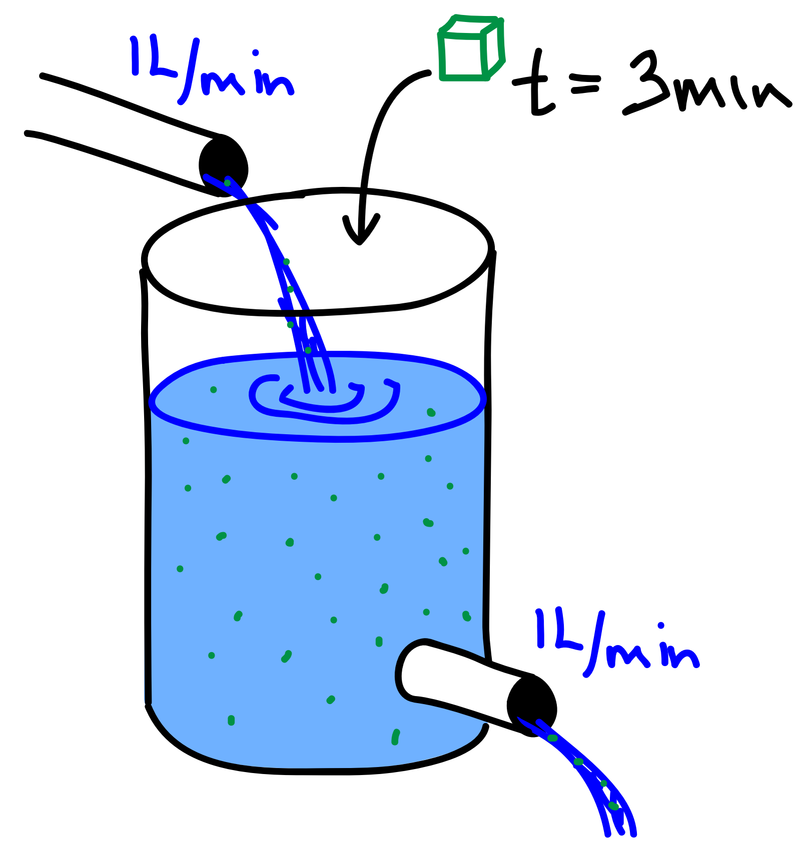
Today

- Modeling with delta-function forcing
- Convolution
- Transfer functions

Delta-function forcing (6.5)

- Water with c_{in} = 2 g/L of sugar enters a tank at a rate of r = 1 L/min. The initially sugar-free tank holds V = 5 L and the contents are well-mixed. Water drains from the tank at a rate r. At t_{cube} = 3 min, a sugar cube of mass m_{cube} = 3 g is dropped into the tank.



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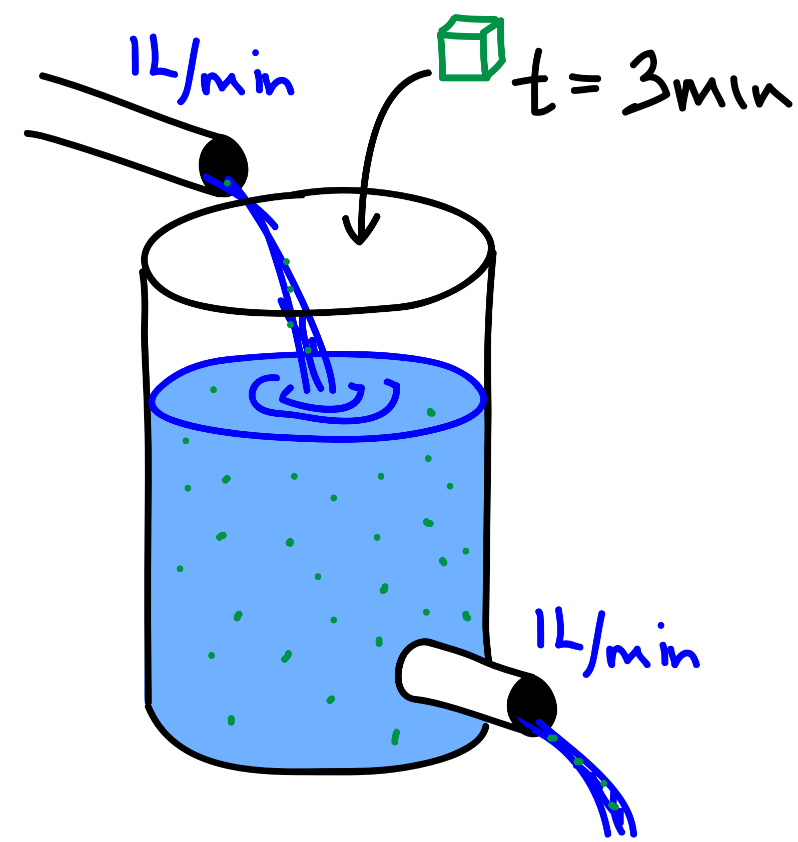

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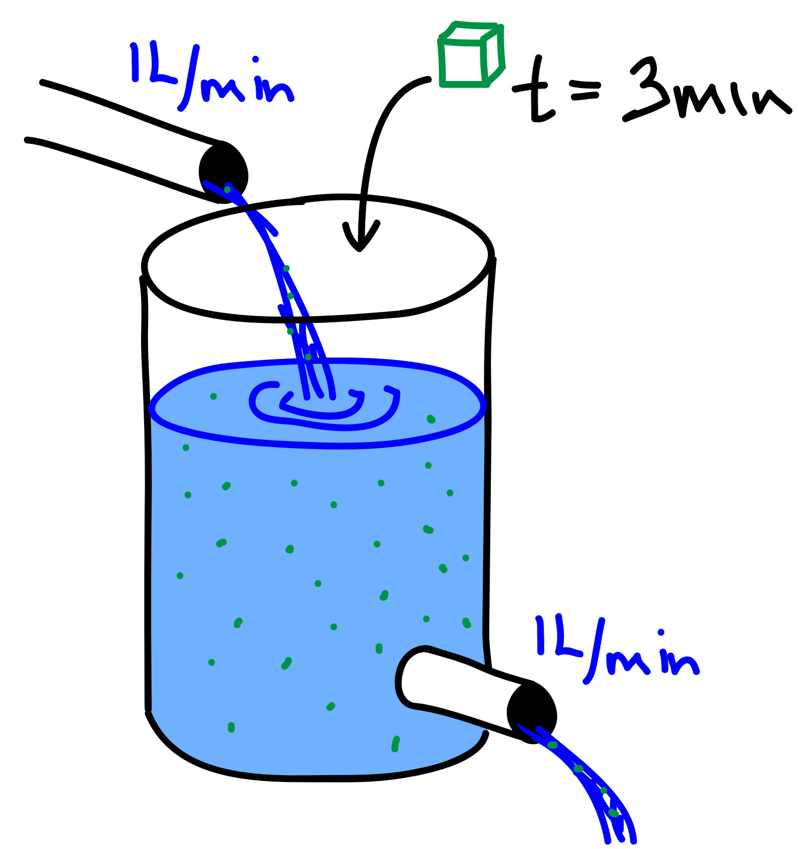

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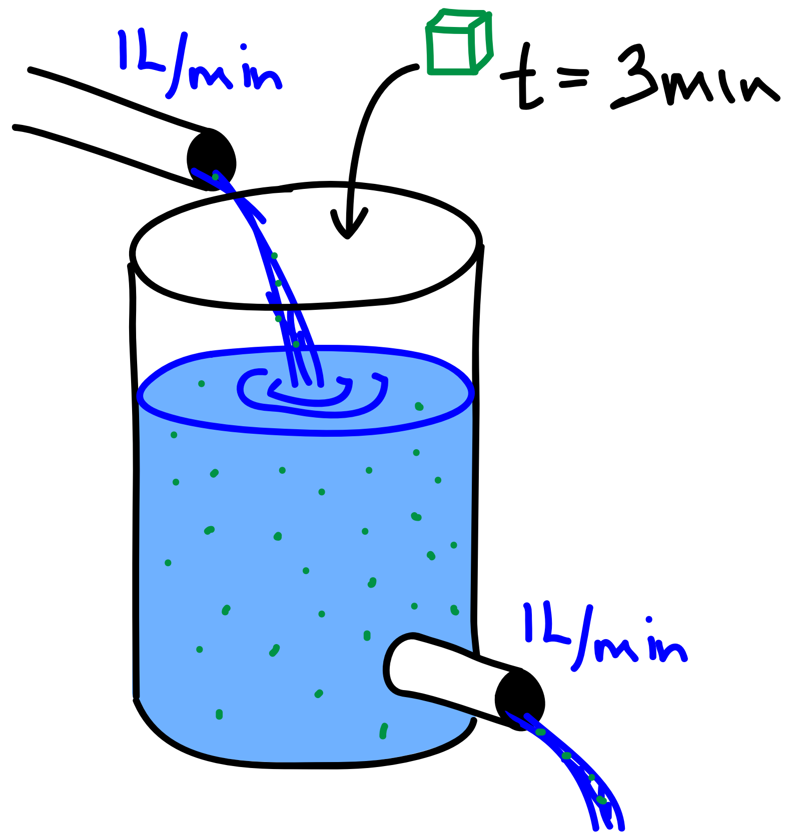

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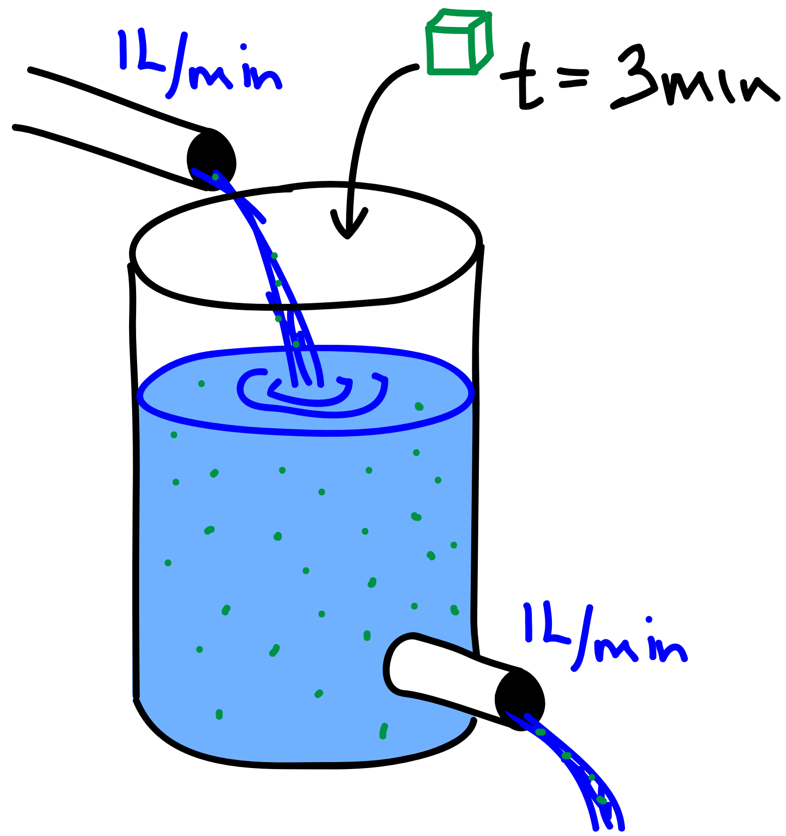

