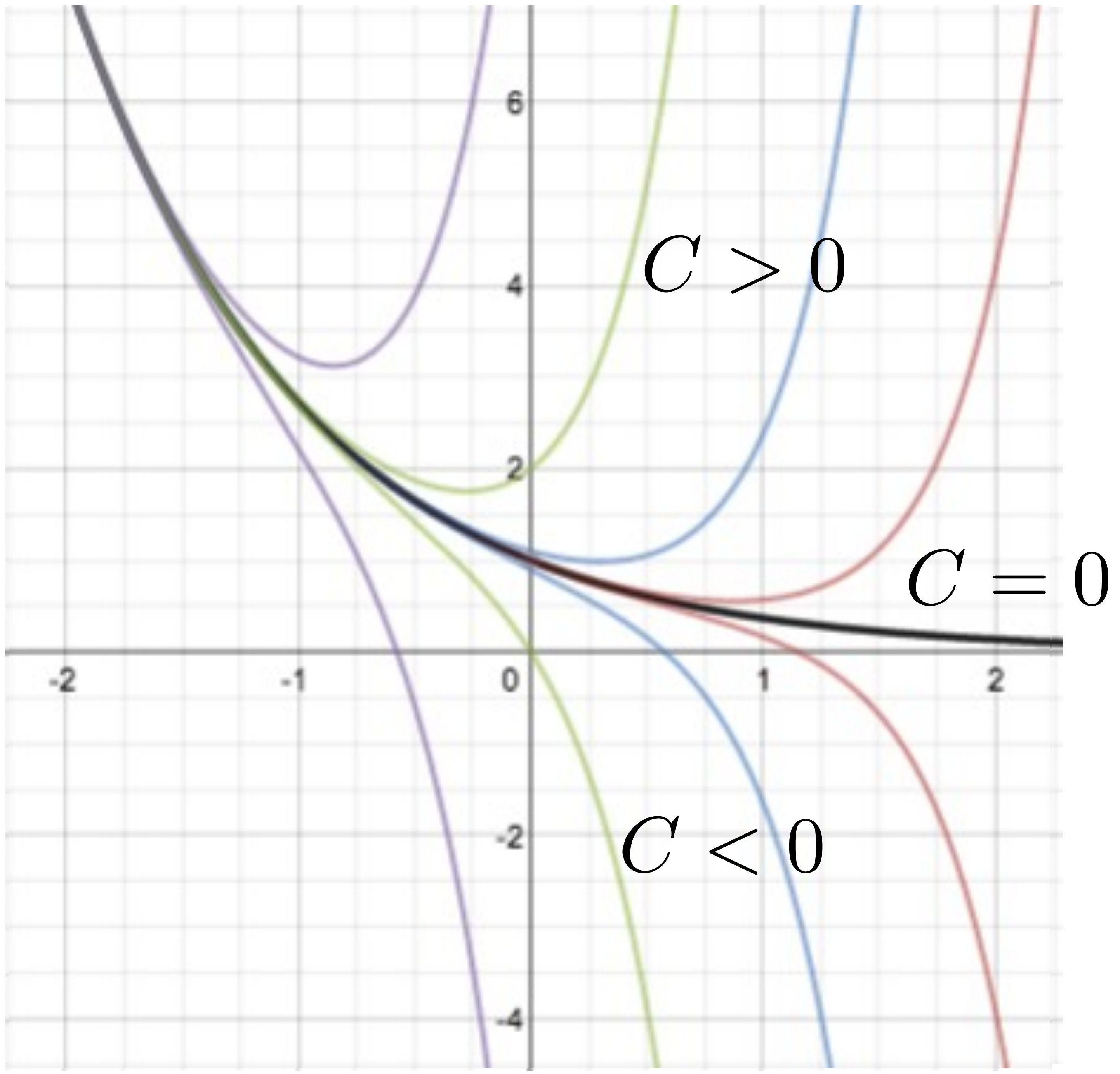
Examples

- Find the general solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- and plot a few of the integral curves.

