Today

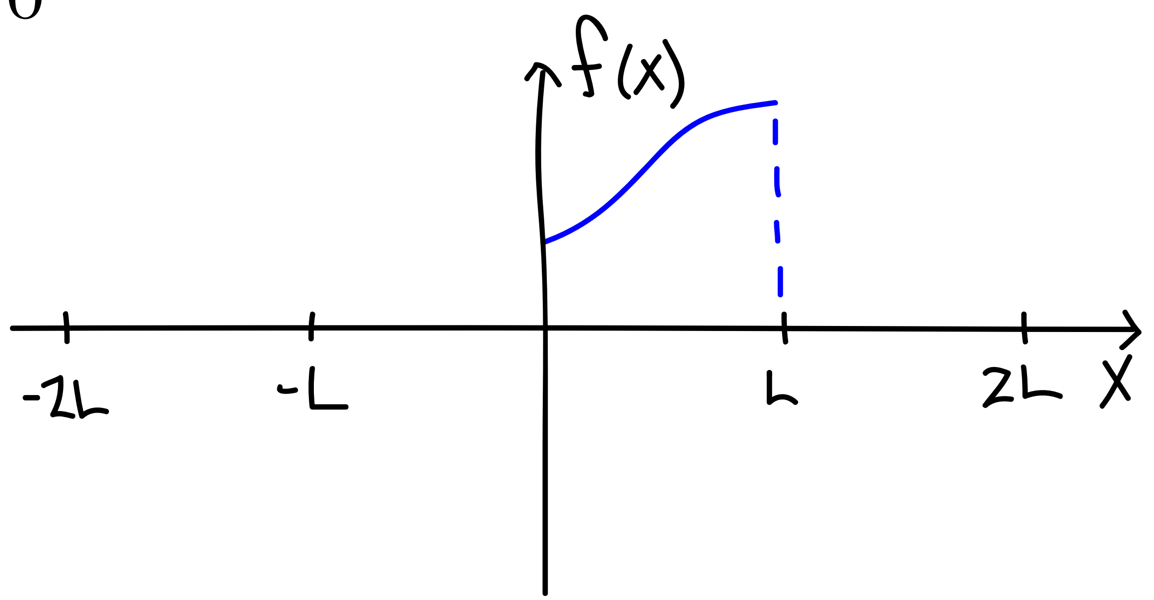
- Review of solutions to the Diffusion Equation with various BCs.
- The Wave Equation.
- Separation of variables.

Review of solutions to the Diffusion Equation

$u_t = Du_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$



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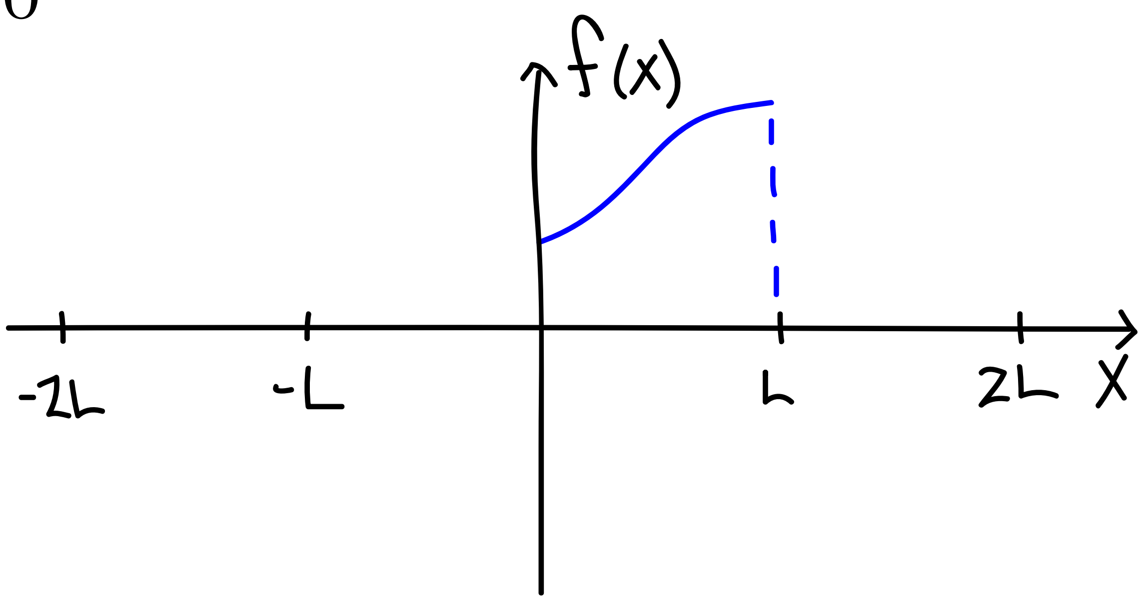
- Extend f(x) to all reals as a periodic function.



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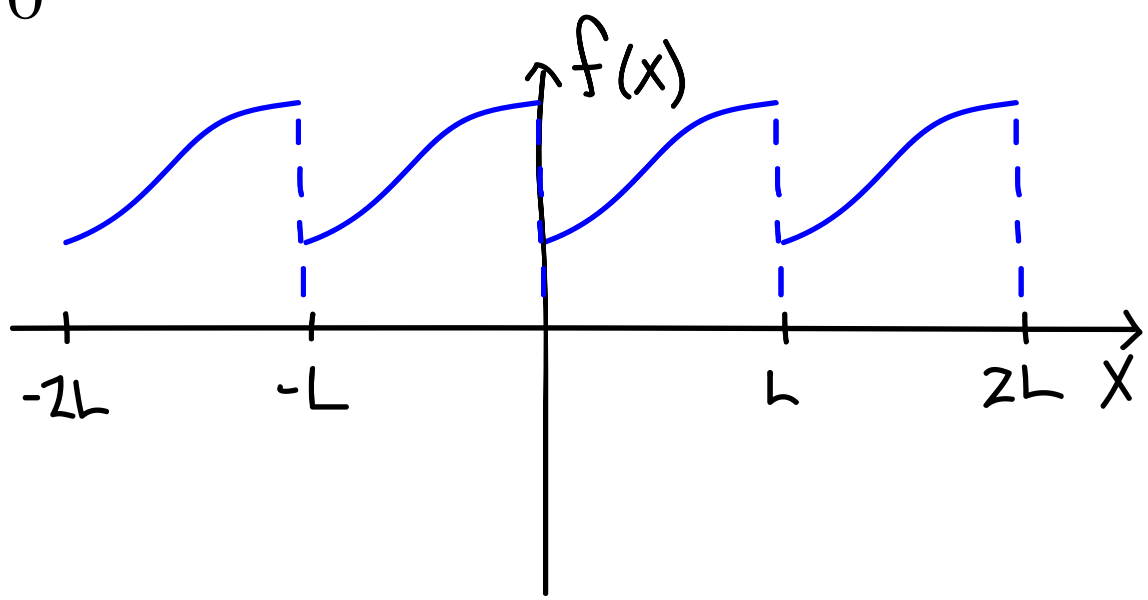

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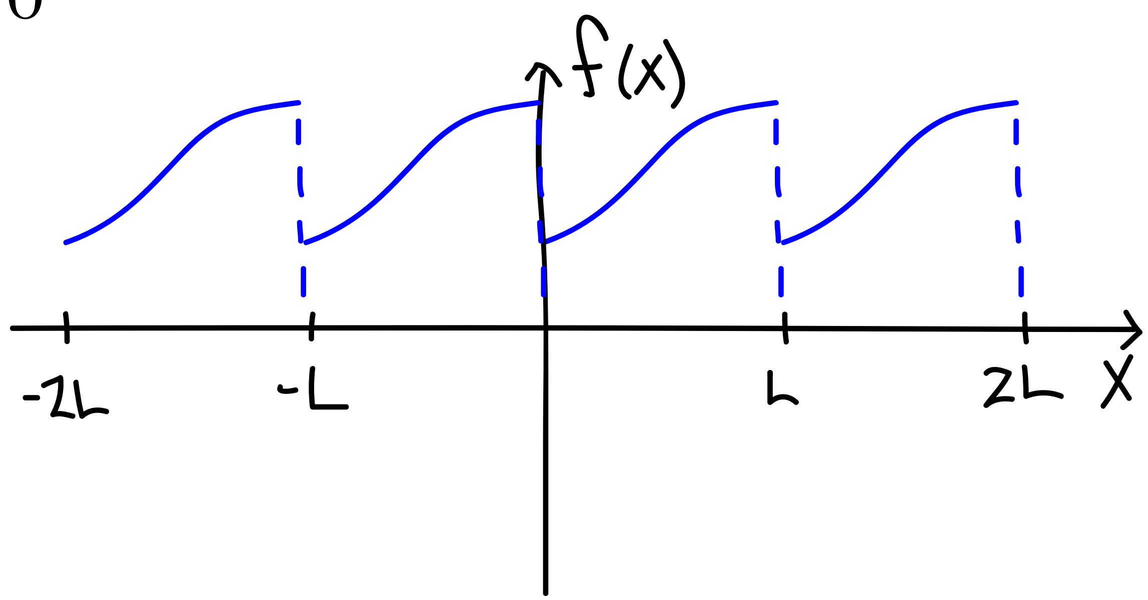

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

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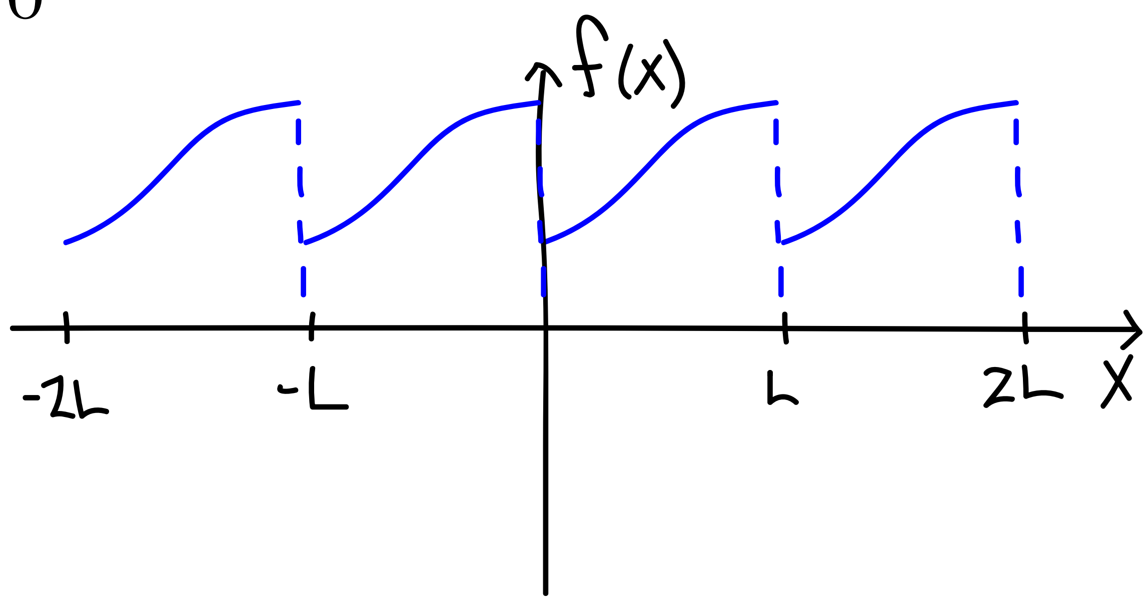

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- All coefficients will be non-zero. Not particularly useful for solving the BCs.