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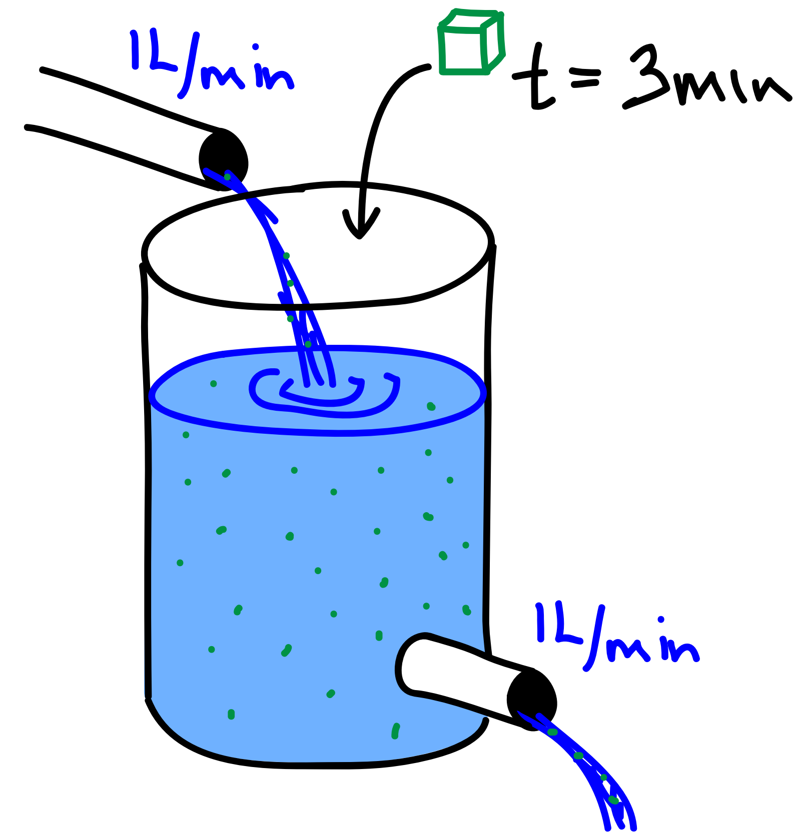

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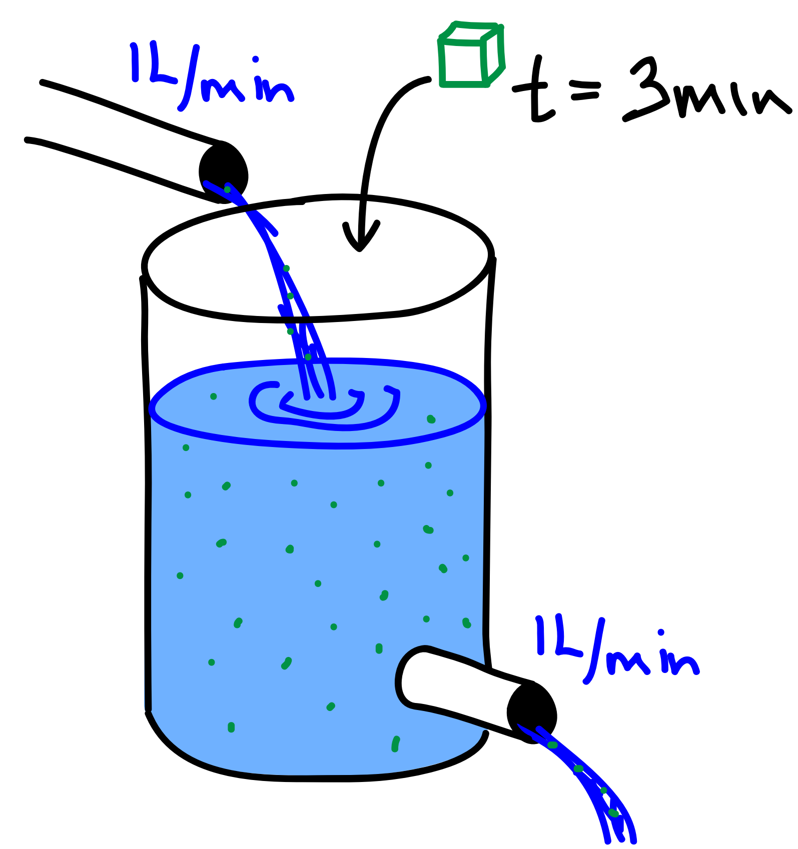

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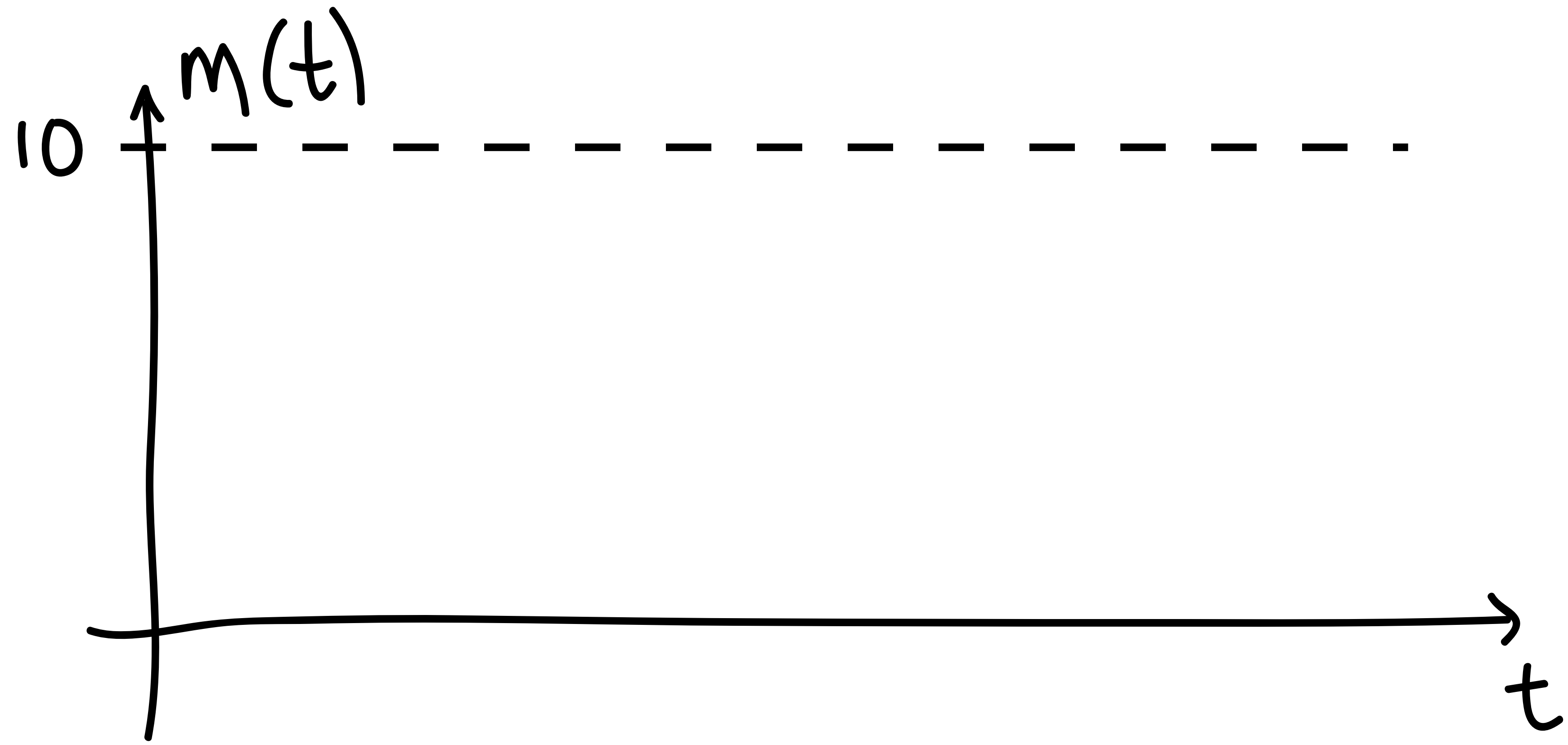

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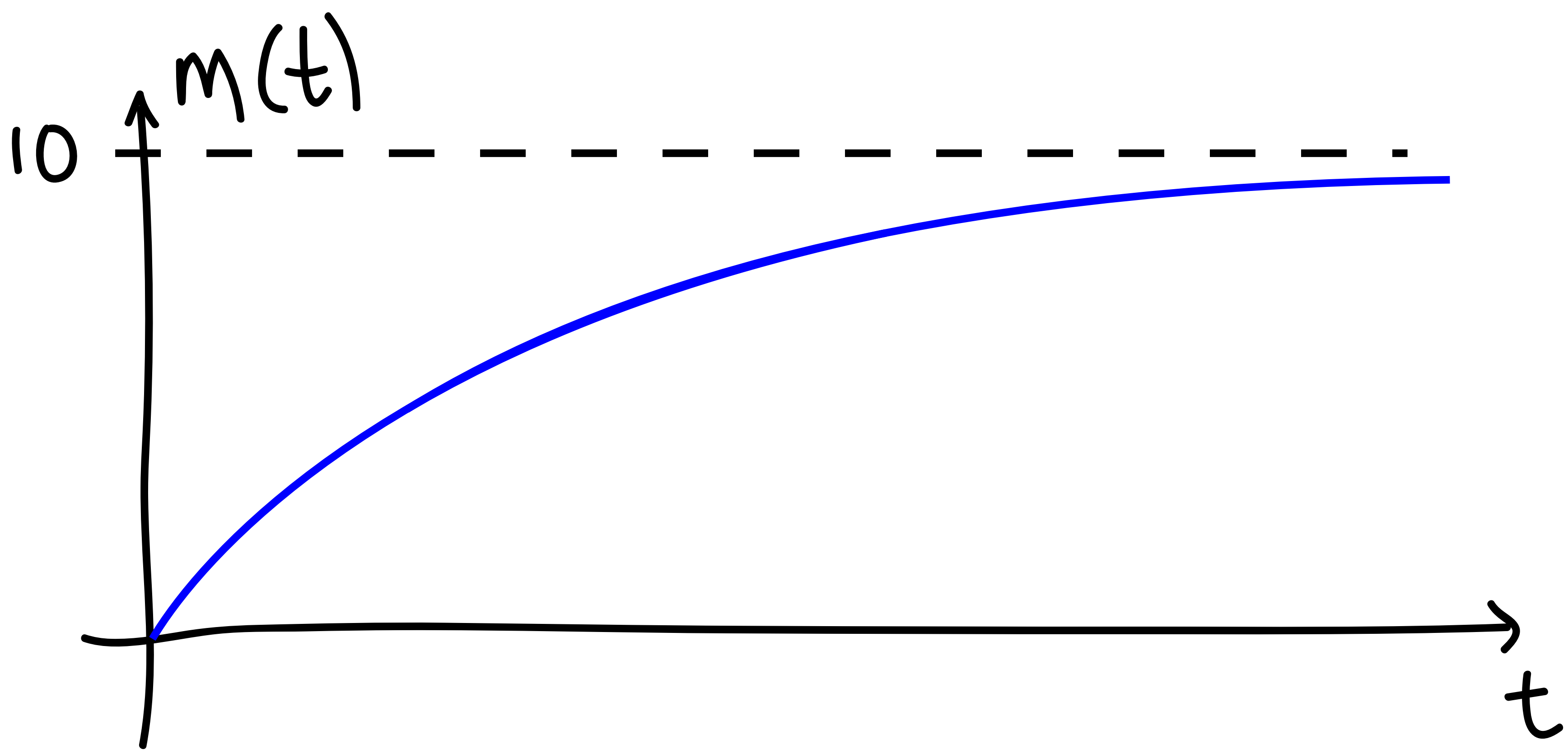