$$y(t) = e^{-t} + Ce^{3t}$$



Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- depending on C, how many different results are possible for

$$\lim_{t\to\infty} y(t) \text{ ?}$$

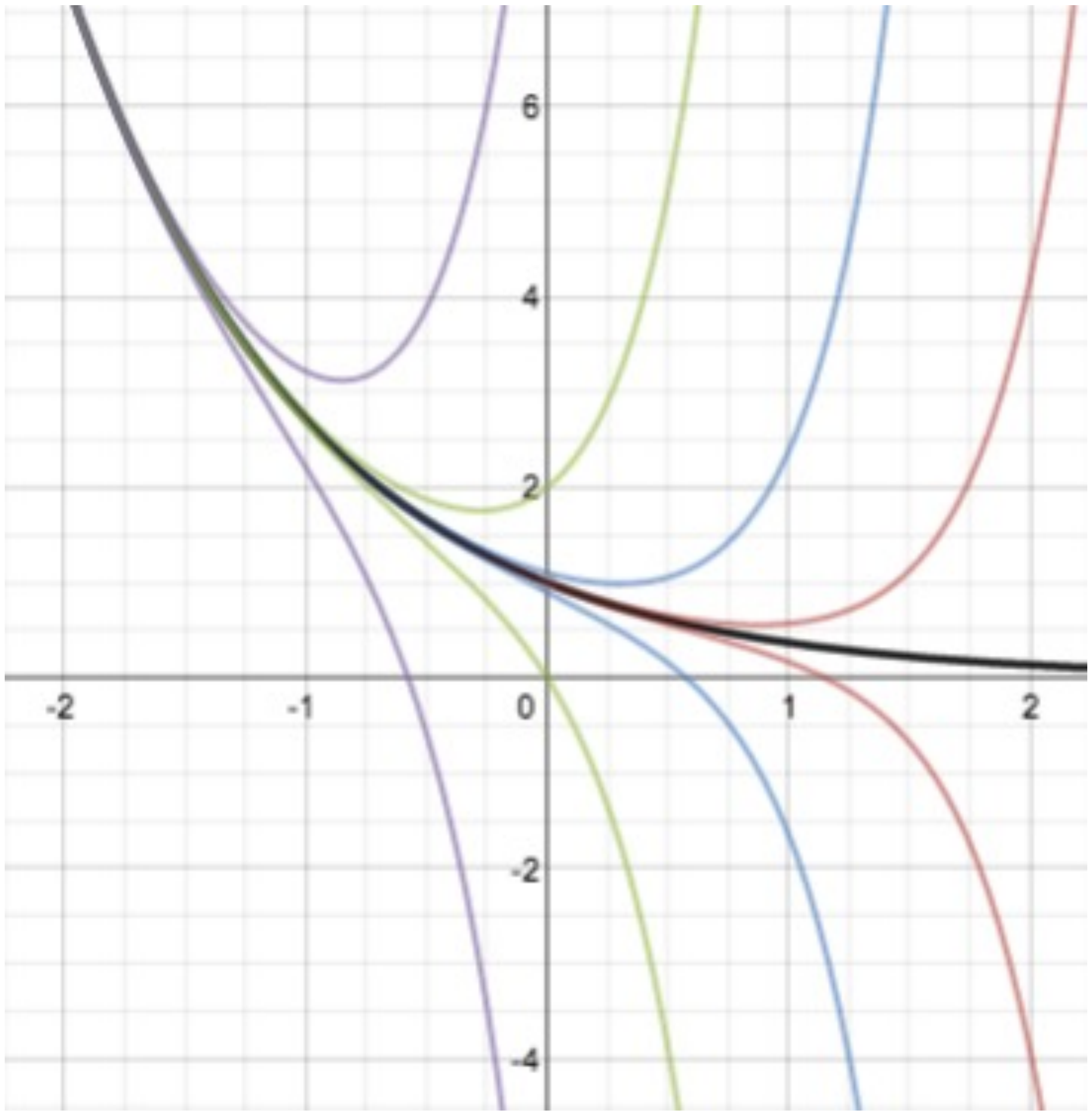
(A) 0

(B) 1

(C) 2

(D) 3

(E) Don't know.