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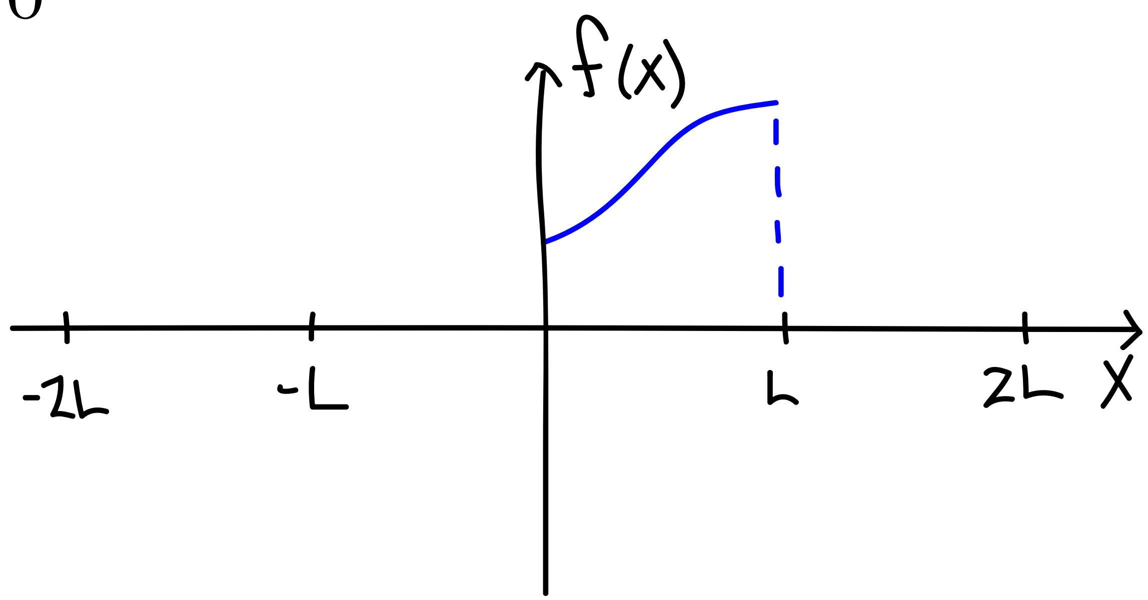
- Extend to -L as an odd function and then to all reals as a periodic function.



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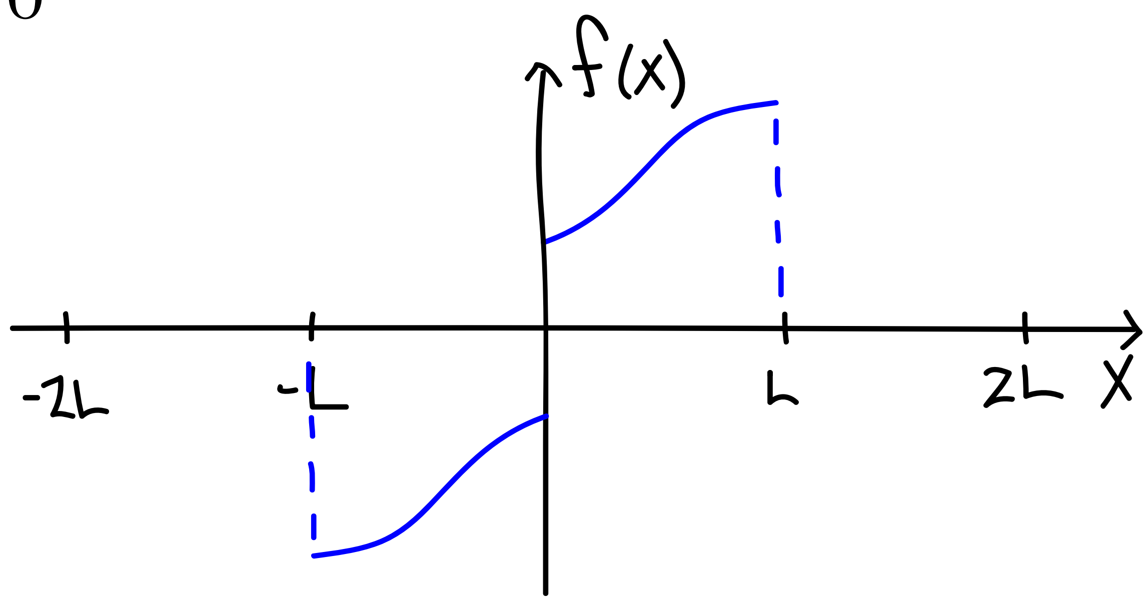

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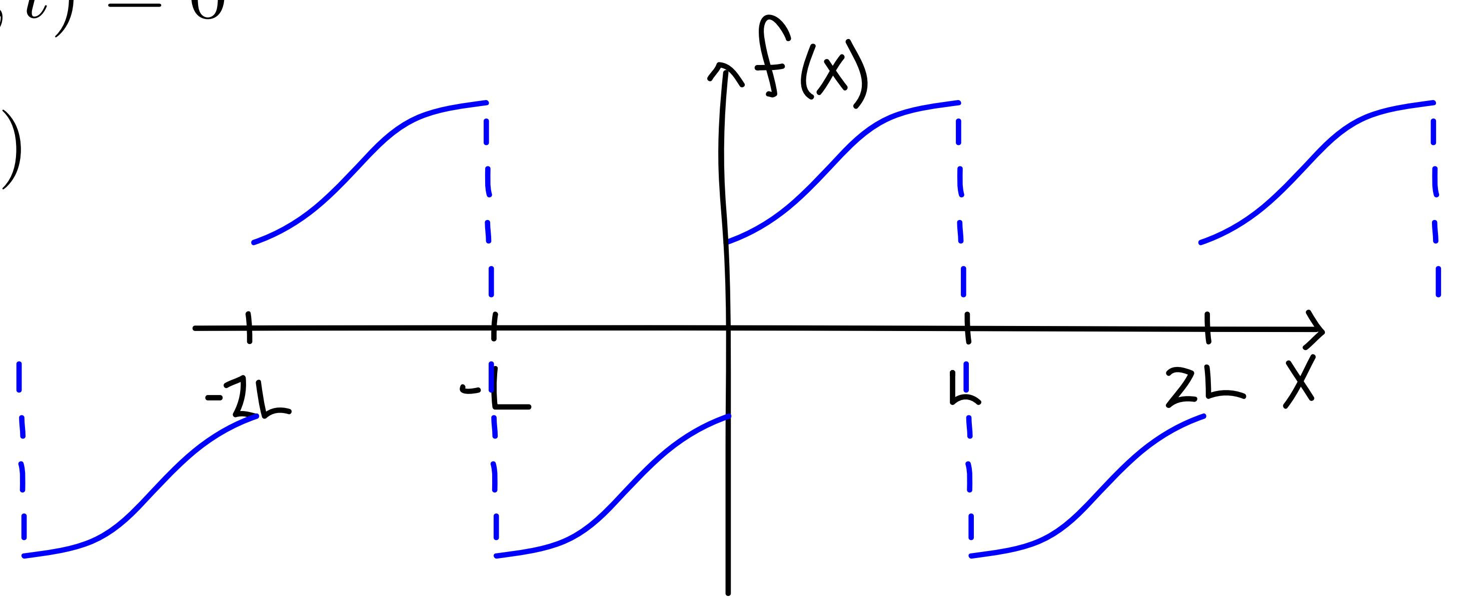

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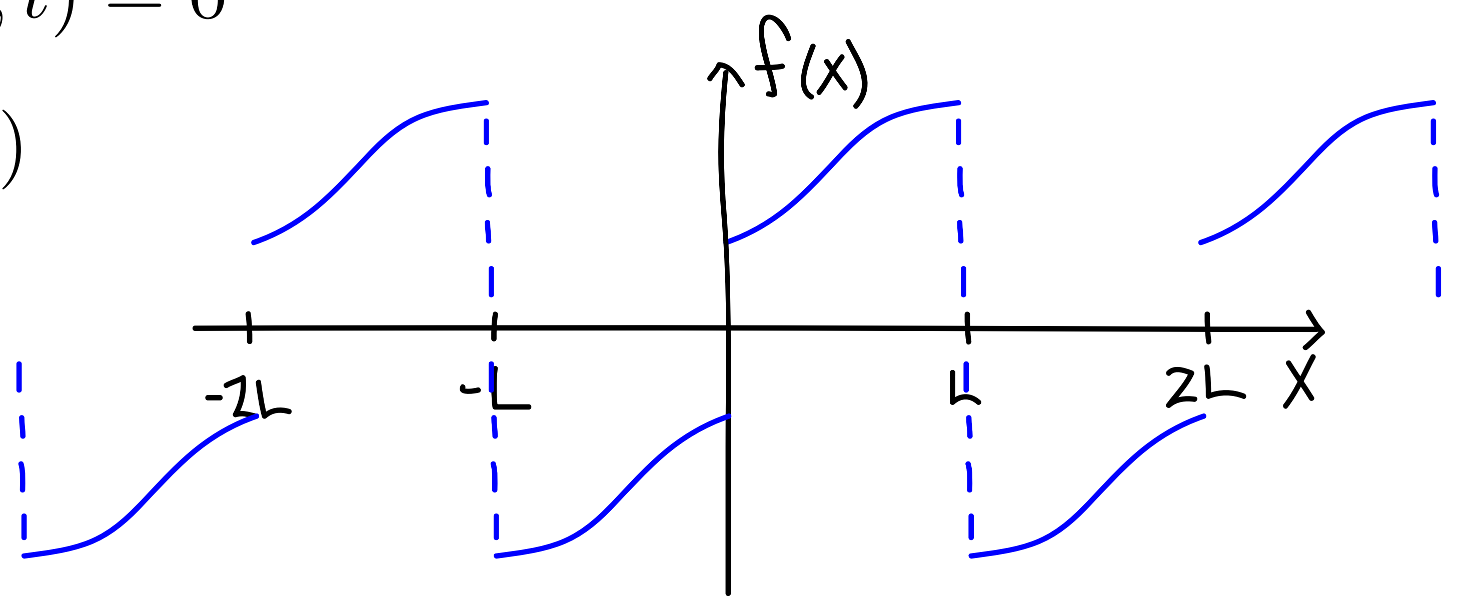

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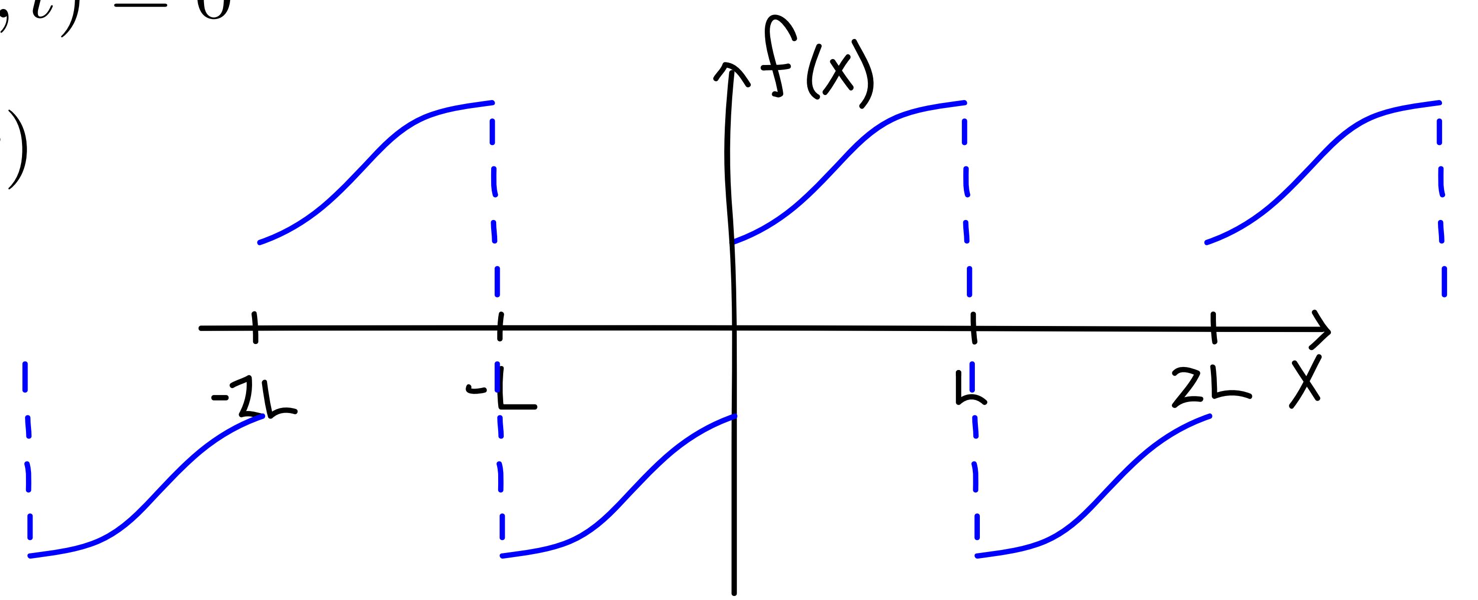

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- Cosine coefficients will be zero because f(x) is odd about x=0 and cosine is even. Useful for solving the Diffusion equation with Dirichlet BCs.