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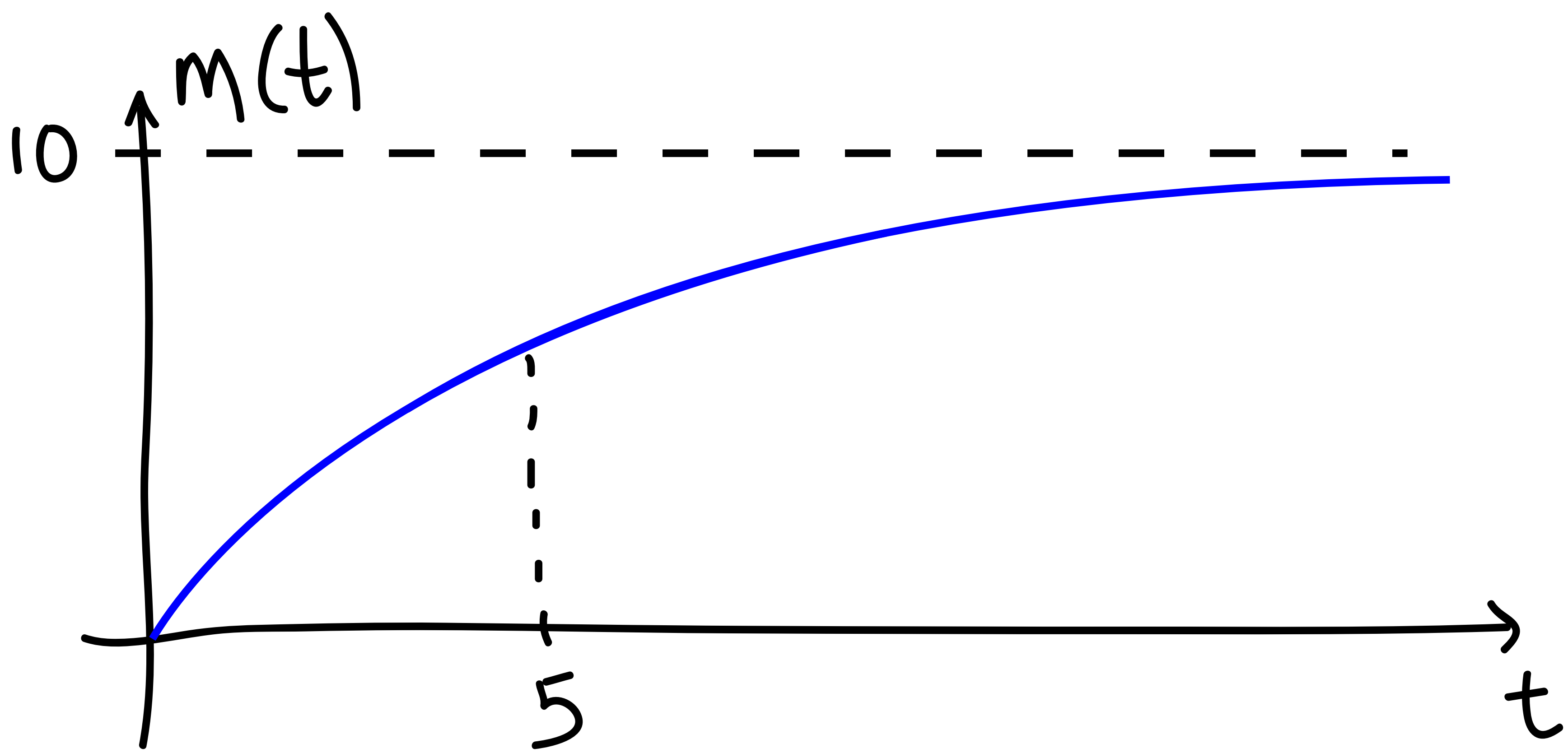

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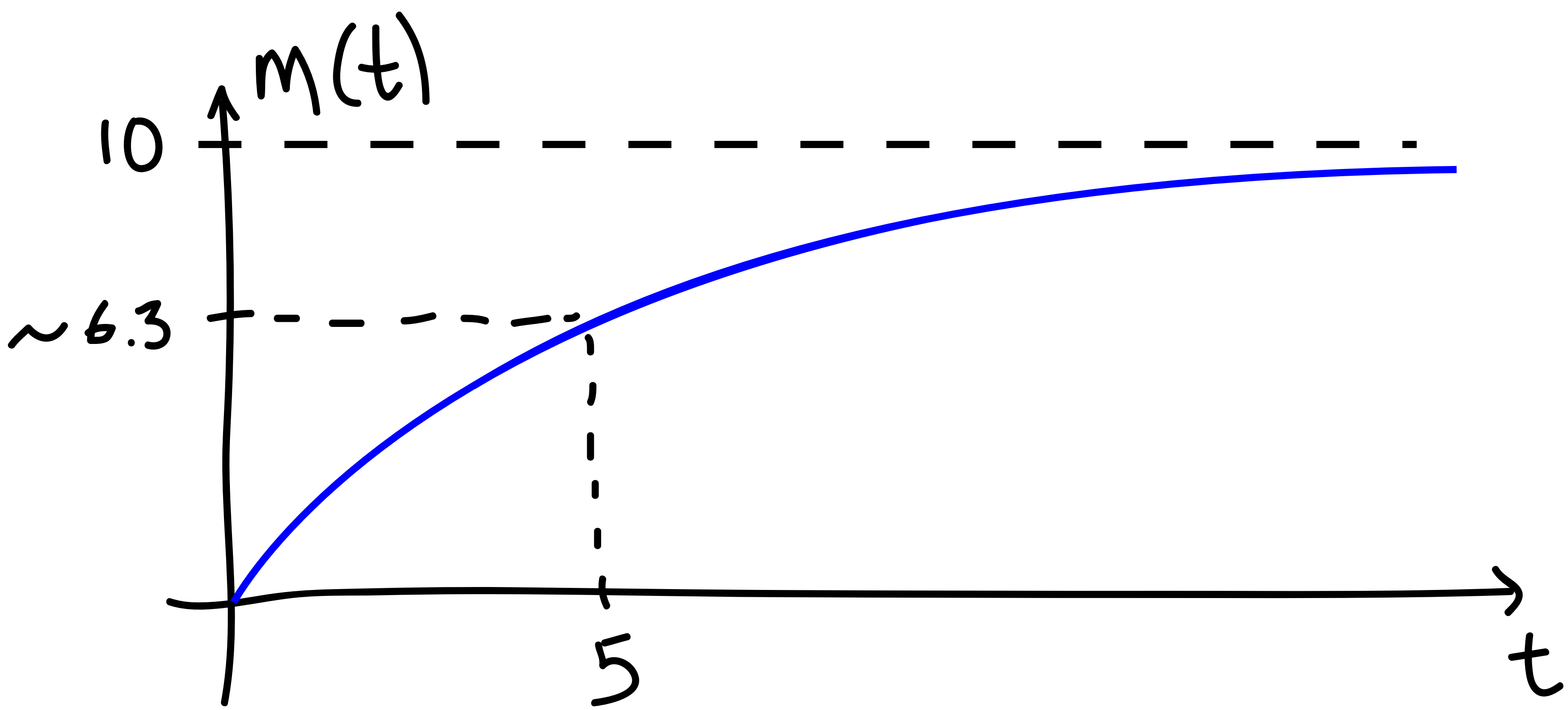

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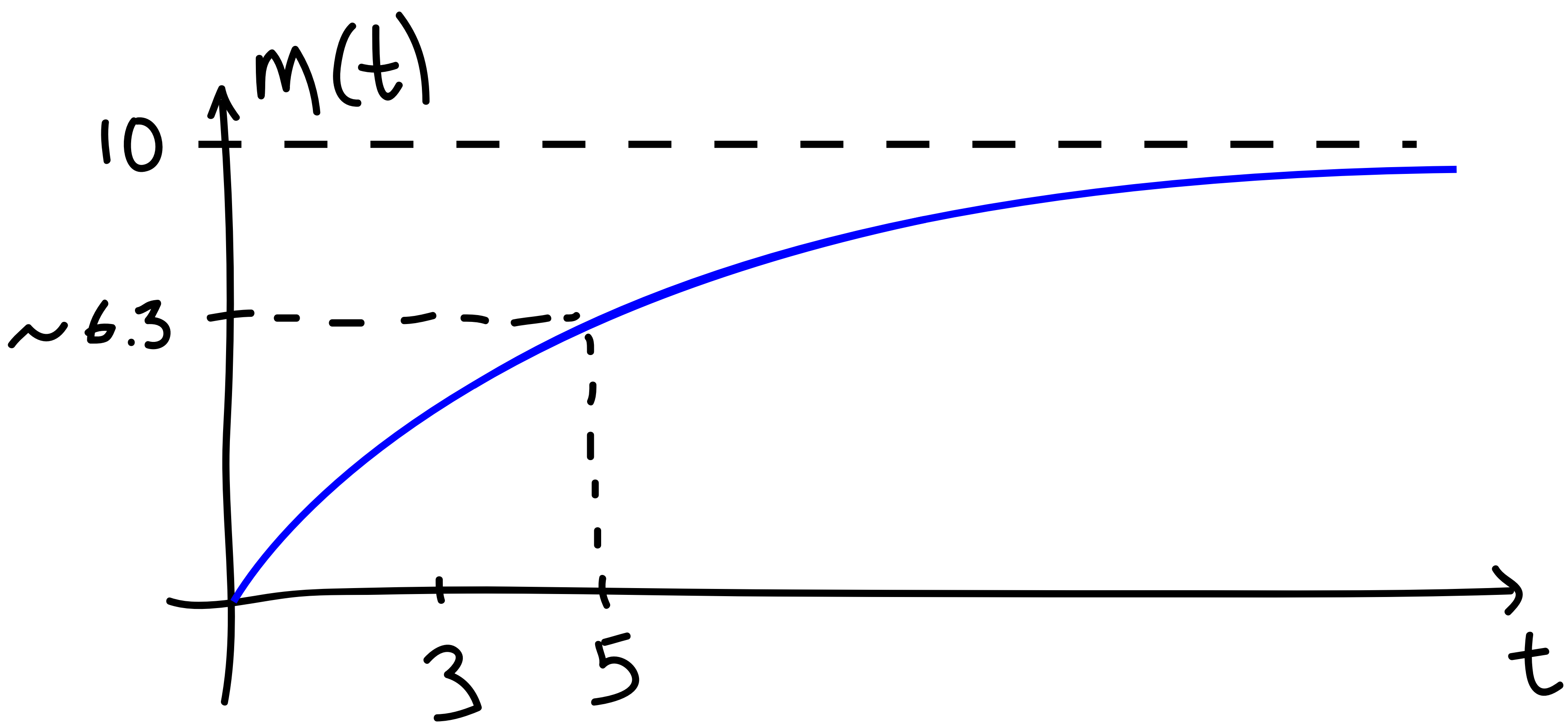

