Today

- Comment on pre-lecture problems
- Finish up with integrating factors
- The structure of solutions
- Separable equations

Pre-lecture assignment comments

Consider the initial value problem

$$\frac{dy}{dt} - 6y = 5e^{2t}, \quad y(0) = A.$$

a. Find the solution.

$y =$ ______

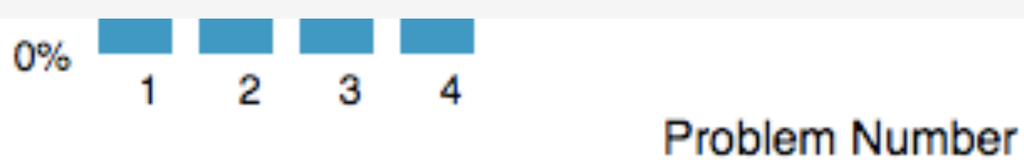
b. For what values of A does the above solution tend to ∞, 0 or $-\infty$ as $t \to \infty$?

As $t \to \infty$,

$y \to \infty$ if $A \in$ ______

$y \to 0$ if $A \in$ ______

$y \to -\infty$ if $A \in$ ______



$$y' - 6y = 5e^{2t}$$

$$(e^{-6t}y)' = 5e^{-4t}$$

$$e^{-6t}y = -\frac{5}{4}e^{-4t} + C$$

$$y(t) = -\frac{5}{4}e^{2t} + Ce^{6t}$$

$$y(0) = -\frac{5}{4} + C = A$$

$$C = A + \frac{5}{4}$$

$$y(t) = -\frac{5}{4}e^{2t} + \left(A + \frac{5}{4}\right)e^{6t}$$

Desmos demo: https://www.desmos.com/calculator/ne9u9c2q3b

Method of integrating factors

- Given that $$\frac{d}{dt}\left(t^2 y(t)\right) = t^2 \frac{dy}{dt} + 2ty$$

- if you're given the equation $$t^2 \frac{dy}{dt} + 2ty = 0$$

- you can rewrite is as $$\frac{d}{dt}\left(t^2 y(t)\right) = 0$$

- so the solution is $t^2 y(t) = C$ or equivalently $y(t) = \frac{C}{t^2}$.

arbitrary constant that appeared at an integration step

Method of integrating factors

- Solve the equation $t^2 \frac{dy}{dt} + 2ty(t) = \sin(t)$ (not brute force checking).

(A) $y(t) = -\cos(t) + C$

(B) $y(t) = \frac{C - \cos(t)}{t^2}$

(C) $y(t) = \sin(t) + C$

(D) $y(t) = -\frac{1}{t^2}\cos(t)$

(E) Don’t know.

Method of integrating factors

- Solve the equation $t^2 \frac{dy}{dt} + 2ty(t) = \sin(t)$ (not brute force checking).

(A) $y(t) = -\cos(t) + C$

(B) $y(t) = \frac{C - \cos(t)}{t^2}$ ← general solution (although that's not obvious)

(C) $y(t) = \sin(t) + C$

(D) $y(t) = -\frac{1}{t^2}\cos(t)$ ← a particular solution

(E) Don't know.

Initial conditions (IC) and initial value problems (IVP)

- An initial condition is an added constraint on a solution.
- e.g. Solve $t^2 \dfrac{dy}{dt} + 2ty(t) = \sin(t)$ subject to the IC $y(\pi) = 0$.

(A) $y(t) = -\dfrac{C + \cos(\pi)}{\pi^2}$

(B) $y(t) = -\dfrac{1 - \cos(t)}{t^2}$

(C) $y(t) = \dfrac{1 + \cos(t)}{t^2}$

(D) $y(t) = -\dfrac{1 + \cos(t)}{t^2}$

(E) Don't know.

- An Initial Value Problem (IVP) is a ODE together with an IC.

Method of integrating factors

- What function should we multiply through by to make the LHS a perfect product rule?

$$t\frac{dy}{dt} + 2y(t) = 1 \qquad \rightarrow f(t) = t$$

$$t^2\frac{dy}{dt} + 4ty(t) = \frac{1}{t} \qquad \rightarrow f(t) = t^2$$

$$\frac{dy}{dt} + y(t) = 0 \qquad \rightarrow f(t) = e^t$$

$$\frac{dy}{dt} + \cos(t)y(t) = 0 \qquad \rightarrow f(t) = e^{\sin(t)}$$

$$\frac{dy}{dt} + g'(t)y(t) = 0 \qquad \rightarrow f(t) = e^{g(t)}$$

Technical definition of integrating factor

- For the general first order linear ODE

$$a(t)y' + b(t)y = g(t)$$

- Divide through by a(t) and define p(t) = b(t) / a(t) and q(t) = g(t) / a(t) :

$$y' + p(t)y = q(t)$$

- The function that, when multiplied through, make the LHS a perfect product rule is called the integrating factor.