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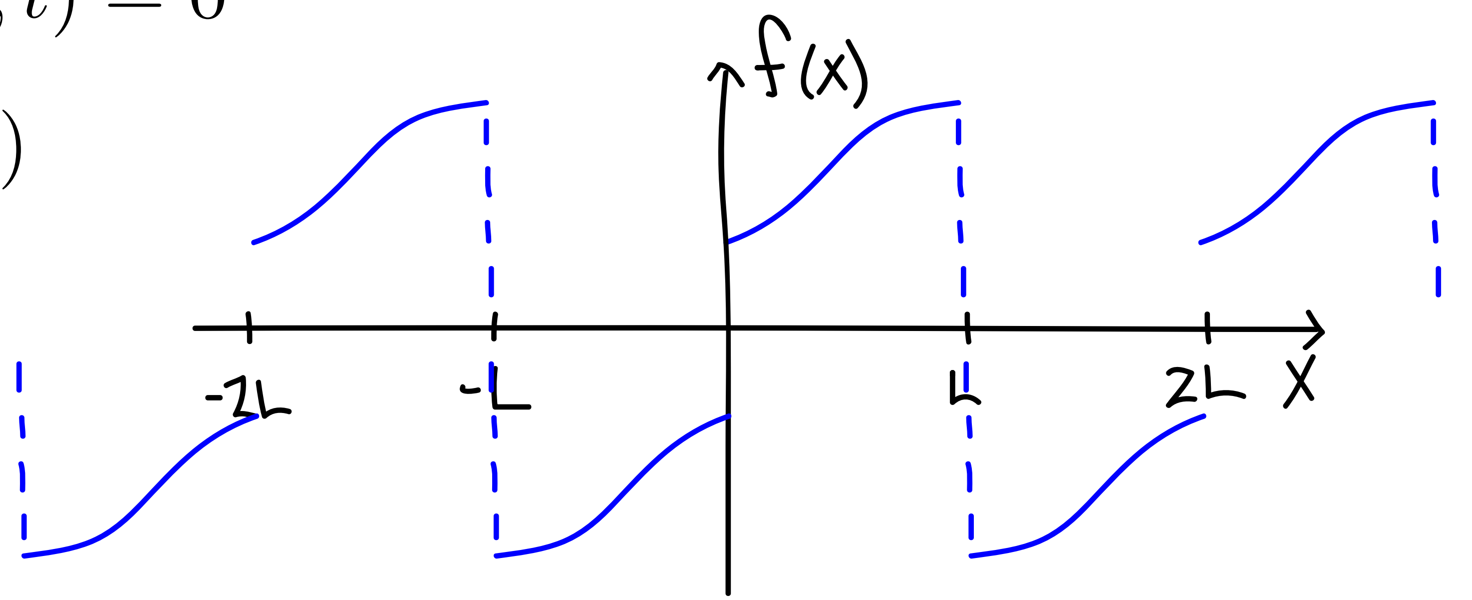

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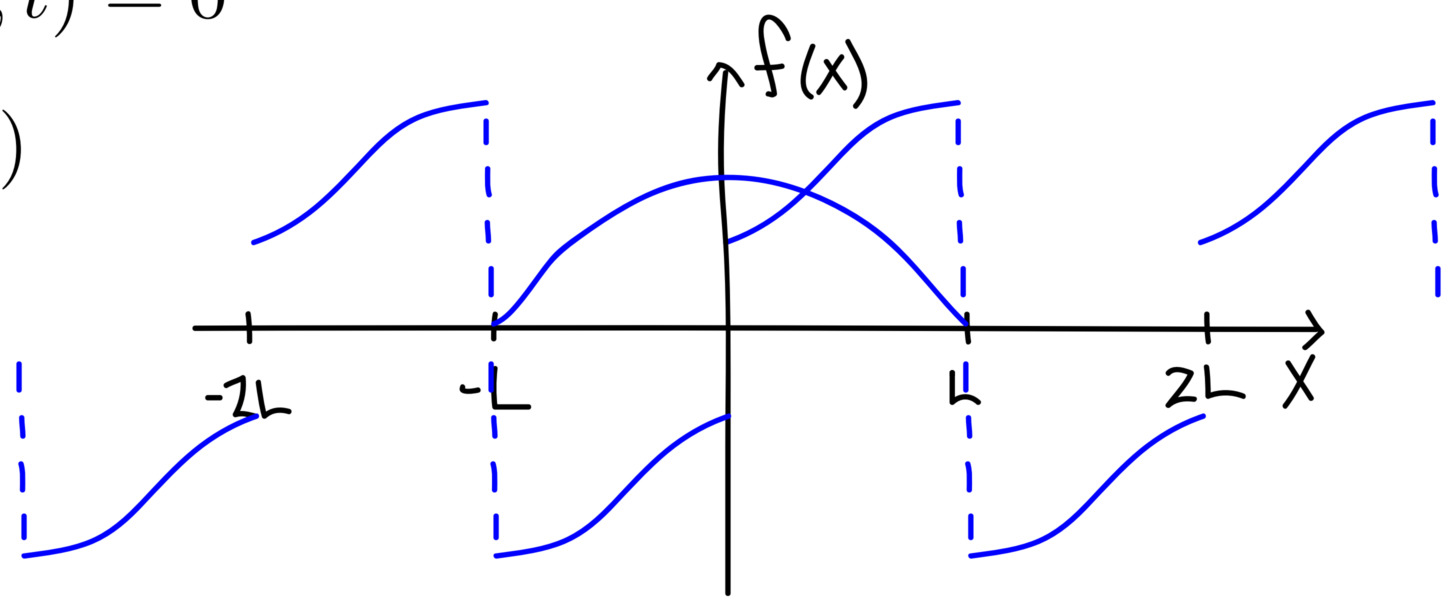

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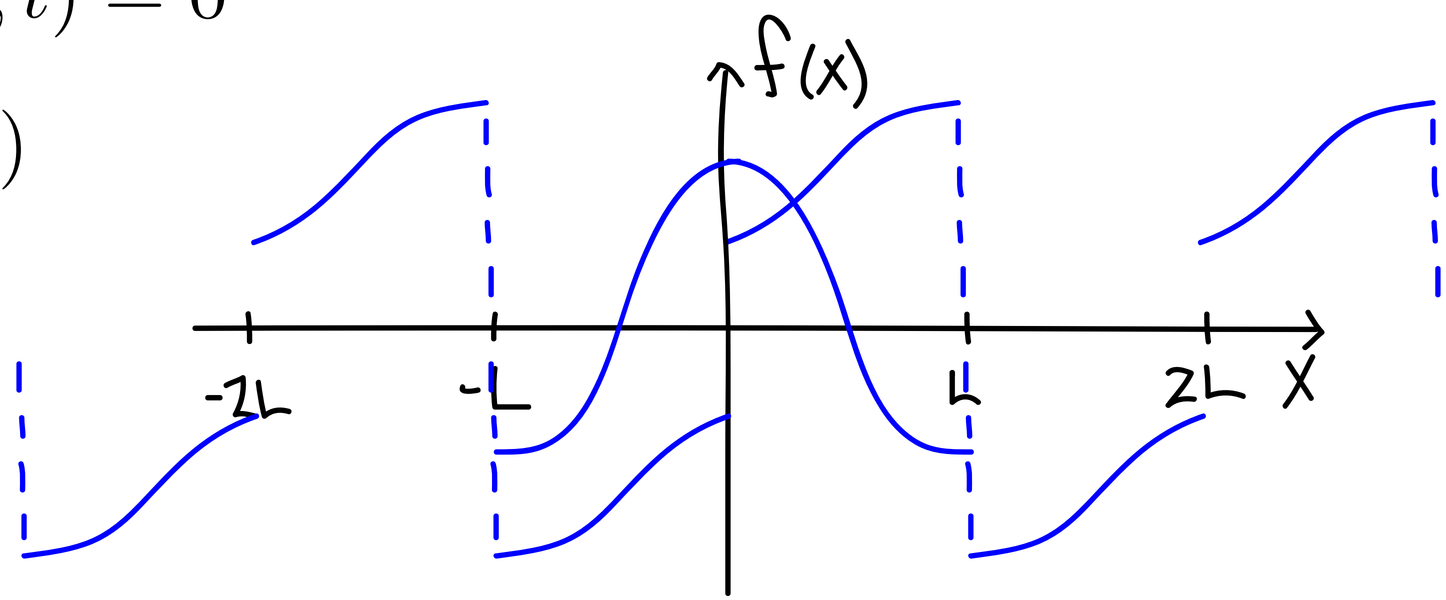

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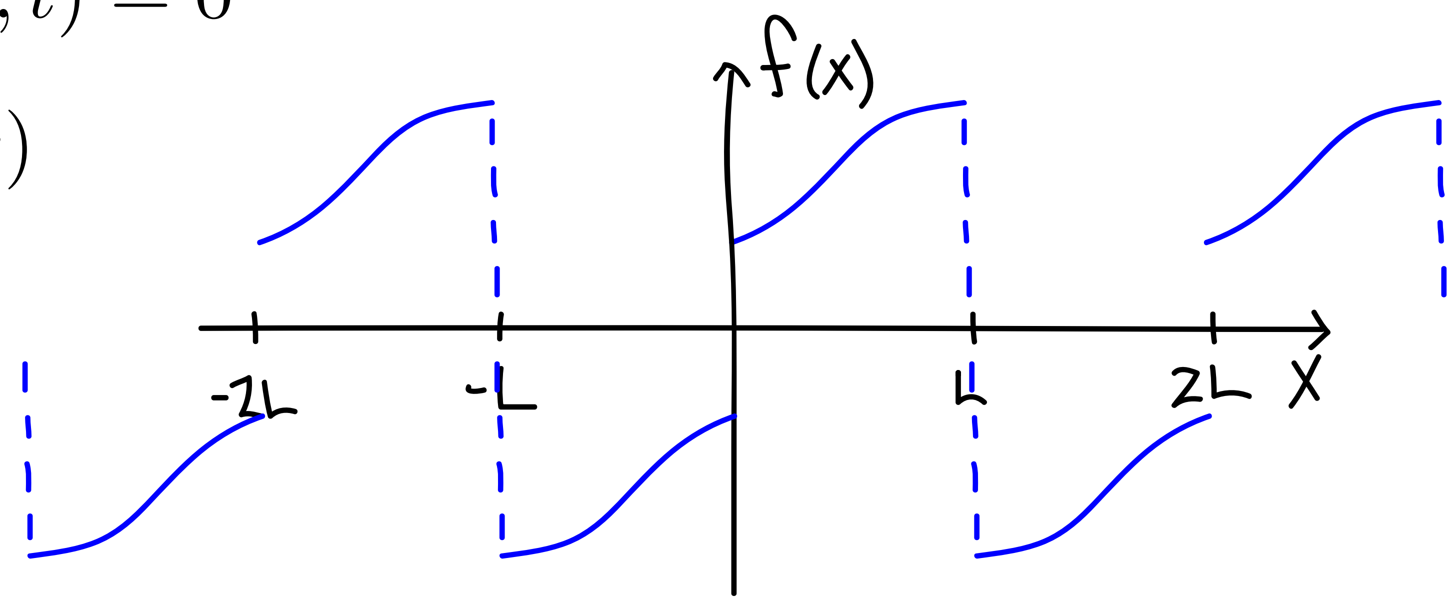