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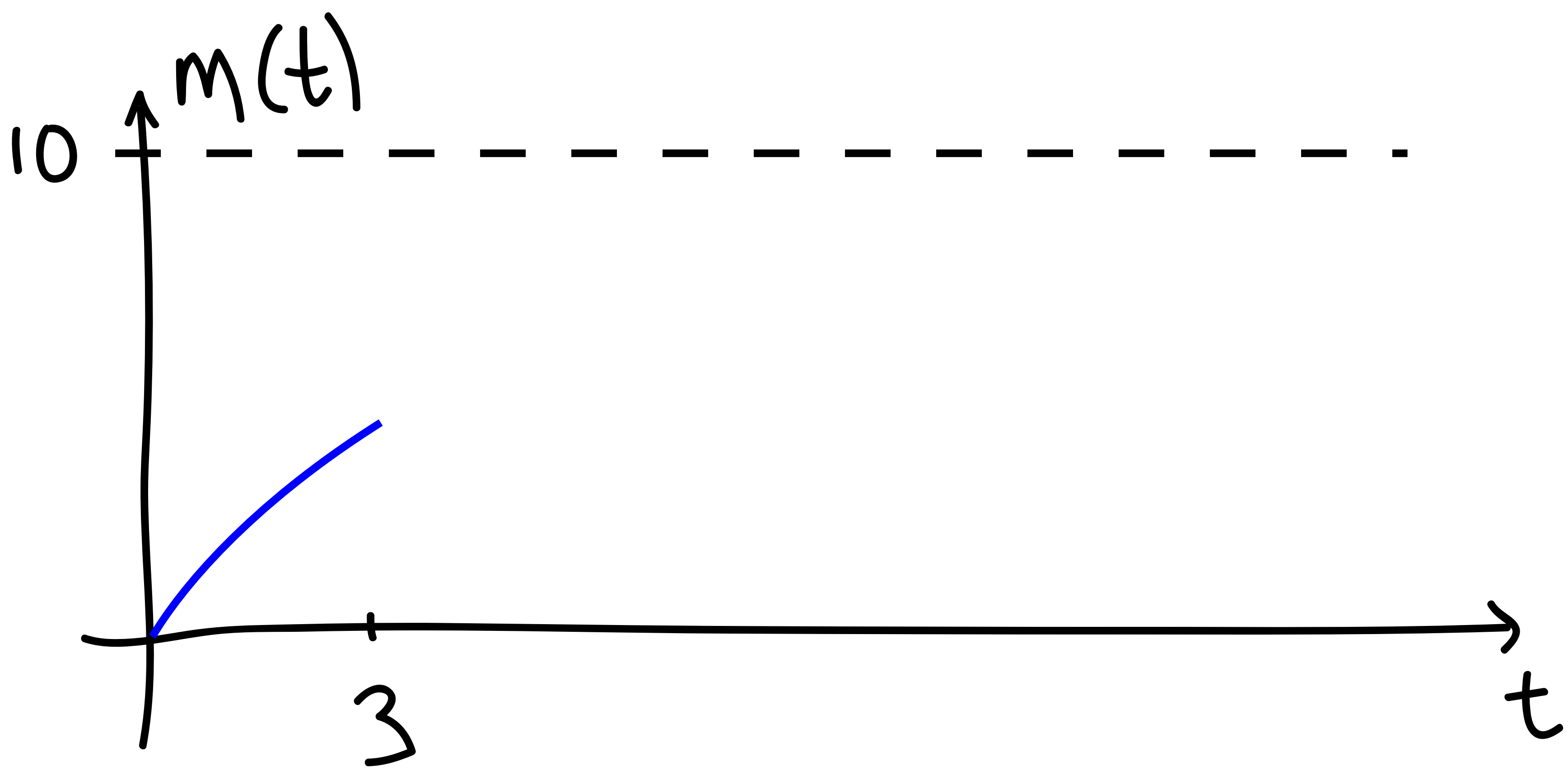

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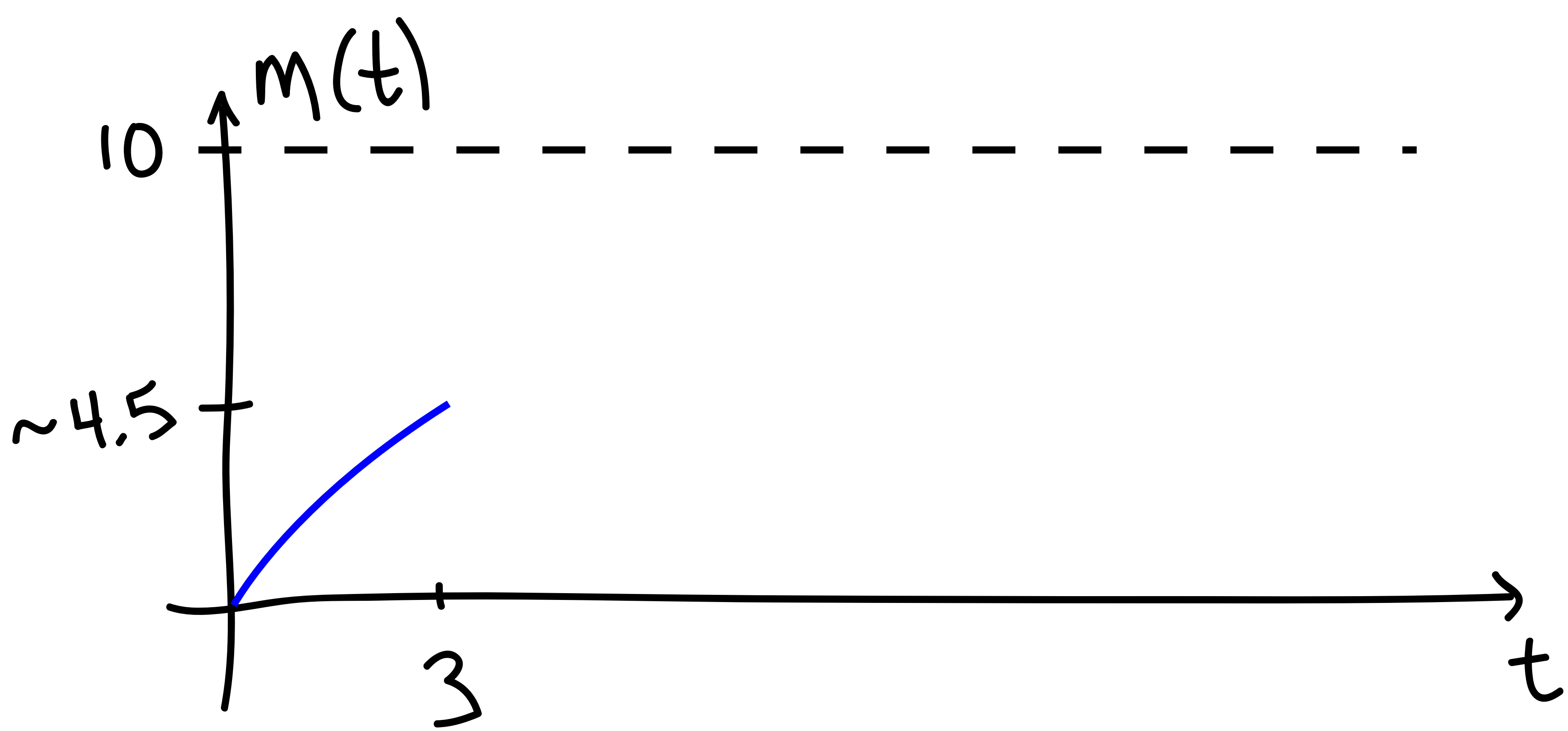

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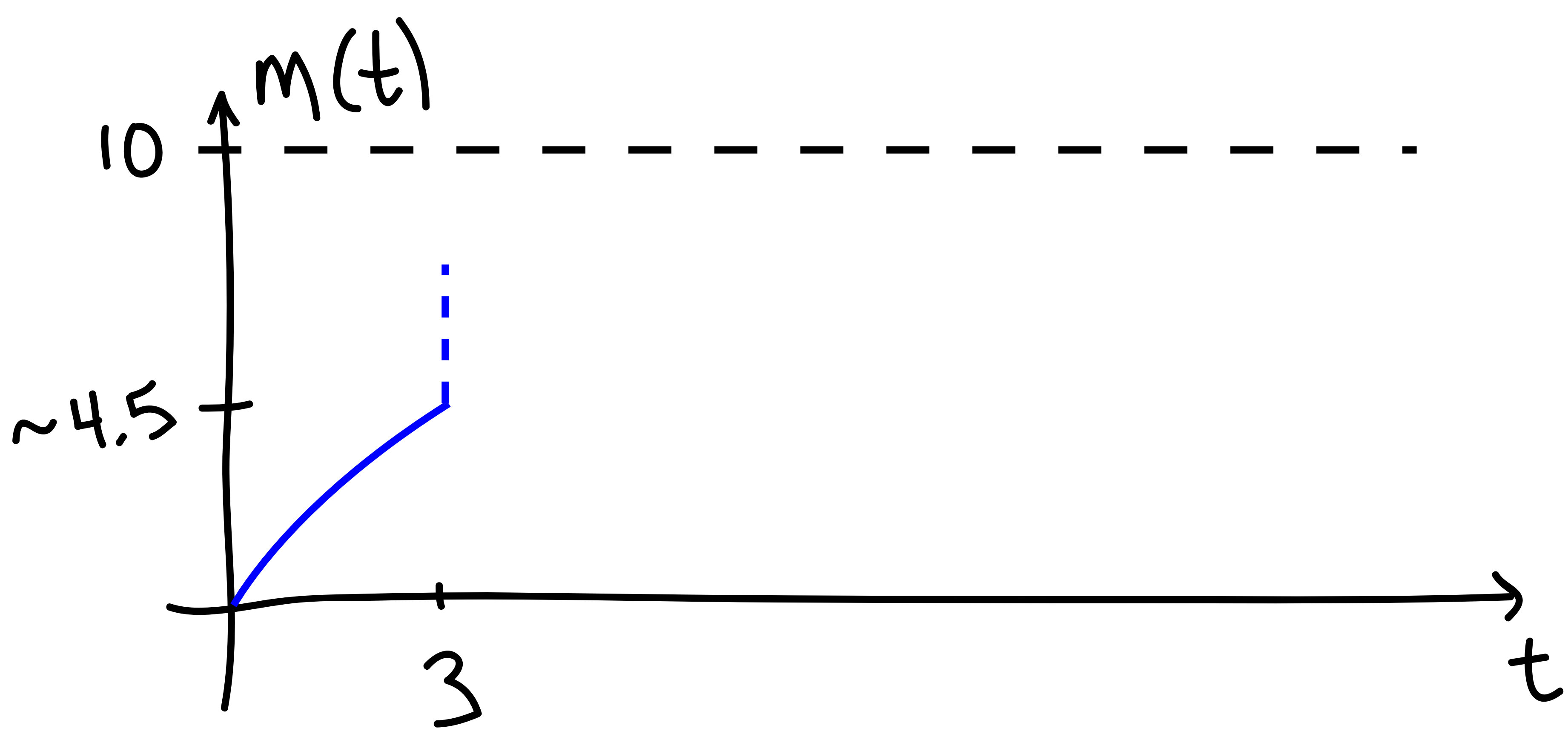