Method of integrating factors

- General case - all first order linear ODEs can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

- The appropriate integrating factor is $e^{\int p(t)dt}$.

- The equation can be rewritten $\frac{d}{dt}\left(e^{\int p(t)dt}y\right) = e^{\int p(t)dt}q(t)$ which is solvable provided you can find the antiderivative of the right hand side.

$$e^{\int p(t)dt}y(t) = \int e^{\int p(t)dt}q(t)dt + C$$

$$y(t) = e^{-\int p(t)dt}\int e^{\int p(t)dt}q(t)dt + Ce^{-\int p(t)dt}$$

The structure of solutions

- When the equation is of the form (called homogeneous)

$$\frac{dy}{dt} + p(t)y = 0$$

- the solution is

$$y(t) = C\mu(t)^{-1}$$

- where

$$\mu(t) = \exp\left(\int p(t)dt\right)$$

- is the integrating factor.

The structure of solutions

- When the equation is of the form (called nonhomogeneous)

$$\frac{dy}{dt} + p(t)y = q(t)$$

- the solution is $y(t) = k(t) + C\mu(t)^{-1}$

- where k(t), as given earlier, involves no arbitrary constants.

- Think about this expression as $y(t) = y_p(t) + y_h(t)$

- Directly analogous to solving the vector equations $A\overline{x} = 0$ and $A\overline{x} = \overline{b}$.