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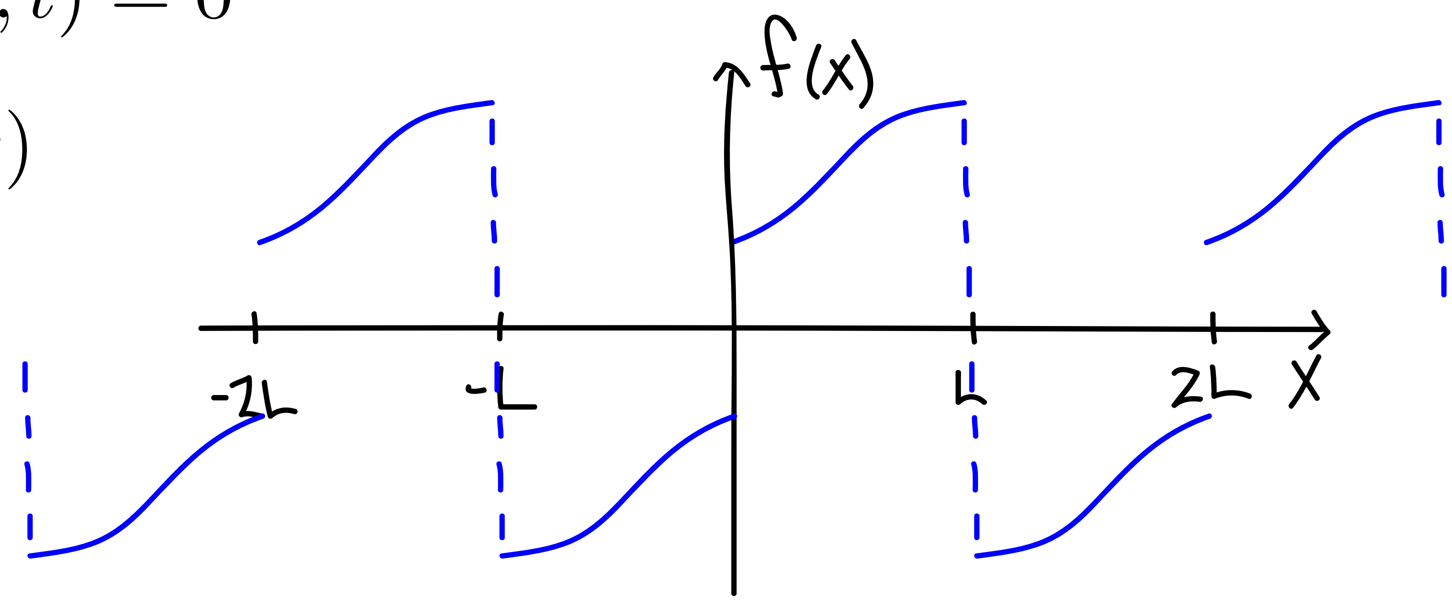

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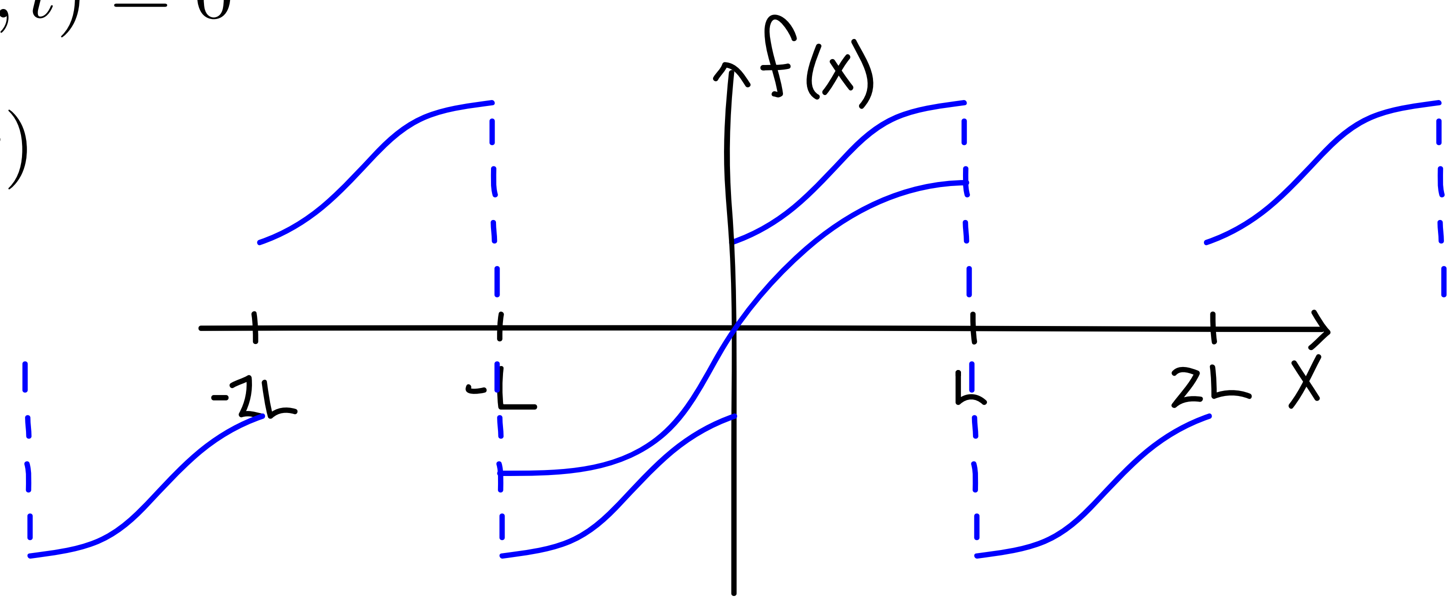

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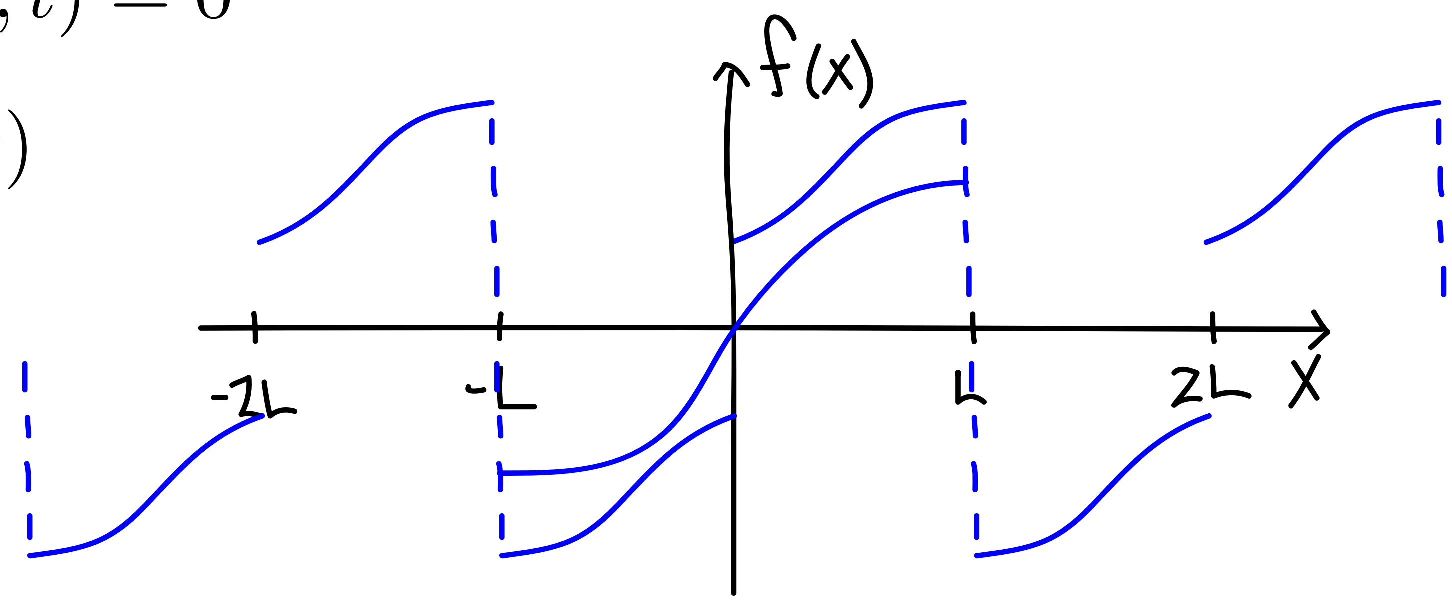

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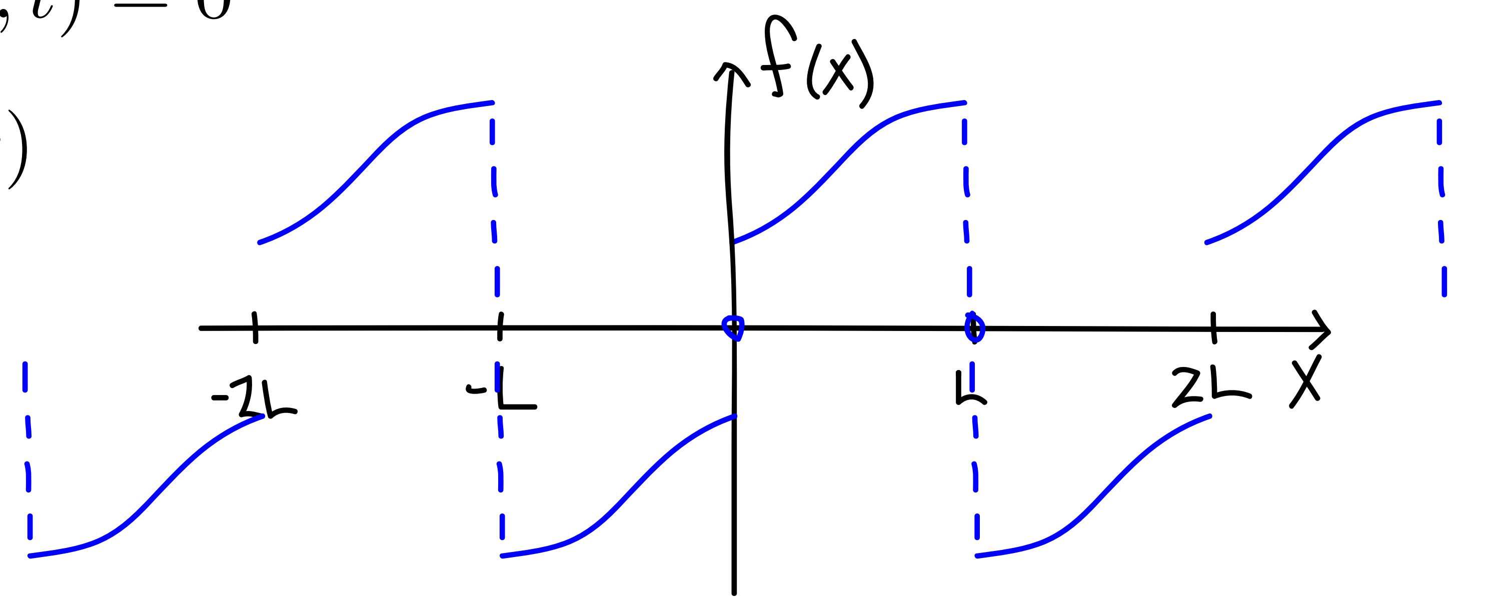

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$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

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Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\; dx$$

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$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x\right)\sin\frac{n\pi x}{L}\, dx$$

- Adding the linear function to the usual solution to the Dirichlet problem ensures that the BCs are satisfied without changing the fact that it satisfies the PDE.