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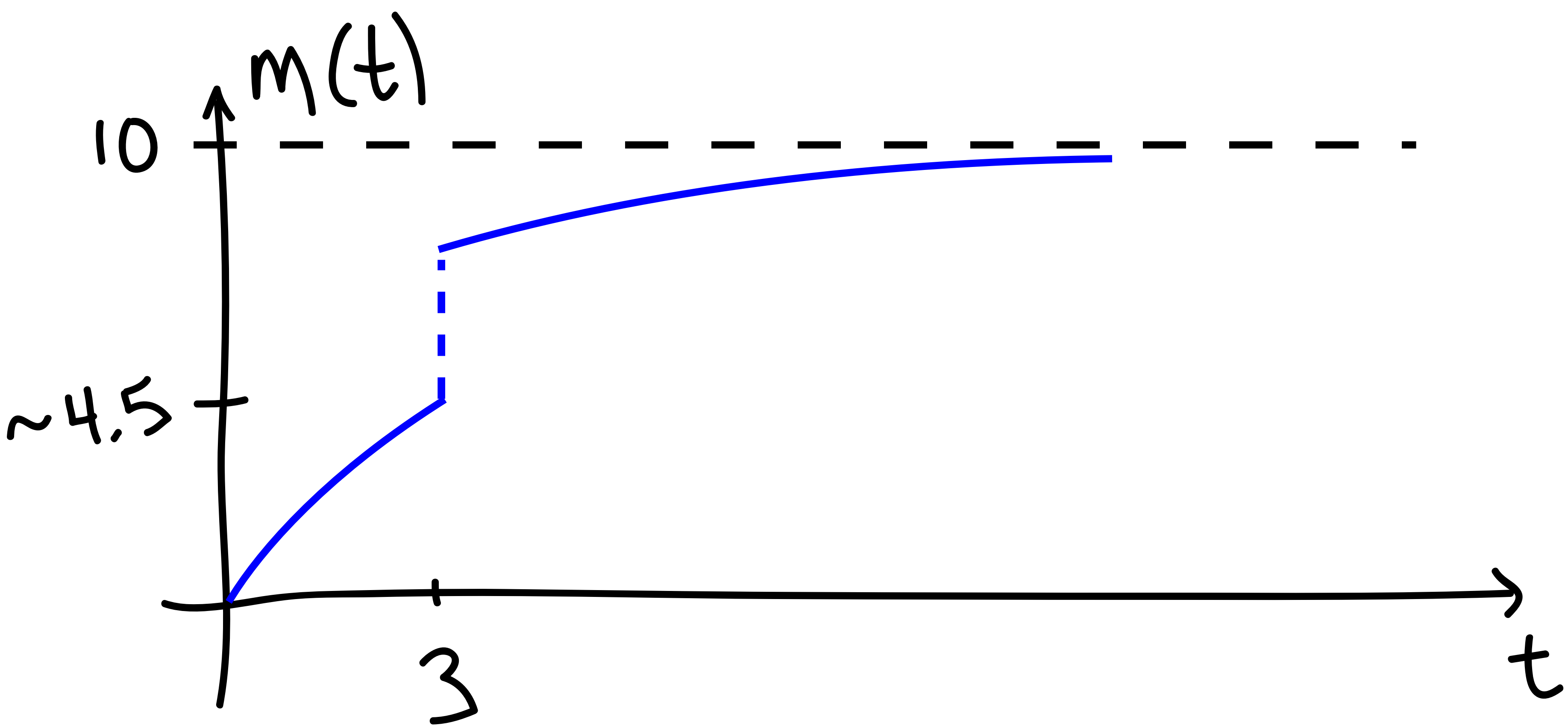

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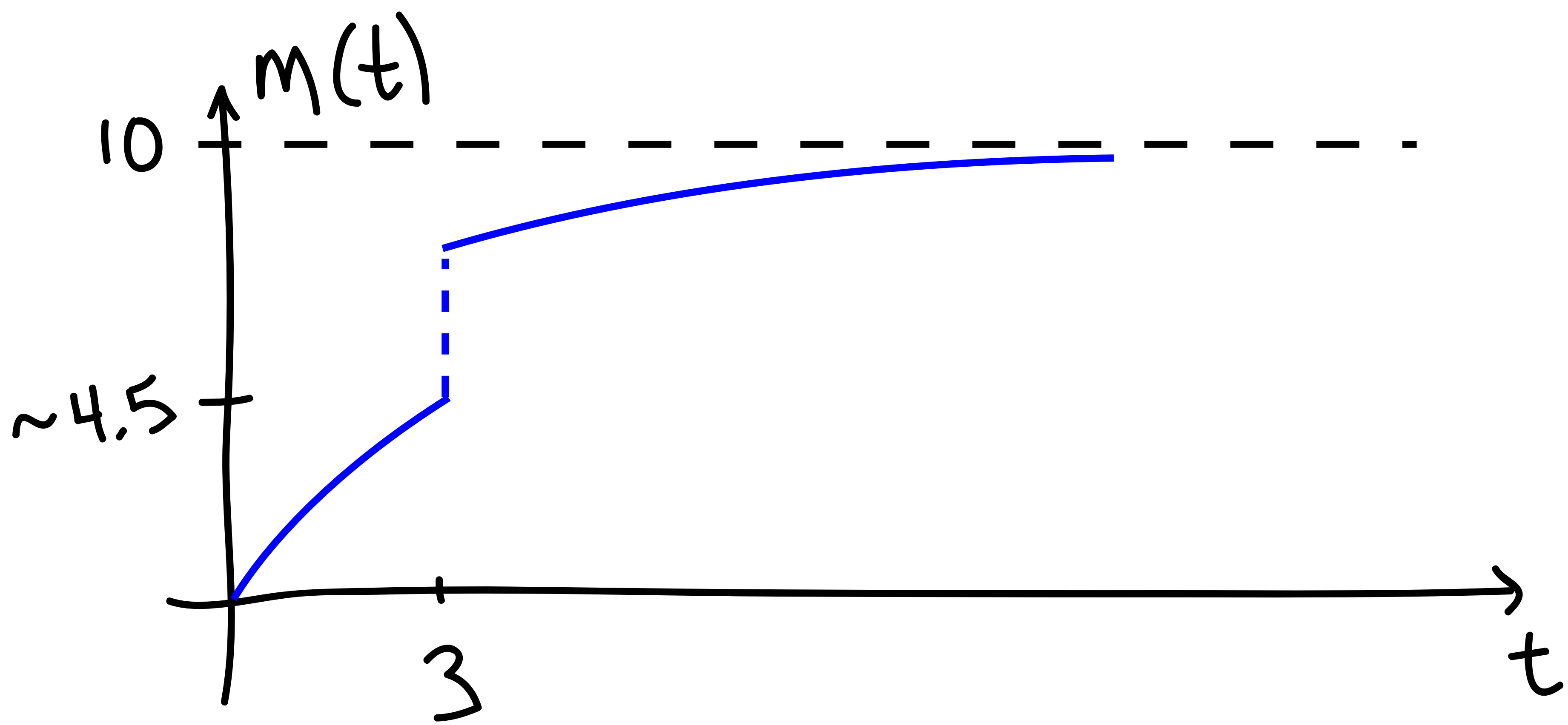

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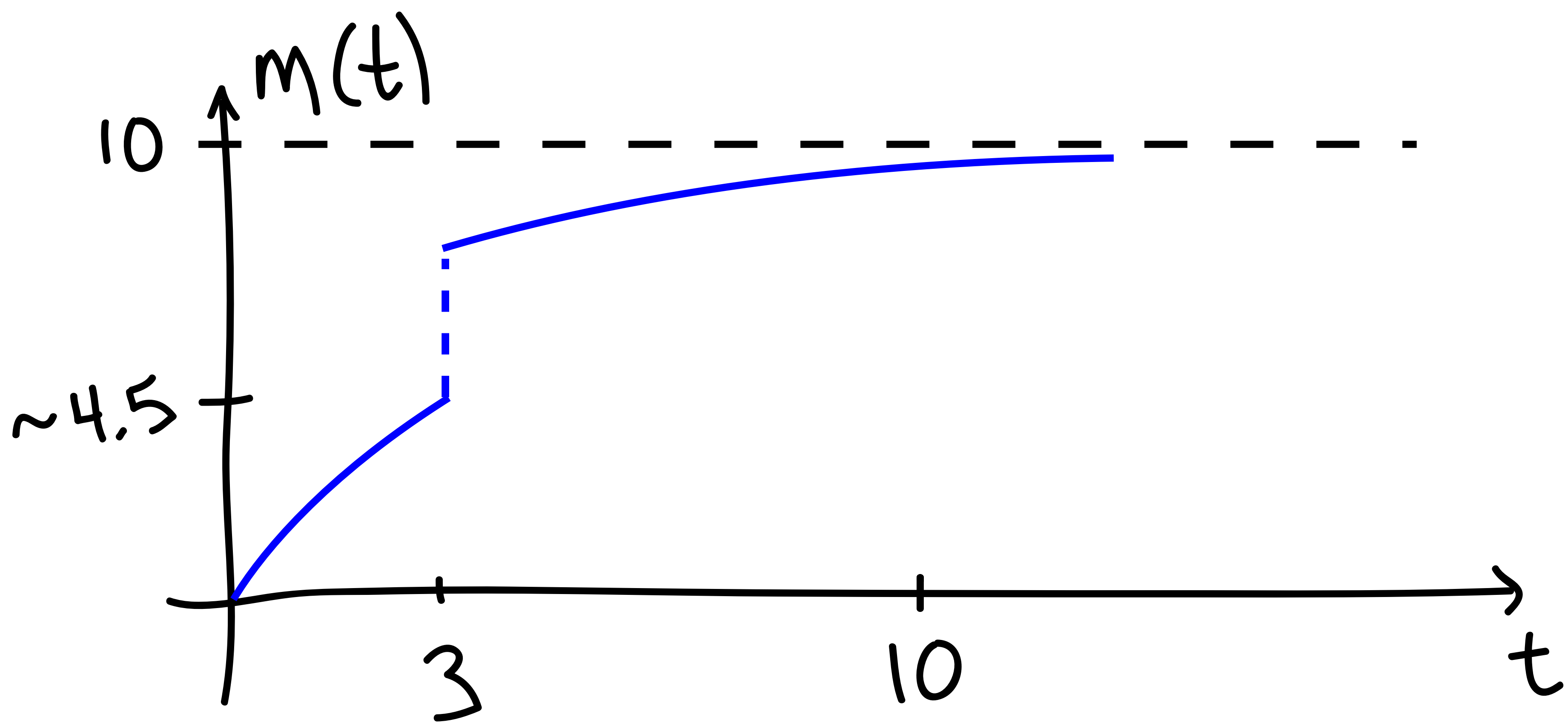