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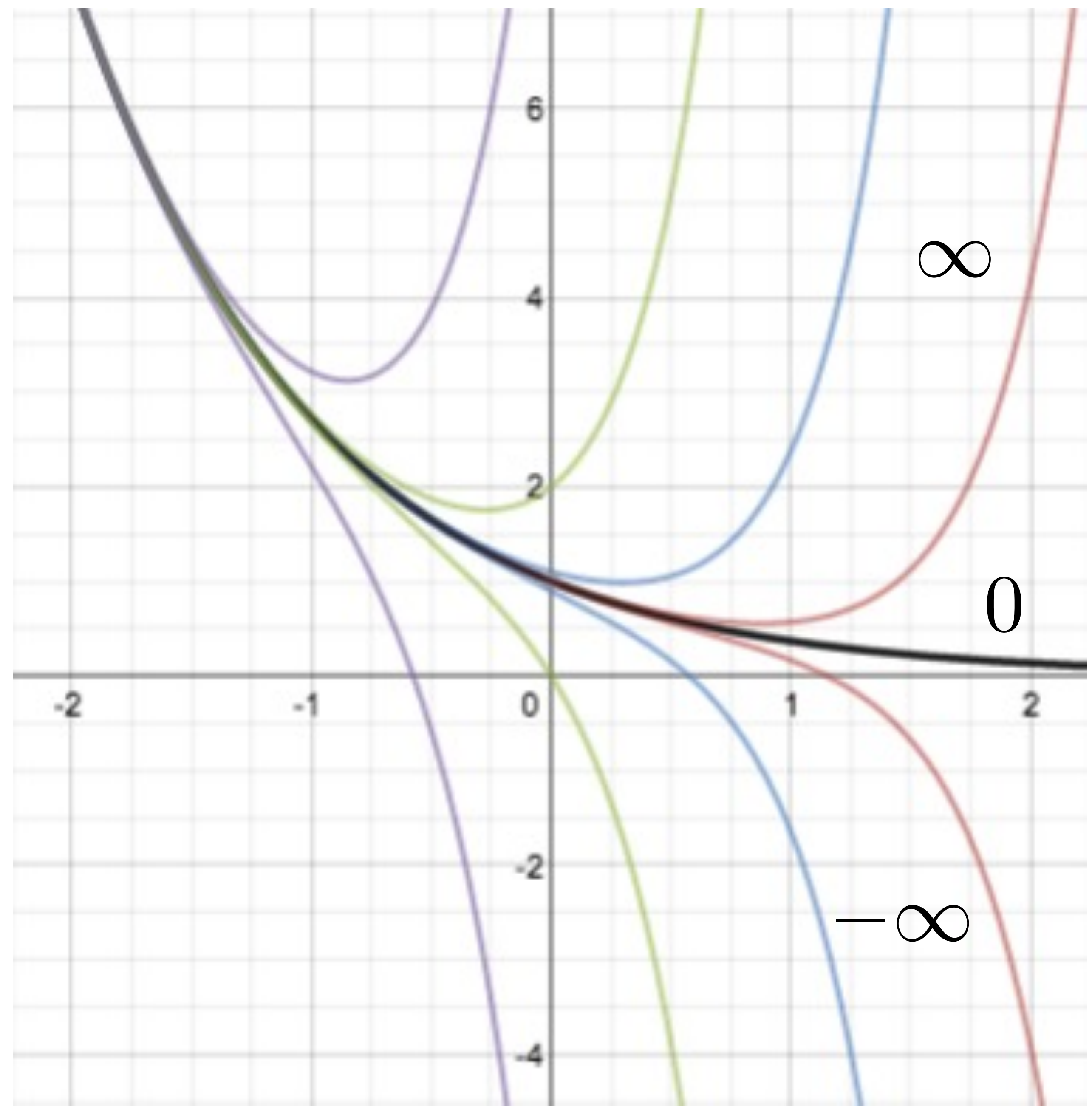