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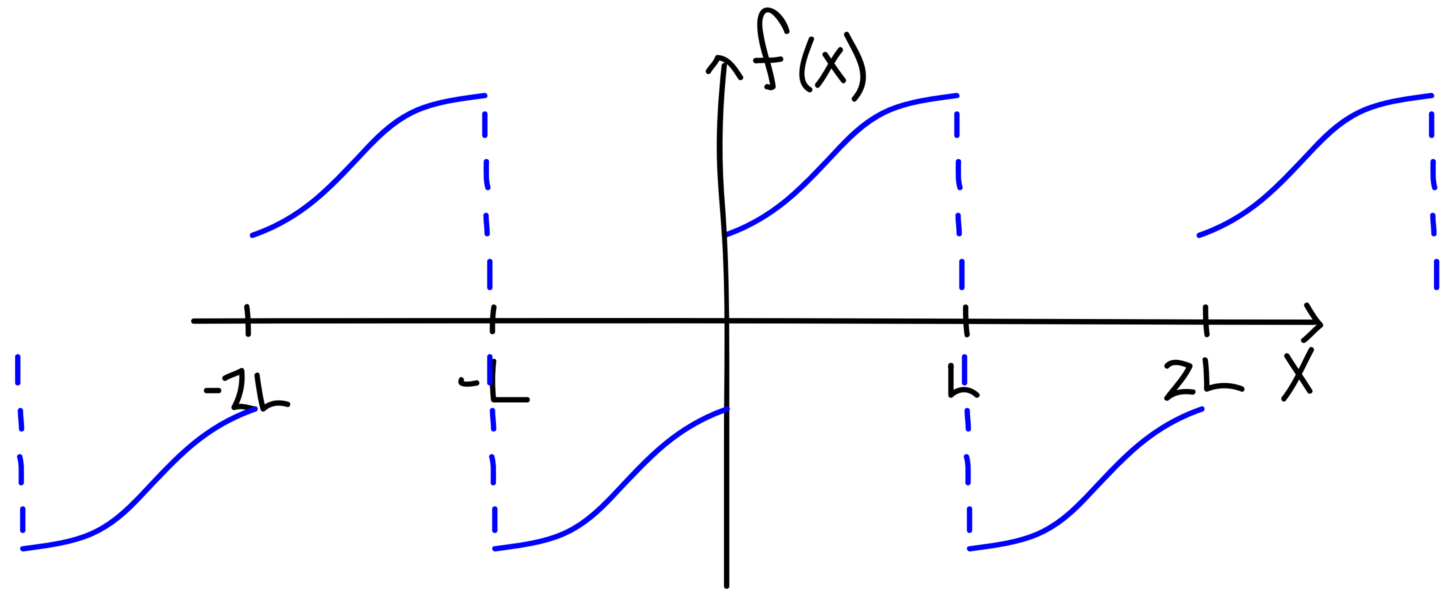

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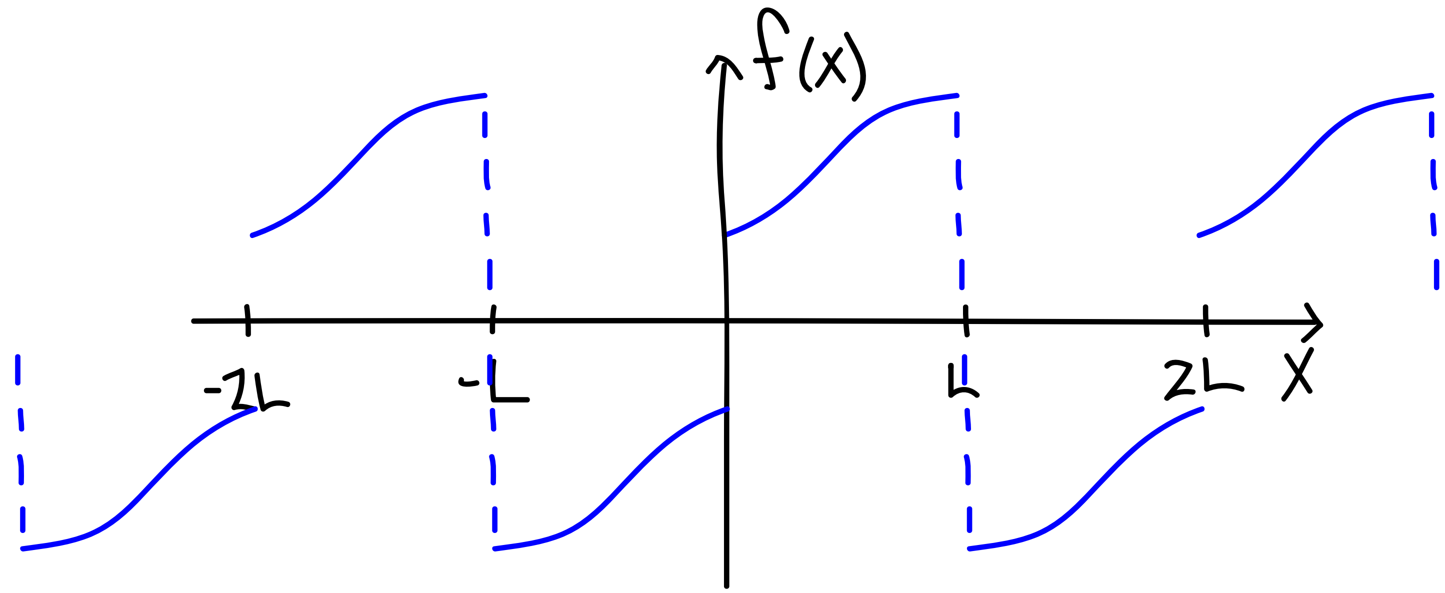

Odd symmetry about x=0

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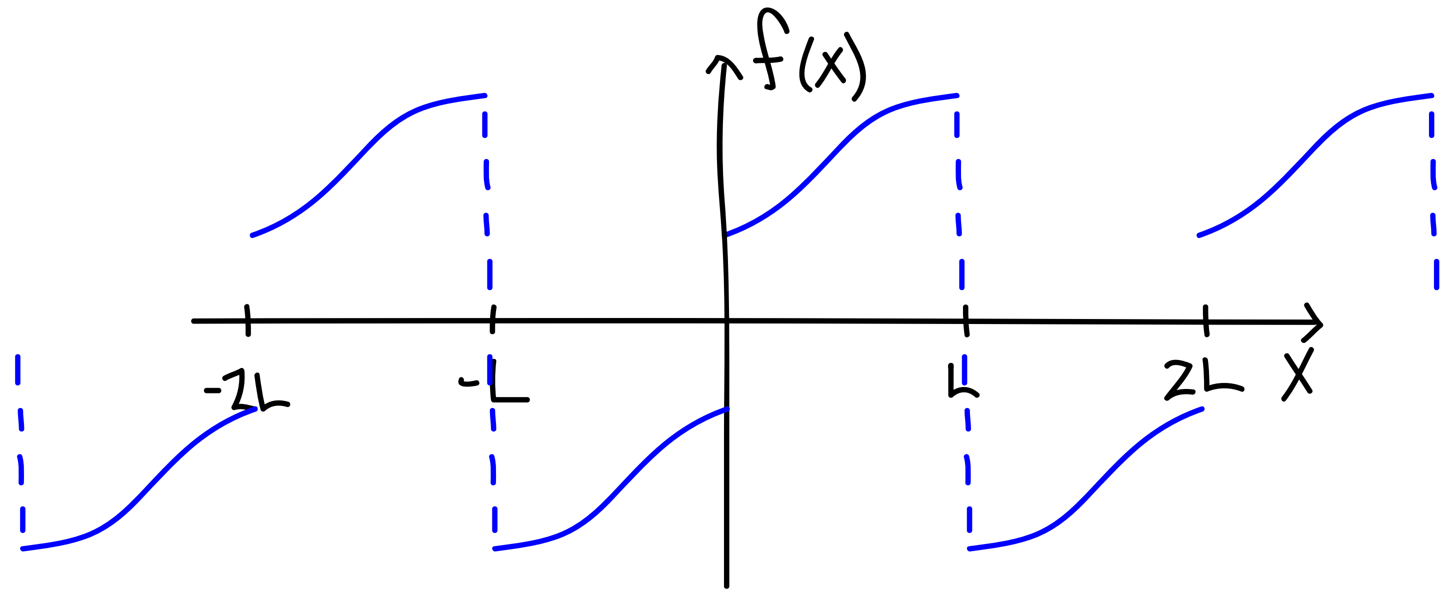

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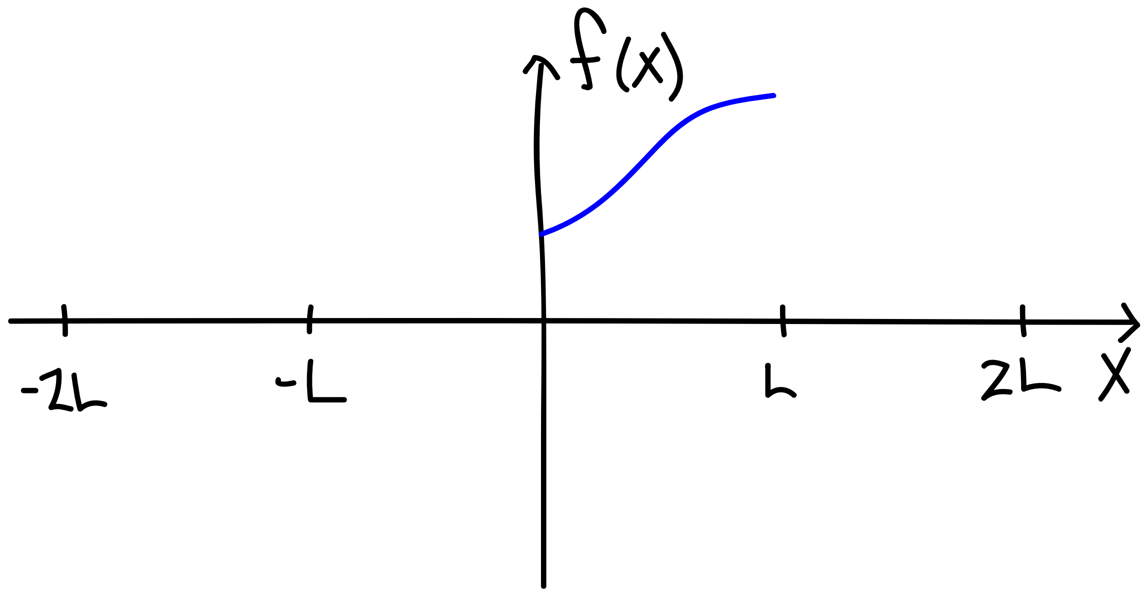

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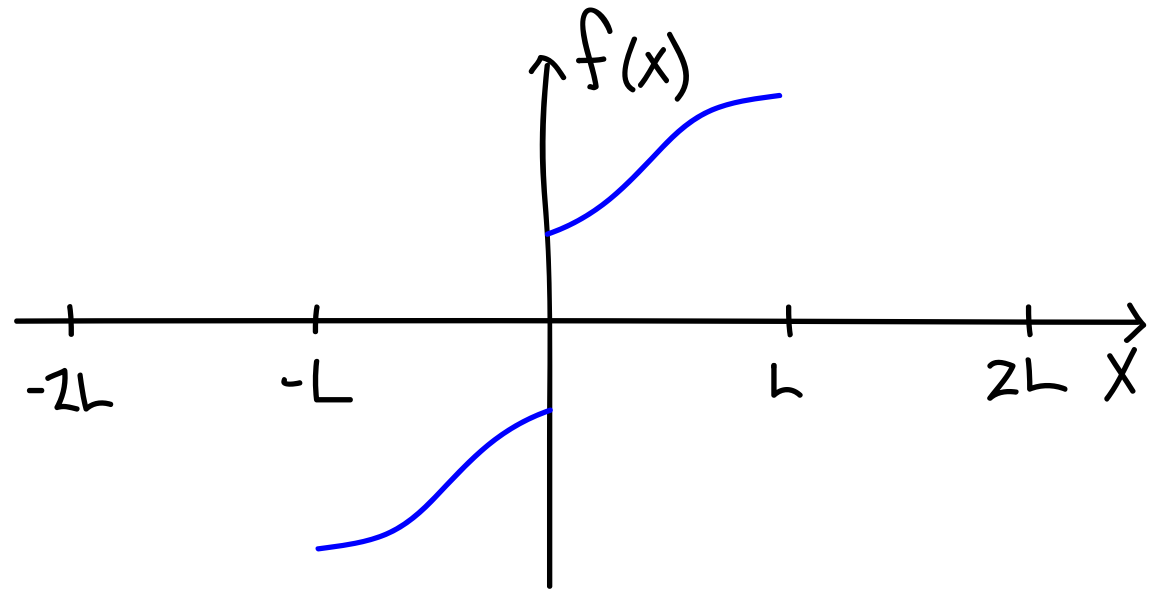