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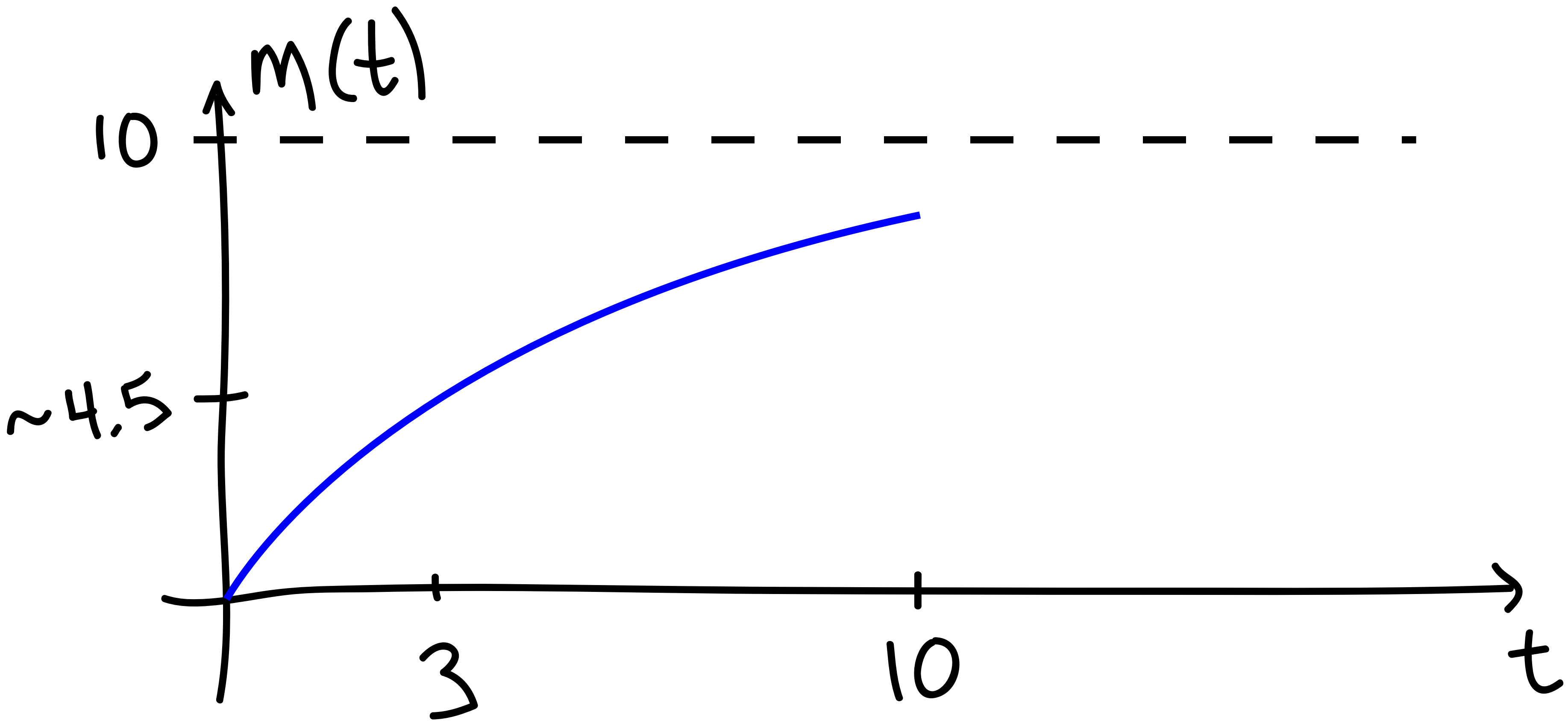

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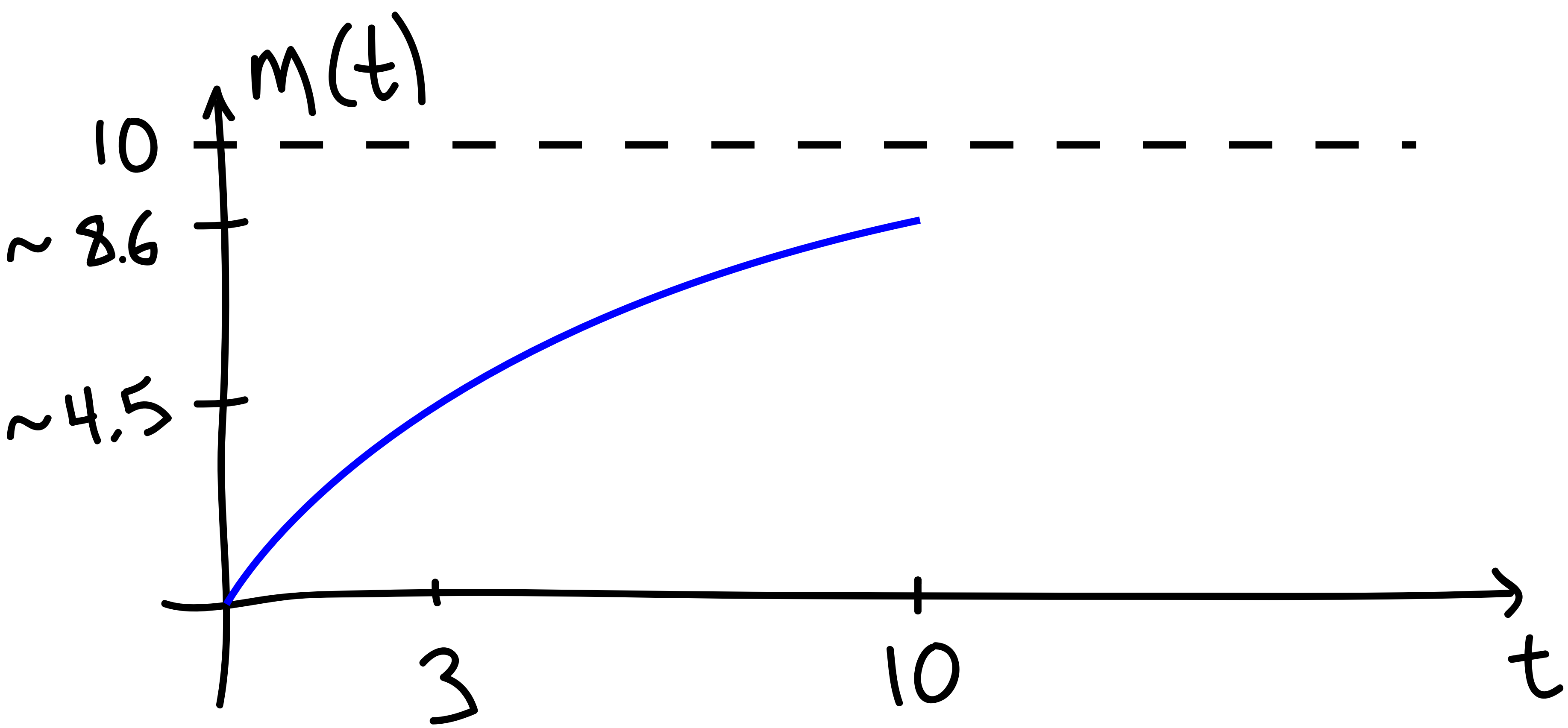

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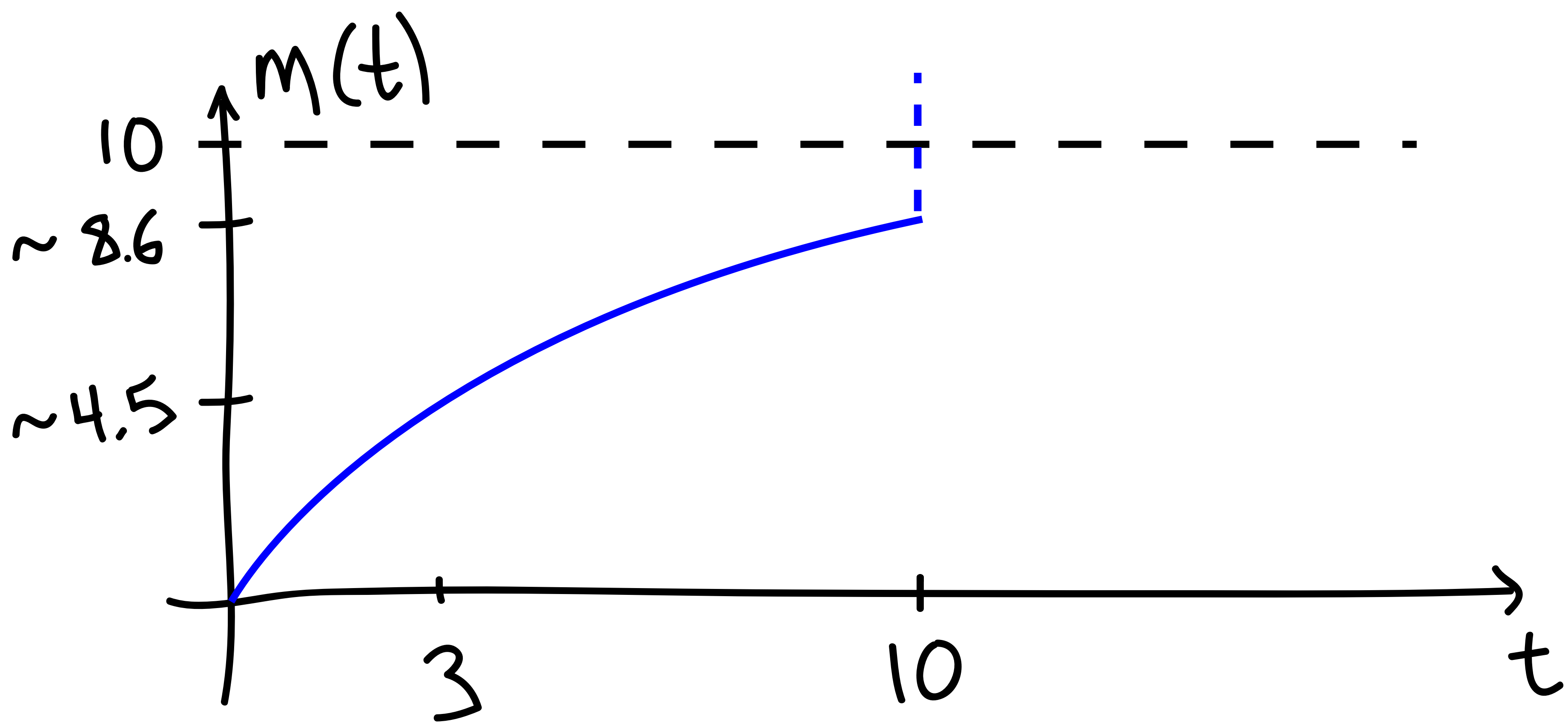