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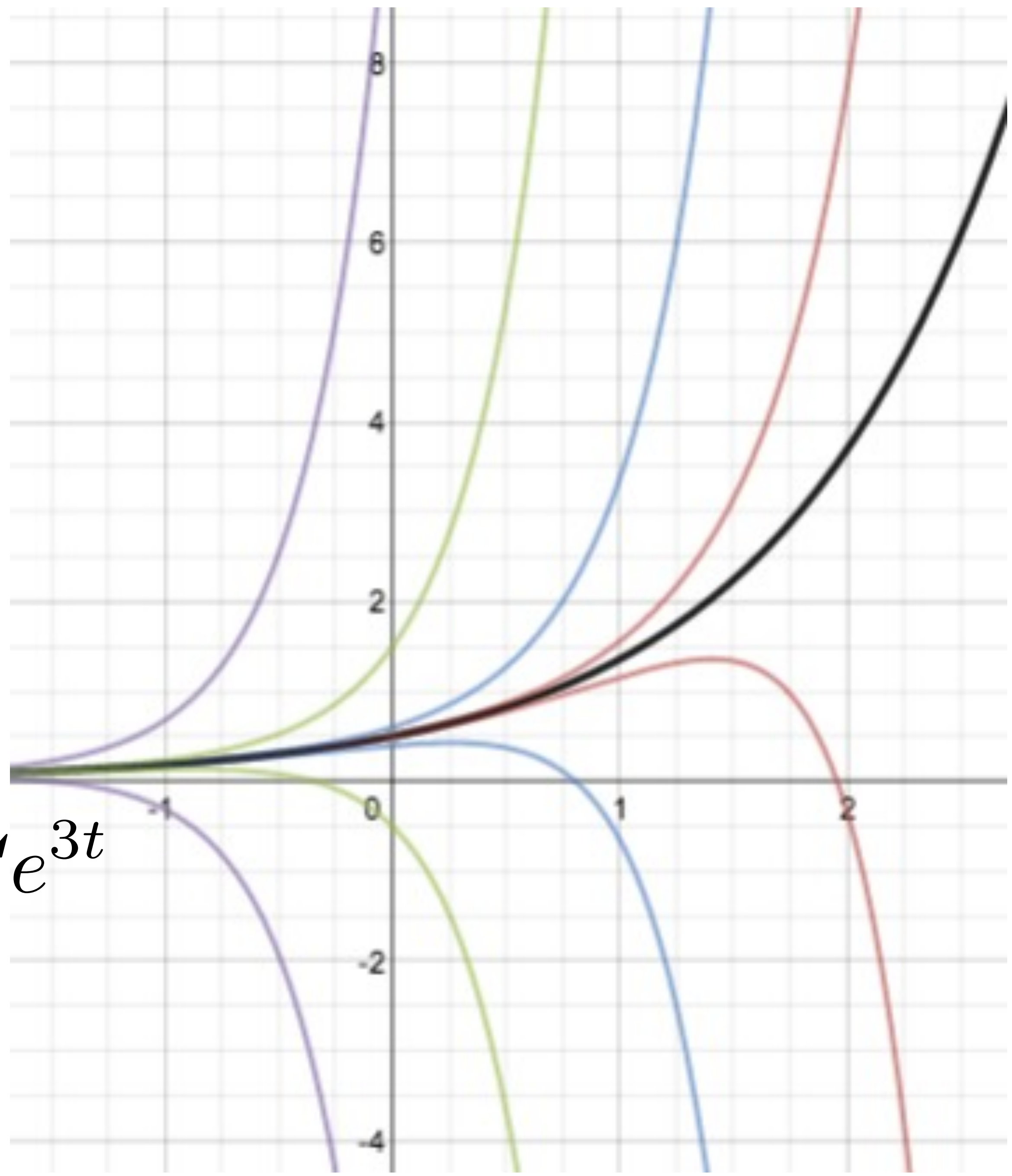
$$y(t) = \frac{1}{2}e^t + Ce^{3t}$$