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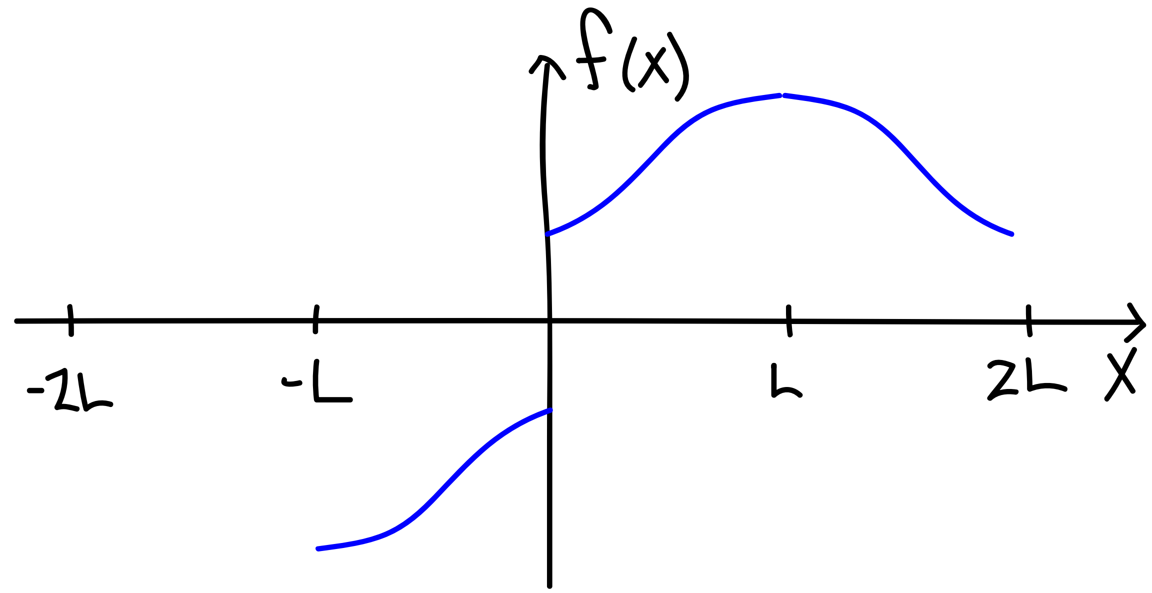

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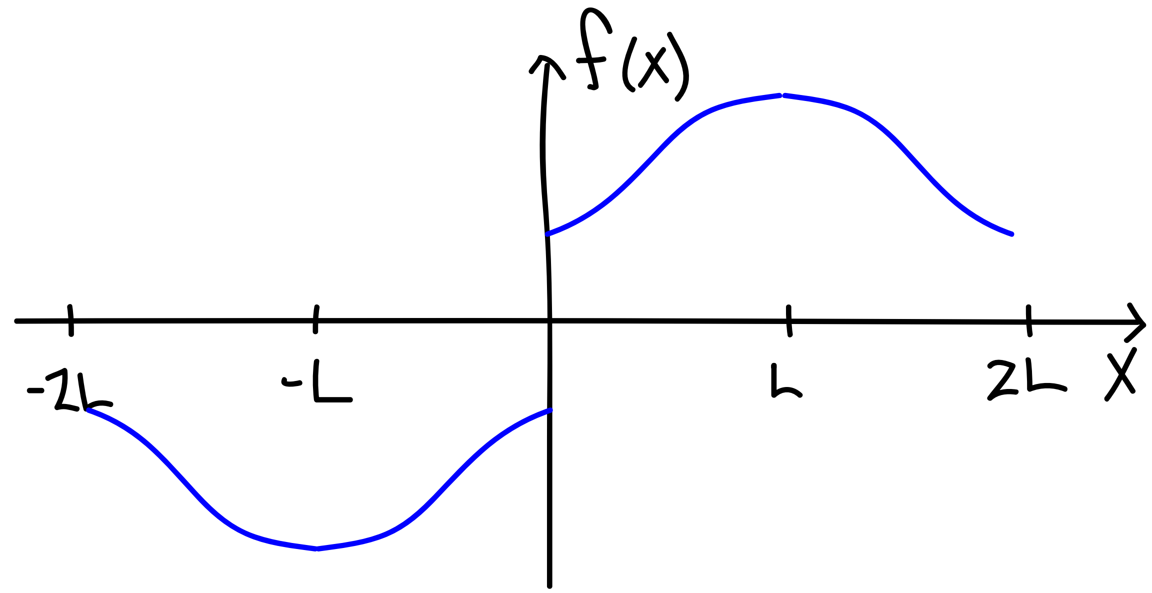

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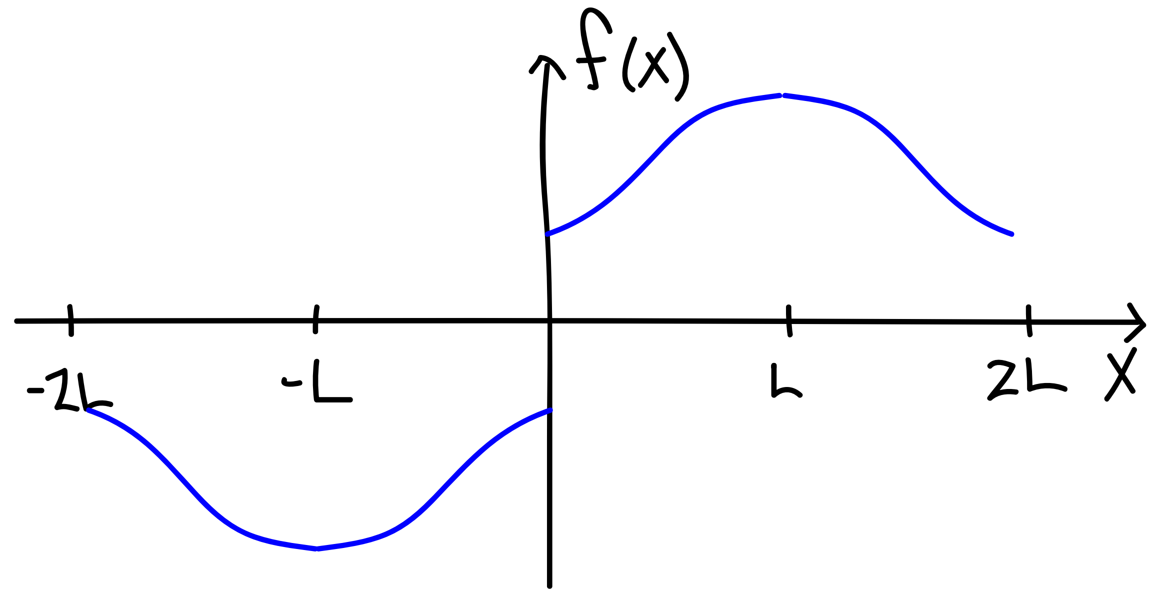

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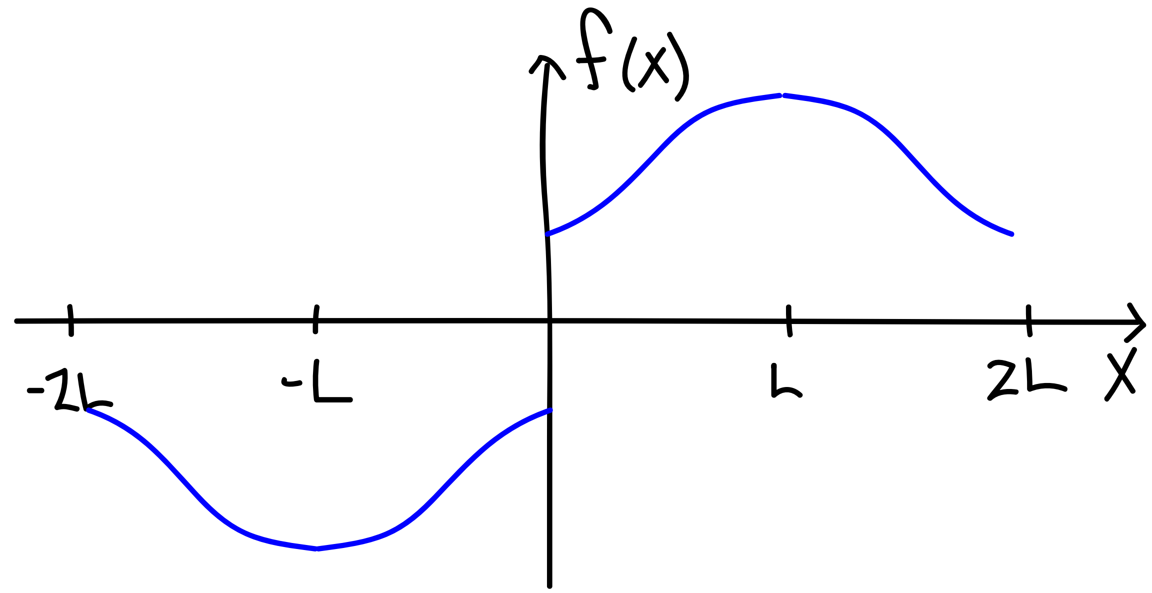

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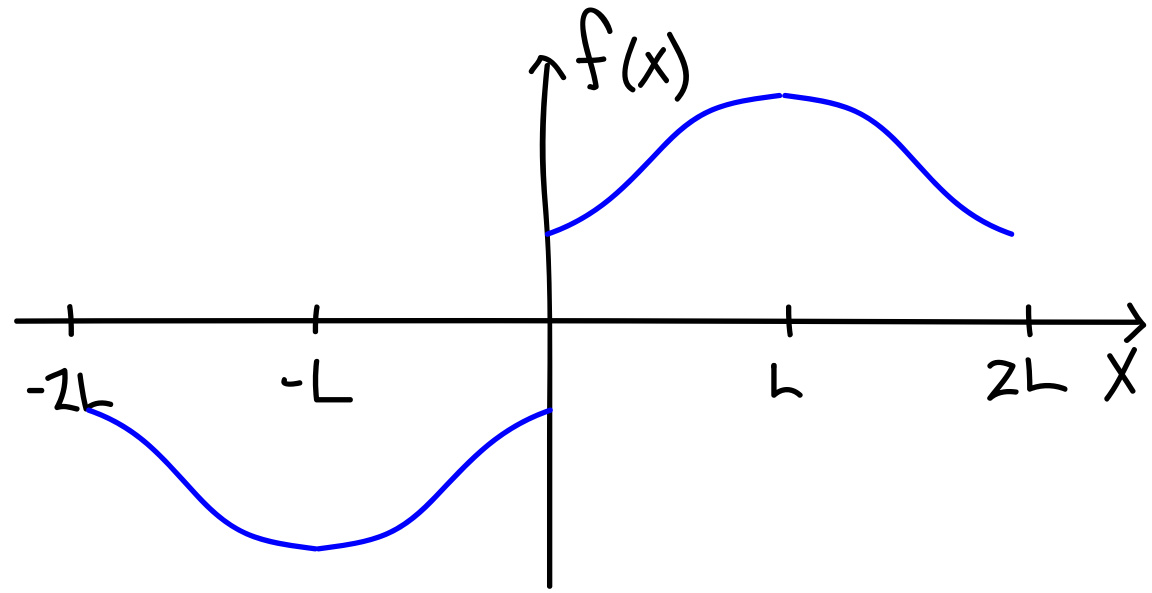