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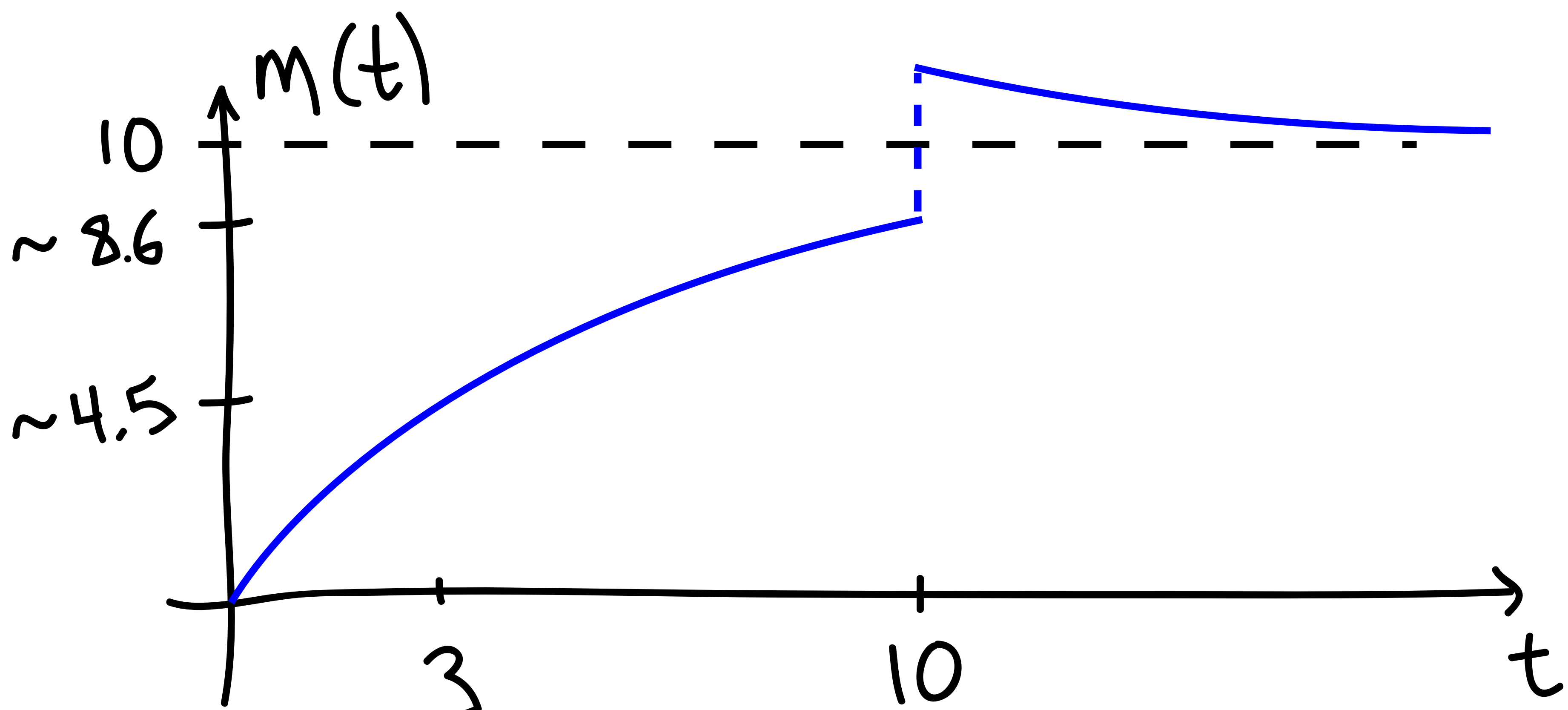

https://www.desmos.com/calculator/eizjtgf3jc

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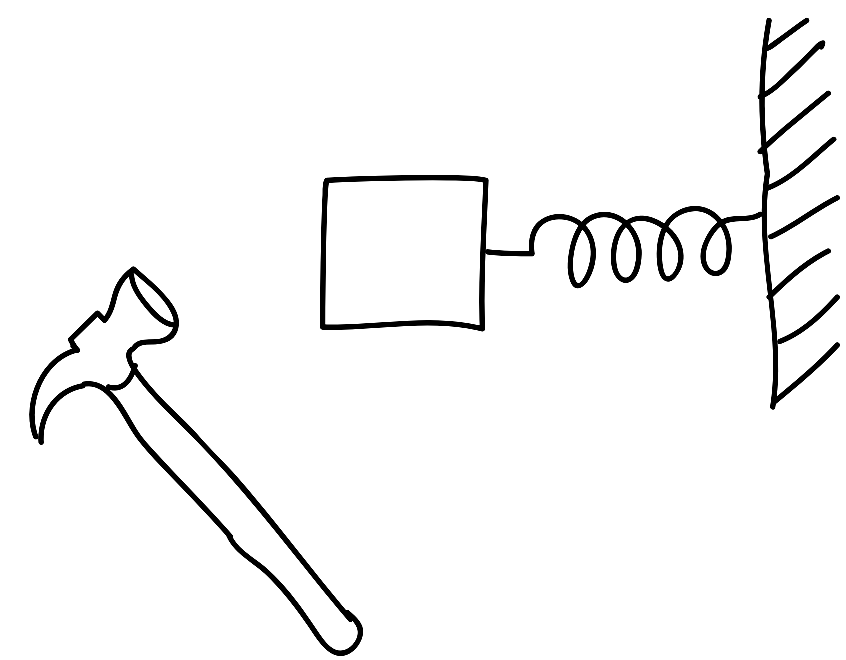

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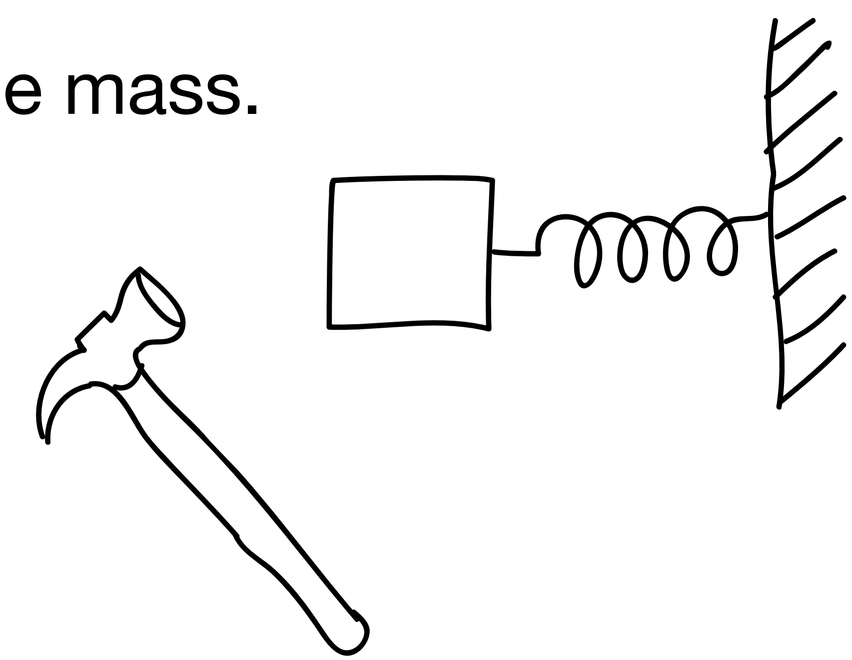

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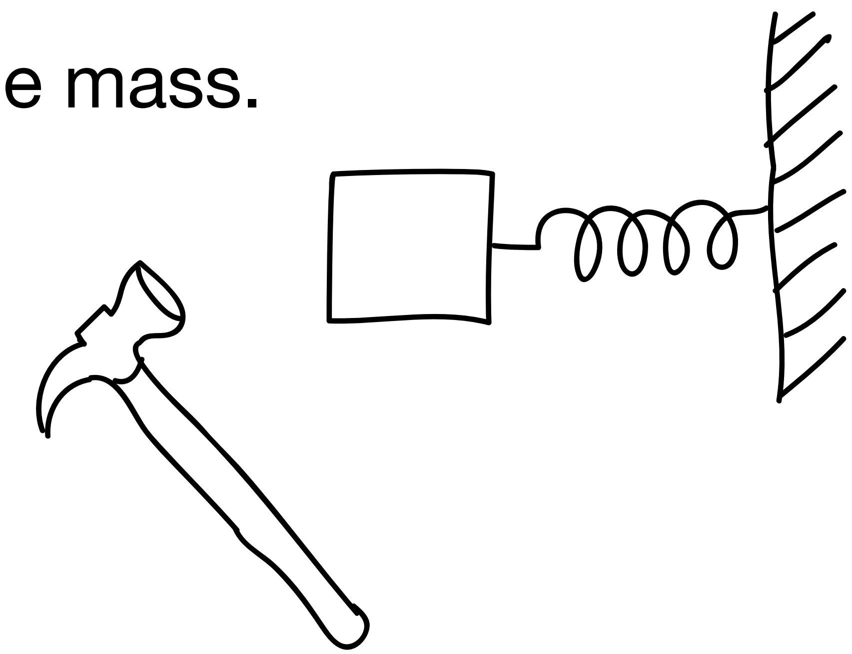

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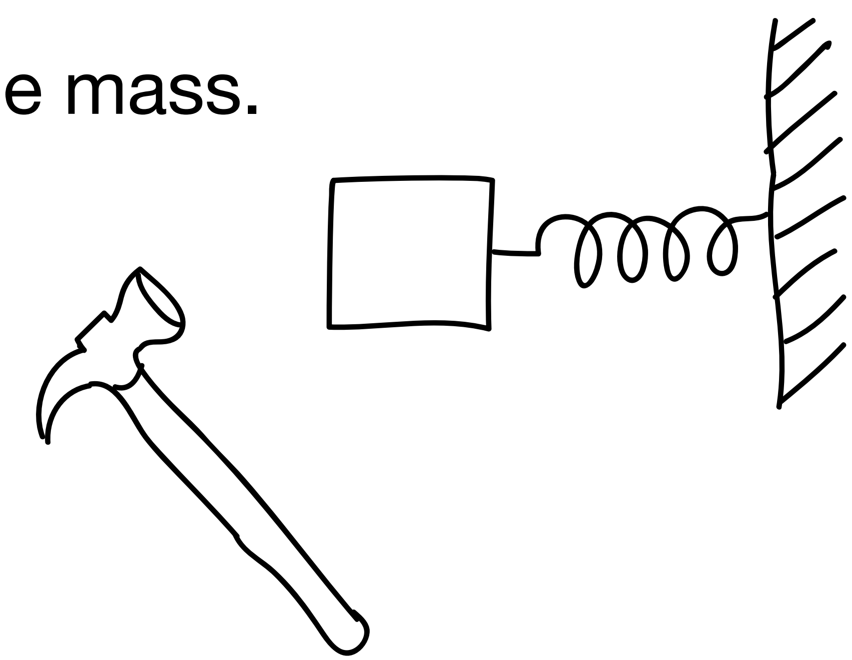

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(B) $y'' + 2y' + 10y = 2\ (u_0(t) - u_5(t))$

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(D) $y'' + 2y' + 10y = 2\ \delta(t-5)$