Examples

- Find the general solution to

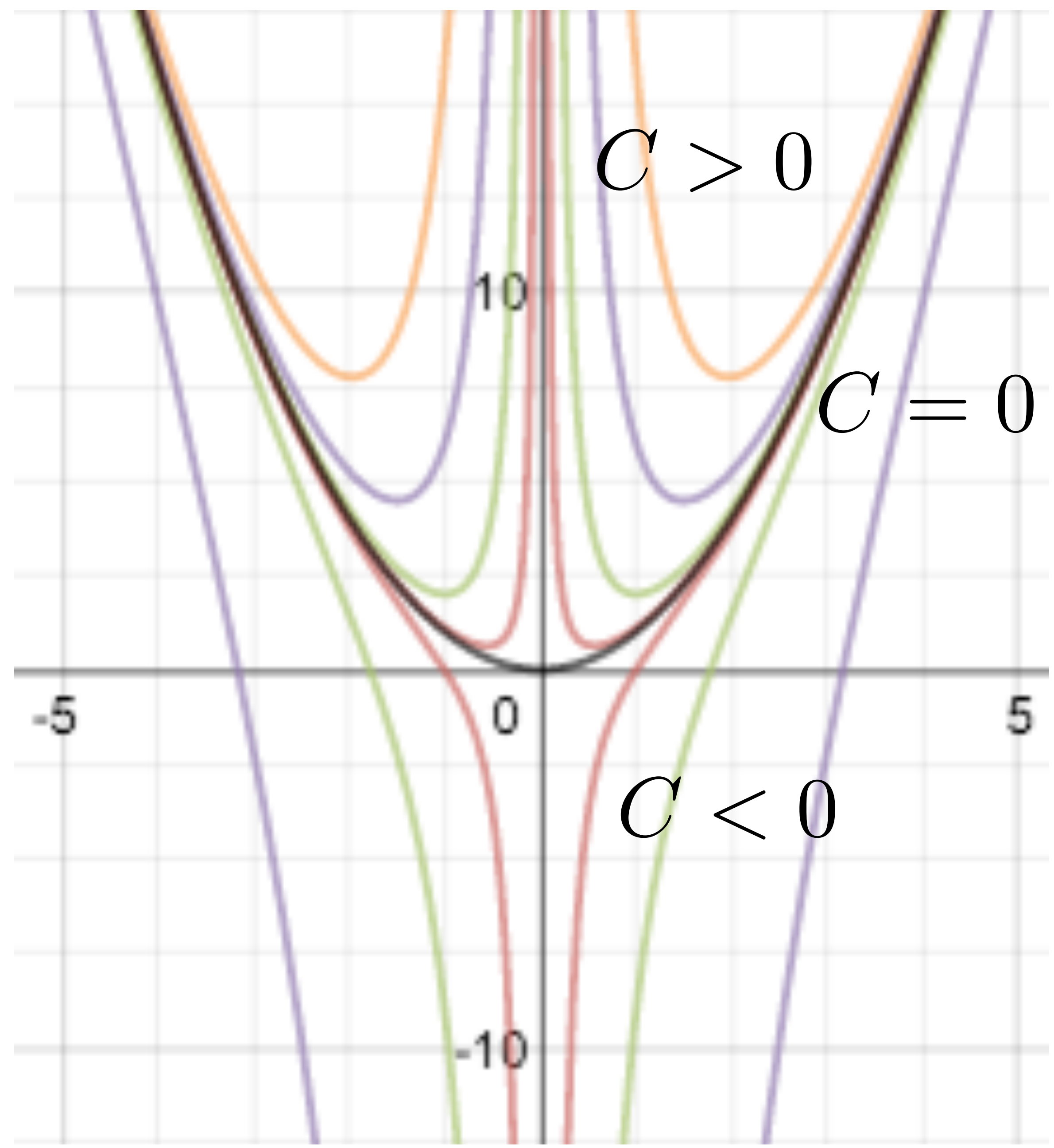
$$t\frac{dy}{dt} + 2y = 4t^2$$

- and plot a few of the integral curves.

- Integral curve - the graph of a solution to an ODE.

$$y(t) = t^2 + C\frac{1}{t^2}$$

- Steps: divide through by t, calculate I(t), take antiderivatives, solve for y. Or shortcut.