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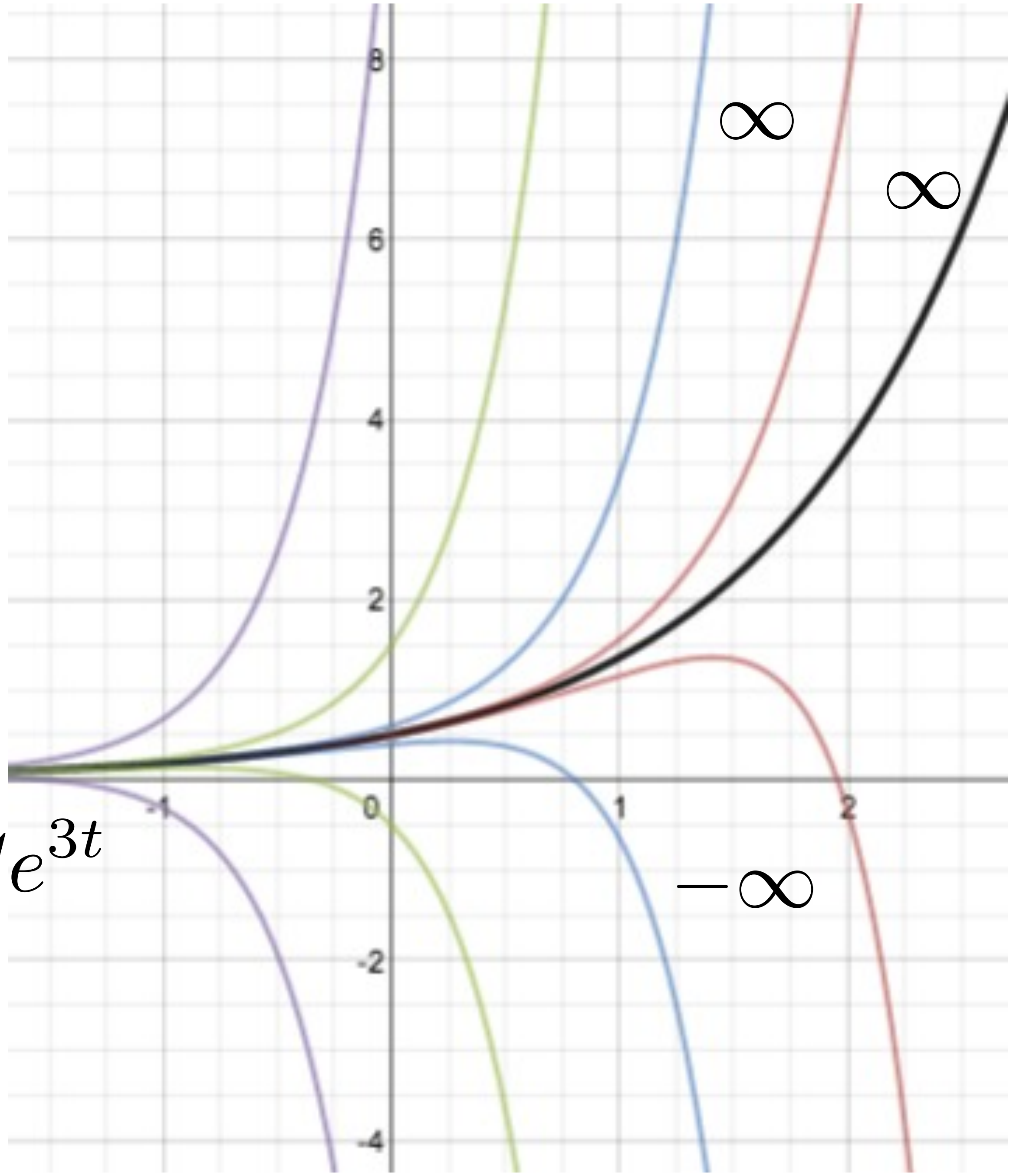

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Separable equations (Section 2.2)

- What is $\frac{d}{dt}e^y$?

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(B) $e^y \frac{dy}{dt}$

(C) ye^{y-1}

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- What is $\dfrac{d}{dt}e^y$?

Hint: rewrite as $e^y \dfrac{dy}{dt} = 1$.

$$\frac{d}{dt}(e^y) = 1$$

$$e^y = t + C$$

(D) ye … dt

(E) Don't know.

- Solve $\dfrac{dy}{dt} = e^{-y}$.

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