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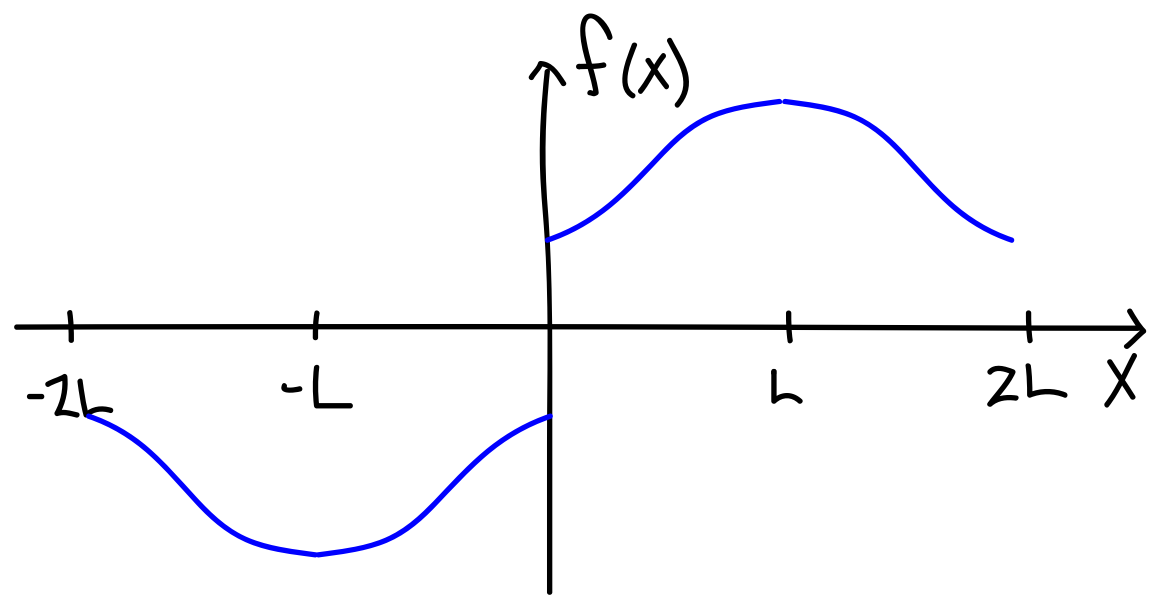

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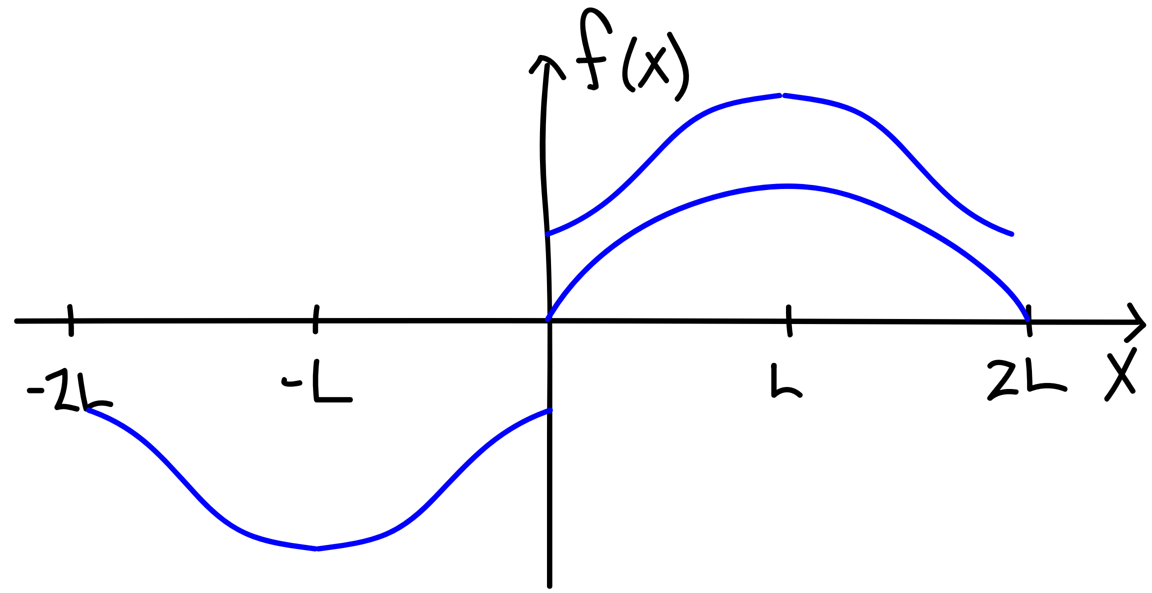

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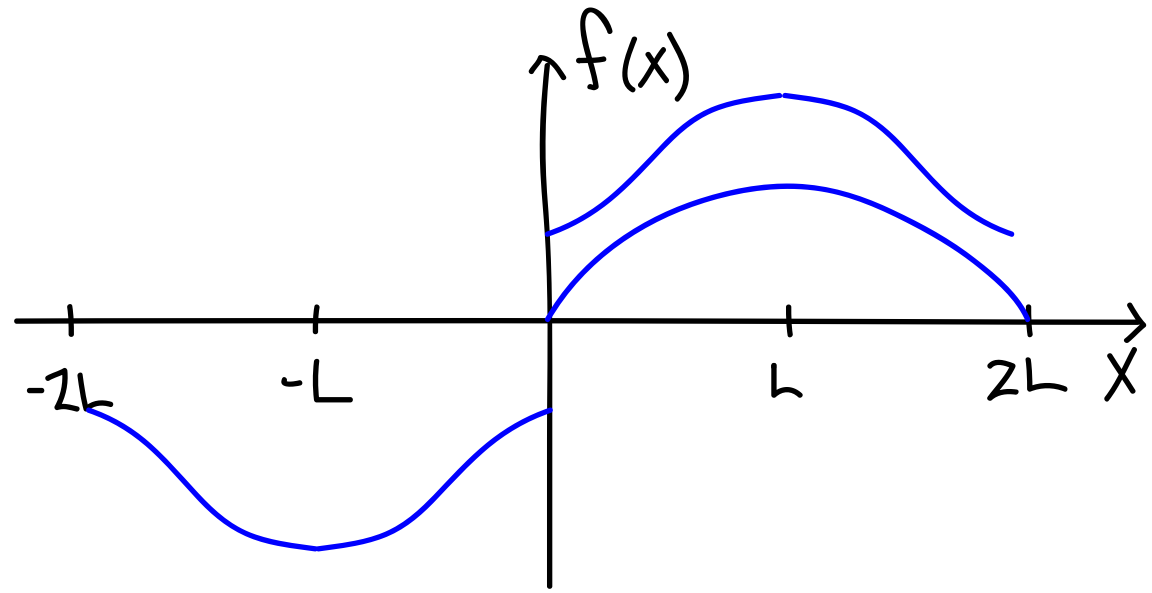

$$f(x) = \cancel{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2L}} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2L}$$

$$b_n = \frac{1}{2L} \int_{-2L}^{2L} f(x) \sin \frac{n\pi x}{2L}\, dx$$

$$= \frac{1}{L} \int_{0}^{2L} f(x) \sin \frac{n\pi x}{2L}\, dx \quad = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{2L}\, dx$$

$$\text{(for } n \text{ odd)}$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

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$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

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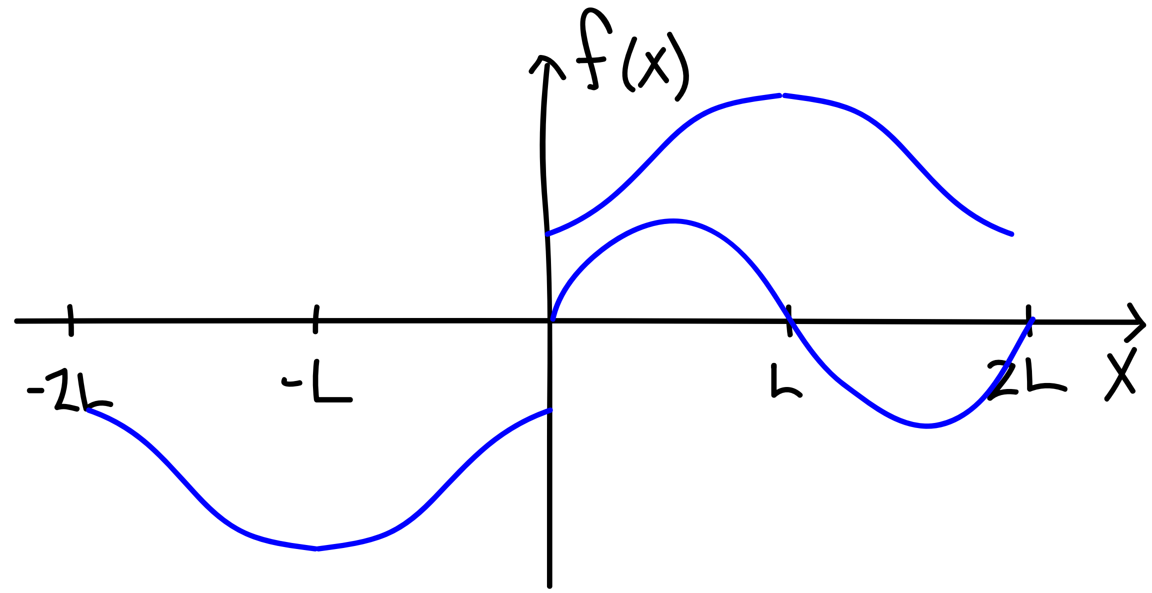

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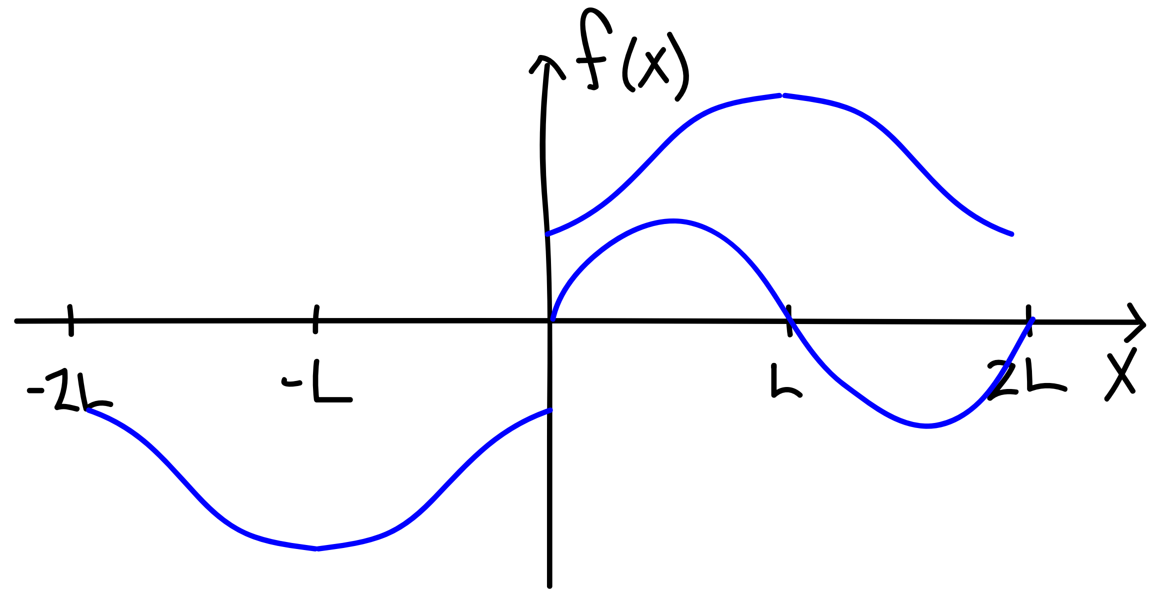

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The Wave Equation

- Motion of a string

The Wave Equation

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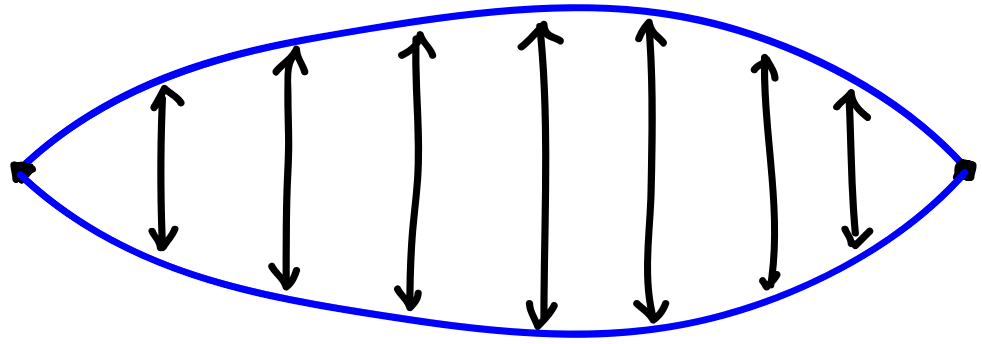