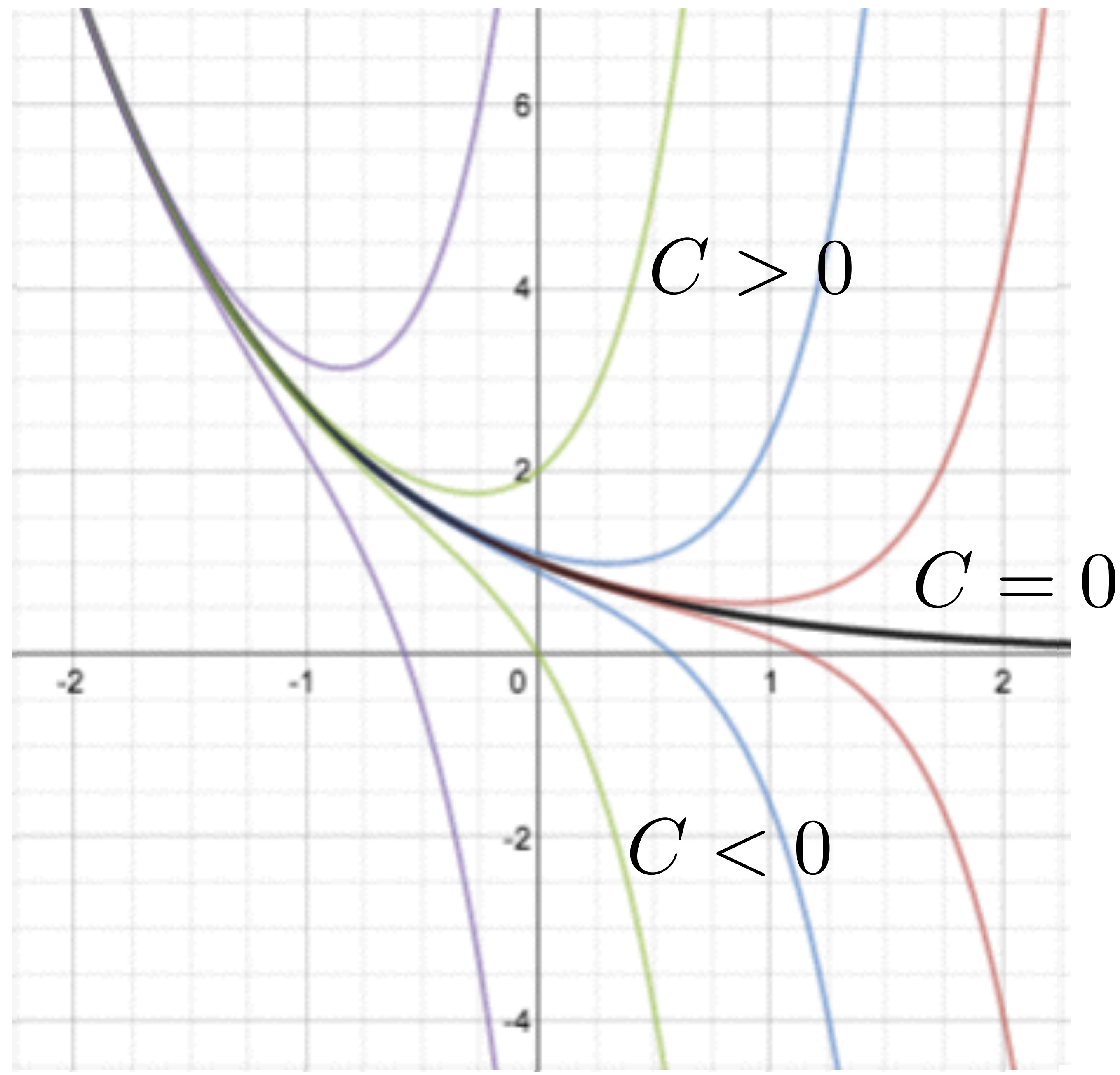
Examples

- Find the general solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- and plot a few of the integral curves.

$$y(t) = e^{-t} + Ce^{3t}$$



Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- depending on C, how many different results are possible for

$$\lim_{t \to \infty} y(t) \text{ ?}$$

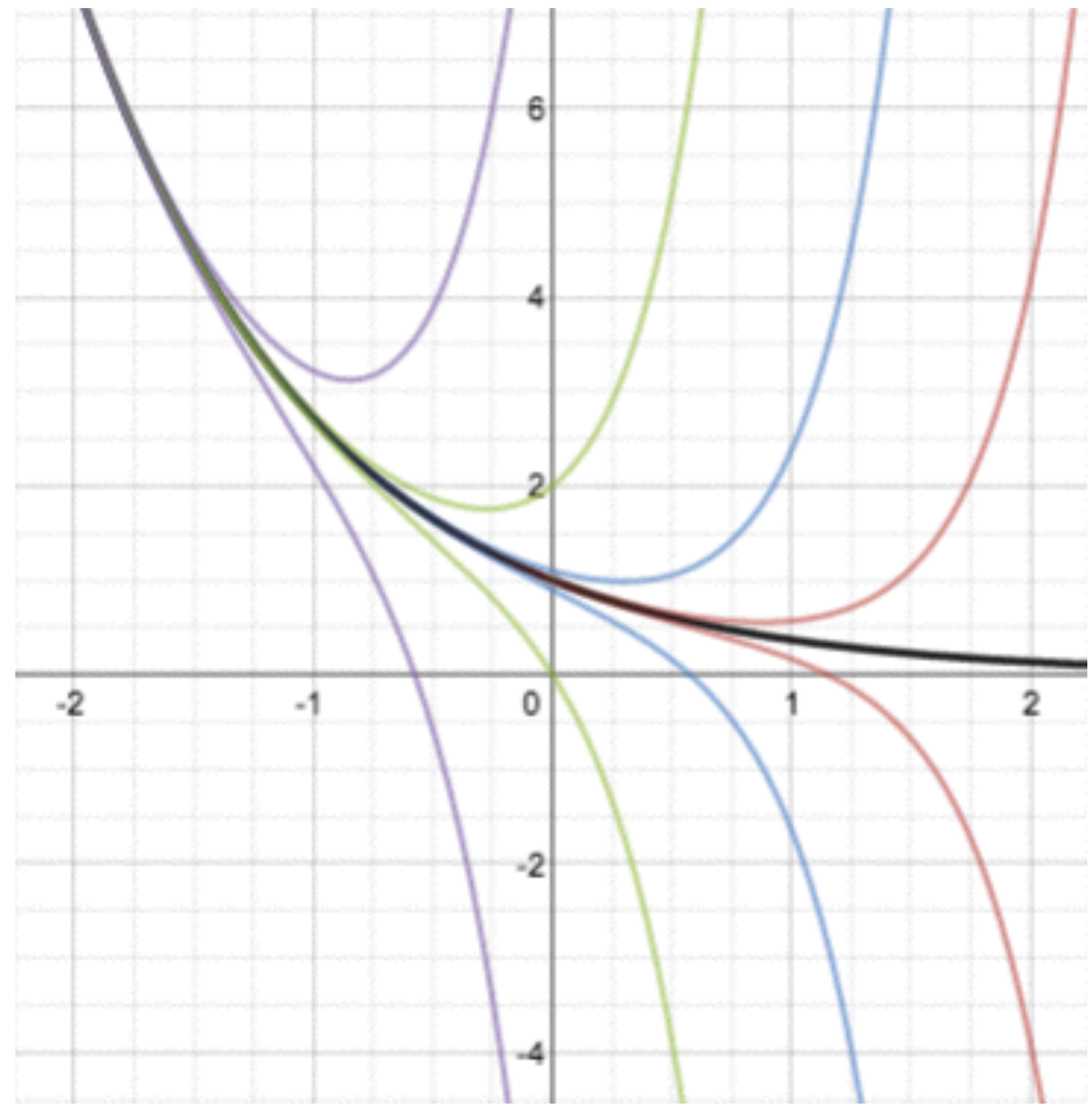
(A) 0

(B) 1

(C) 2

(D) 3

(E) Don't know.