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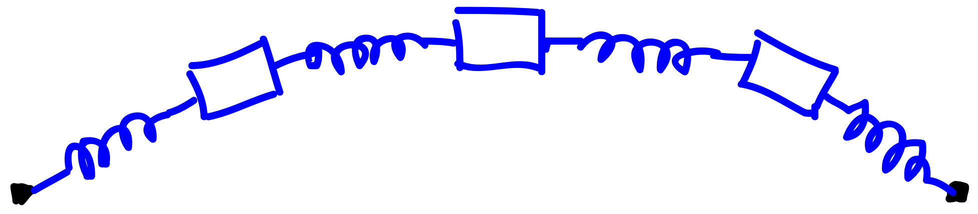

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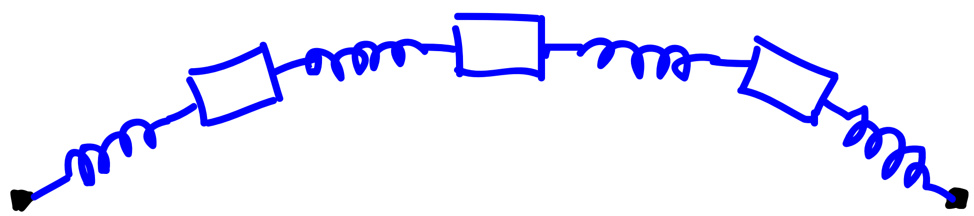

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The Wave Equation

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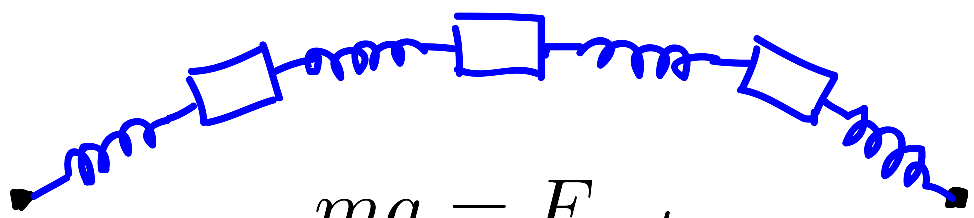

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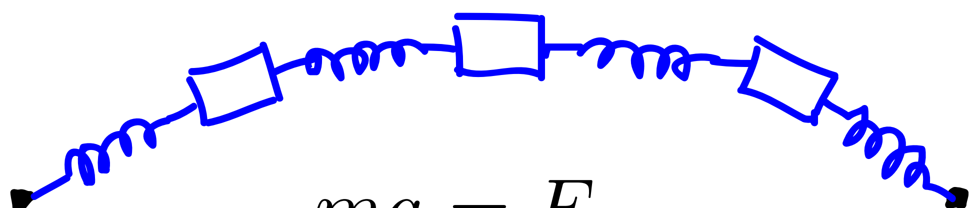

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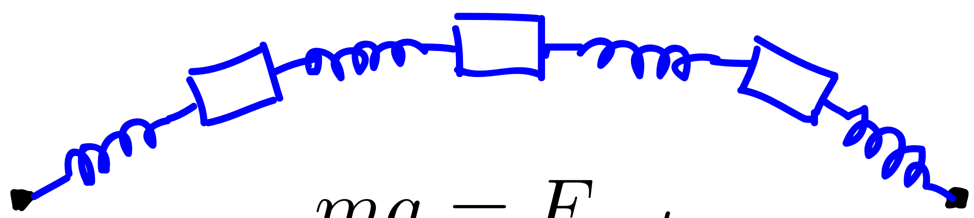
(not obvious - requires more detail)

The Wave Equation

- Motion of a string



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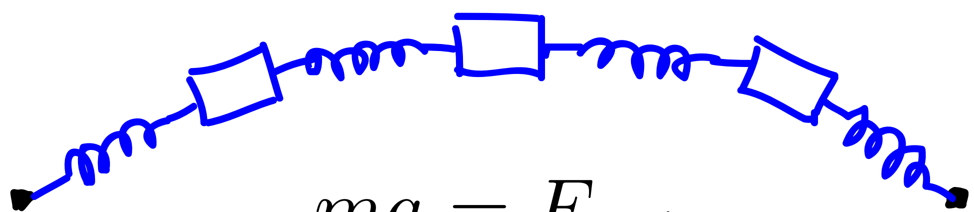

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The Wave Equation

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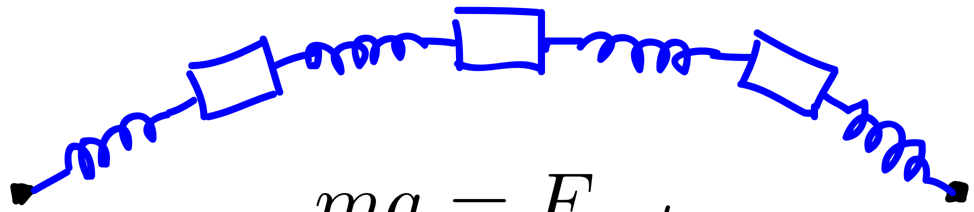

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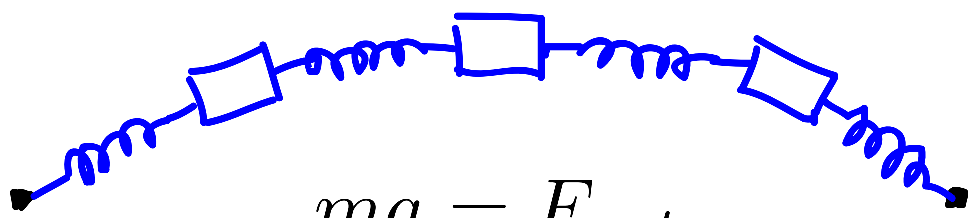

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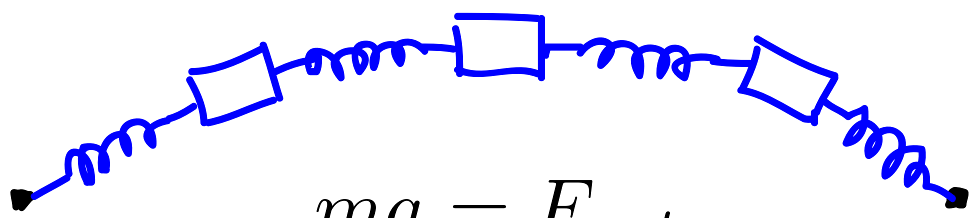

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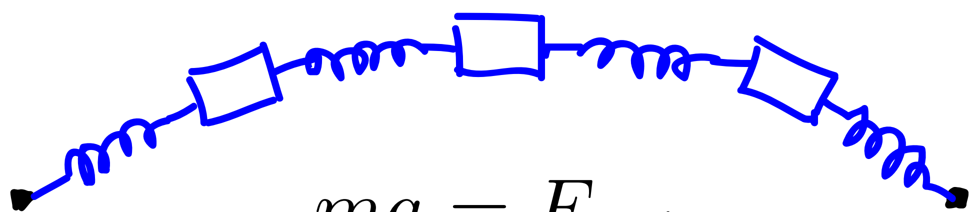

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The Wave Equation

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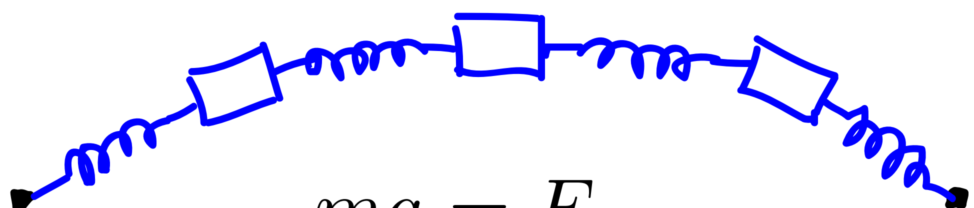

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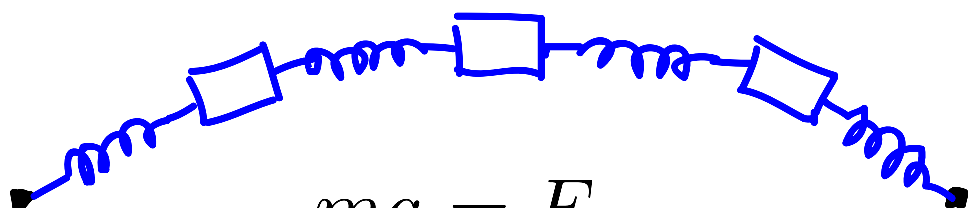

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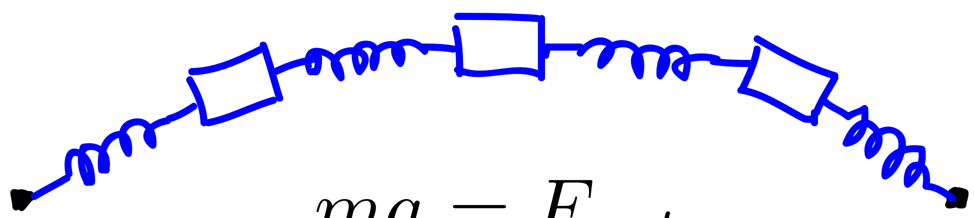

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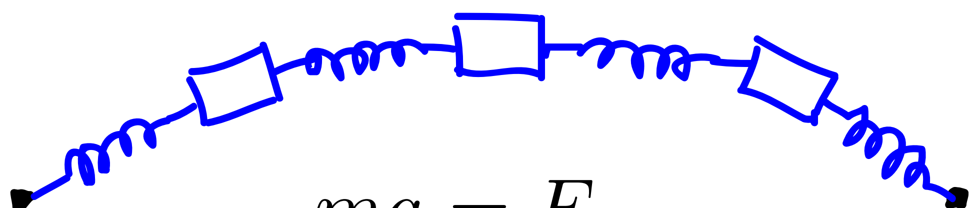
The Wave Equation

- Motion of a string



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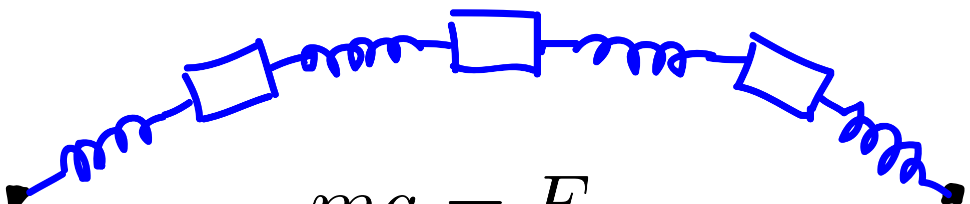

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$$u(0,t) = 0$$

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ICs: $u(x,0) = f(x), \; \dfrac{\partial}{\partial t}u(x,0) = g(x)$

The Wave Equation

- Other physical systems described by the wave equation:
- Sound waves - u(x,t) is the air pressure, c is speed of sound.
- Water waves - u(x,t) is water height, c is wave speed.
- Electromagnetic waves - u(x,t) is field intensity, c is speed of light ...

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Separation of variables

- There is a way to calculate eigenvalues as we did for the Diffusion Equation but that requires rewriting it as a first order system:

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

find evalues/vectors of this operator

- Easier approach - separation of variables...

- Inspired by the fundamental solutions to the Diffusion Equation, assume

$$u_n(x,t) = T(t)v(x)$$

$$u_n(x,t) = e^{\lambda_n t}\sin\frac{n\pi x}{L}$$

$$T(t) = e^{\lambda t}$$

Expect oscillations so

$$T(t) = e^{i\beta t}$$

(or sines and cosines)

Separation of variables

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https://www.desmos.com/calculator/dmfijt0e6q

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$$u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

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