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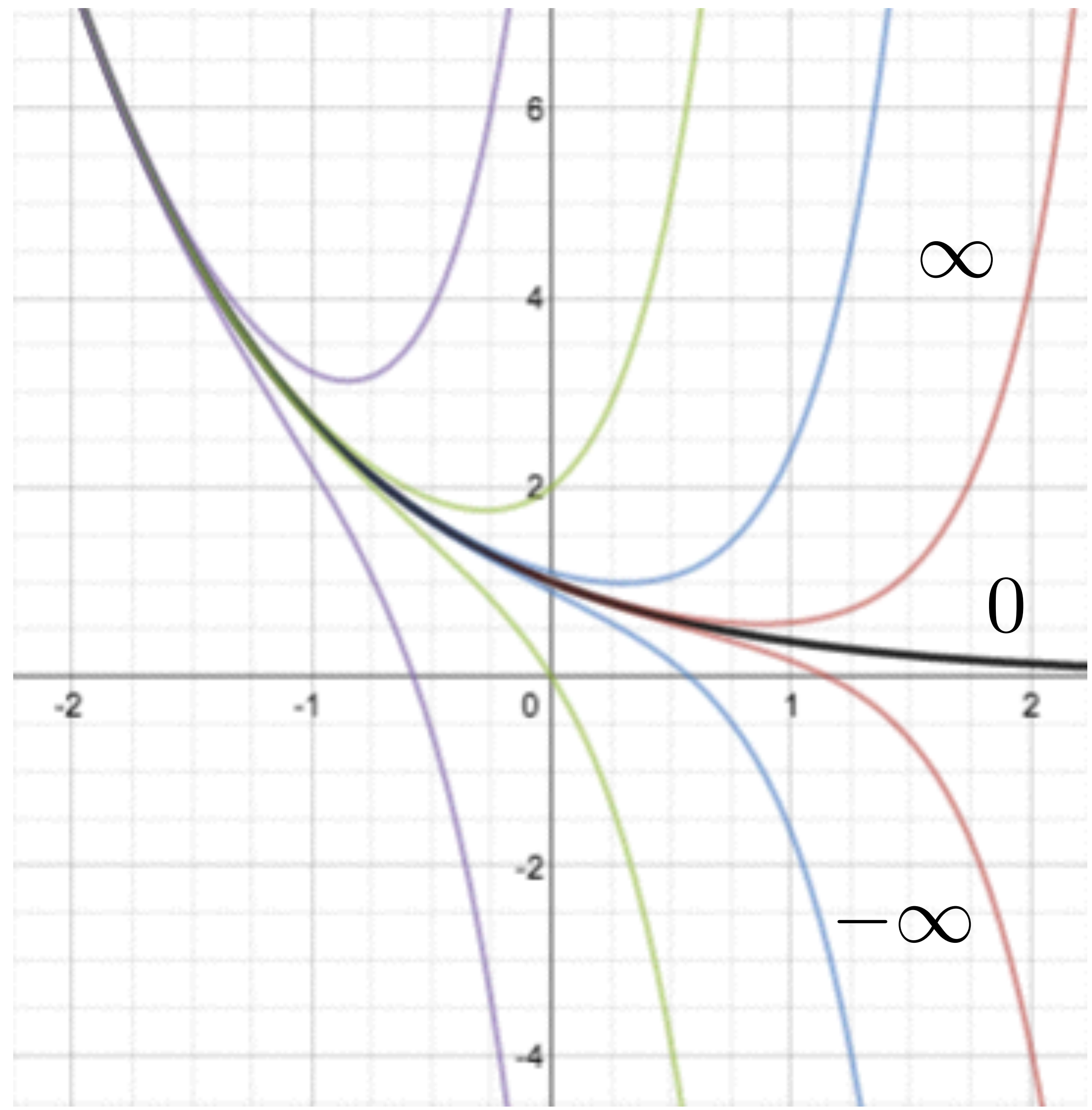

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