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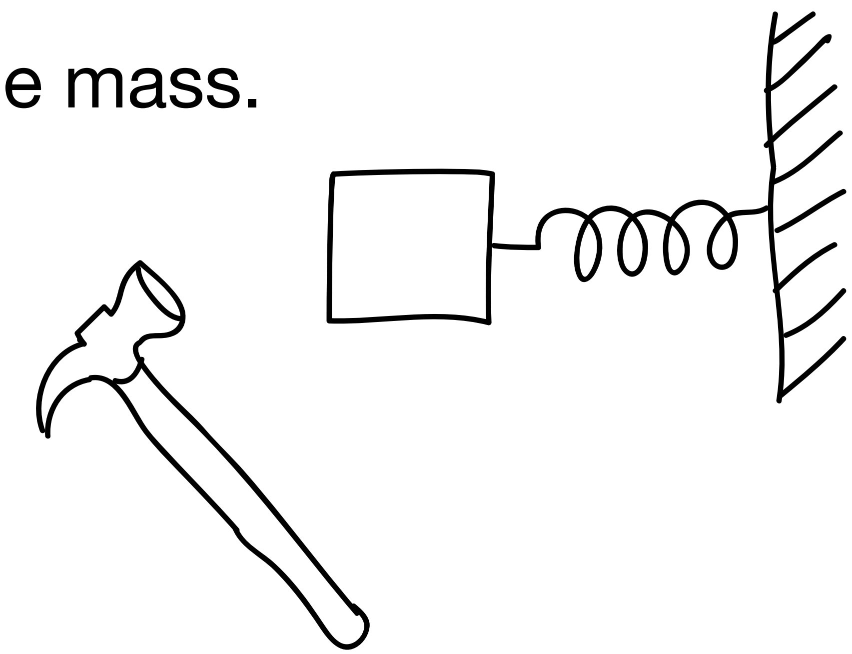

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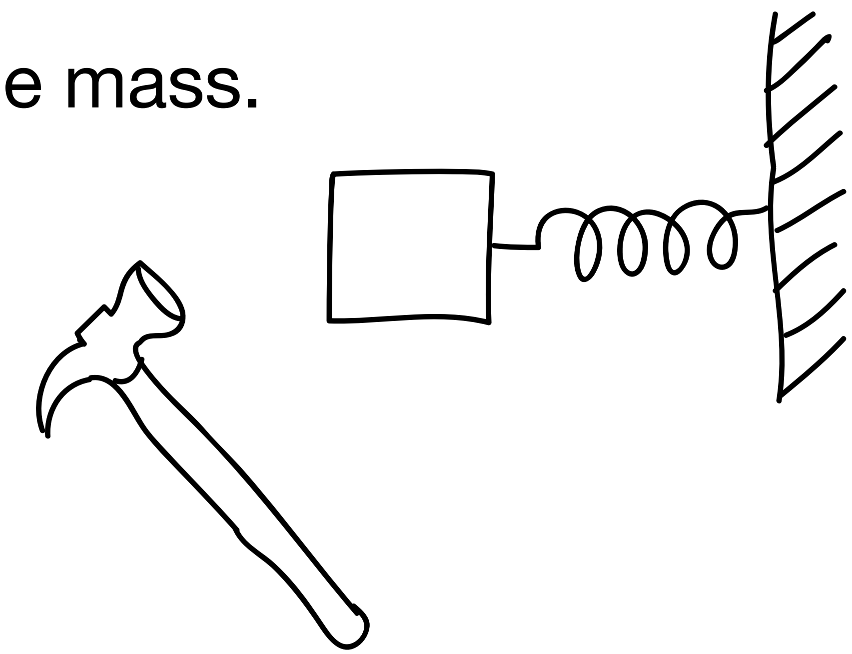

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$$s^2Y - 2s + 2sY - 4 + 10Y = 2e^{-5c}$$

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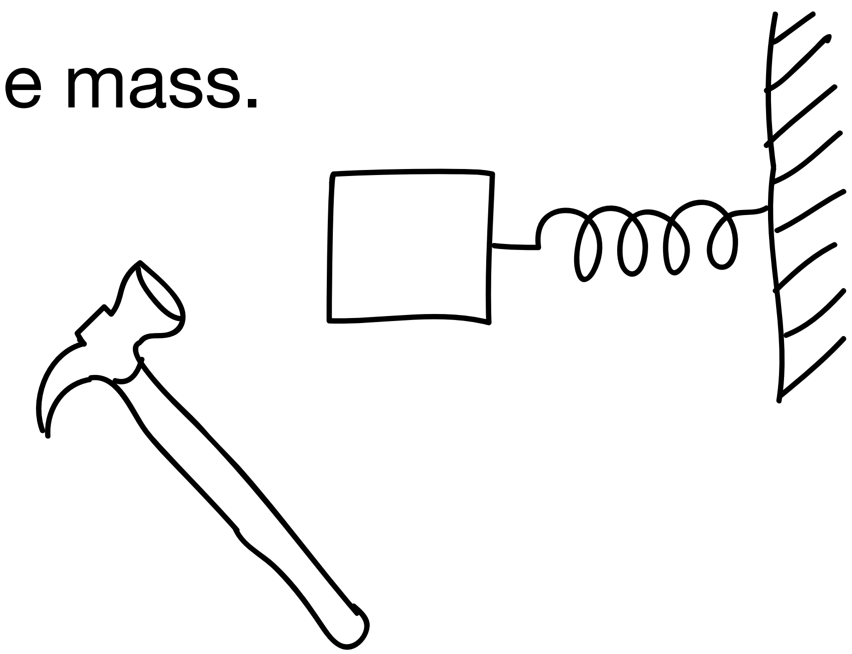

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$$s^2Y - 2s + 2sY - 4 + 10Y = 2e^{-5c}$$

$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10}$$

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- Inverting Y(s)... (go through this on your own)

$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10} = \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+2}{s^2 + 2s + 10}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+2}{(s+1)^2 + 9}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{(s+1)^2 + 9}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{3}\frac{3}{(s+1)^2 + 9}$$

$$= \frac{2}{3}\frac{3e^{-5s}}{(s+1)^2 + 9} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{3}\frac{3}{(s+1)^2 + 9}$$

$$y(t) = \frac{2}{3}u_5(t)e^{-(t-5)}\sin(3(t-5)) + 2e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)$$

Convolution (6.6)

- We often end up with transforms to invert that are the product of two known transforms. For example,

$$Y(s) = \frac{2}{s^2(s^2+4)} = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$$

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Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

(on the board)

Convolution (6.6)

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw \\
&= \int_0^\infty e^{-sw} g(w) \int_0^\infty e^{-s\tau} f(\tau)\ d\tau\ dw \\
&= \int_0^\infty g(w) \int_0^\infty e^{-s(\tau+w)} f(\tau)\ d\tau\ dw
\end{aligned}
$$

Replace τ using $u = \tau + w$.

$$
\begin{aligned}
&= \int_0^\infty g(w) \int_w^\infty e^{-s(u)} f(u-w)\ du\ dw \\
&= \int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw \\
&= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du
\end{aligned}
$$

Convolution (6.6)

- What are the correct values for a, b, c and d?

$$\int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

(C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

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where $h(u) = \int_0^u g(w) f(u-w)\ dw$

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where $h(u) = \int_0^u g(w) f(u-w)\ dw$

This is called the convolution of f and g.
Denoted $f * g$.

Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \int_0^\infty e^{-sw} g(w)\, dw$$

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$$= \int_0^\infty e^{-su} \int_0^u g(w) f(u-w)\, dw\, du$$

$$= \int_0^\infty e^{-su} h(u)\, du \ = H(s)$$

The transform of a convolution is the product of the transforms.

$$h(t) = f * g(t) = \int_0^u g(w) f(t-w)\, dw$$

$$\Rightarrow H(s) = F(s)G(s)$$

where $h(u) = \int_0^u g(w) f(u-w)\, dw$

This is called the convolution of f and g. Denoted $f * g$.

Convolution (6.6)

- To invert $Y(s) = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$, we can use the fact that the inverse is the convolution of the inverses of the two pieces (instead of PFD...).

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} =$$

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$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \quad t$$

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$y(t) =$

(A) $\displaystyle\int_0^t (t-w)\sin(2w)\ dw$

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$$f * g = g * f$$

$$\int_0^t f(t-w)g(w)\ dw = \int_0^t f(t)g(t-w)\ dw$$

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Convolution (6.6)

- Transfer functions

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0$$

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- Interpreting the transfer function in a model of memory.
- Suppose your contact list got deleted and you are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget number at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.

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$$n' = -kn + g(t)$$

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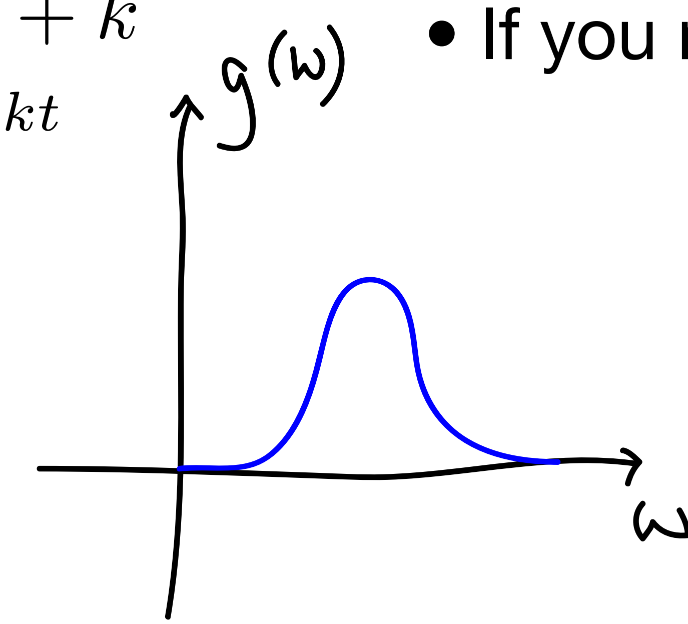

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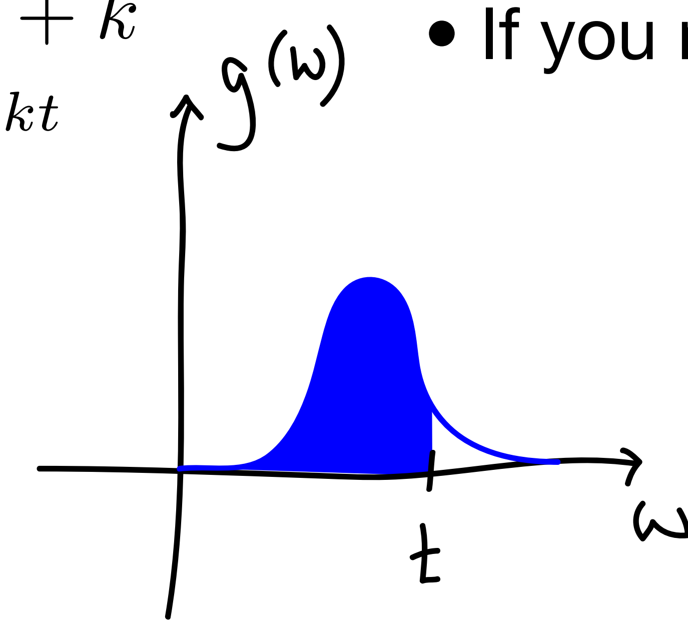

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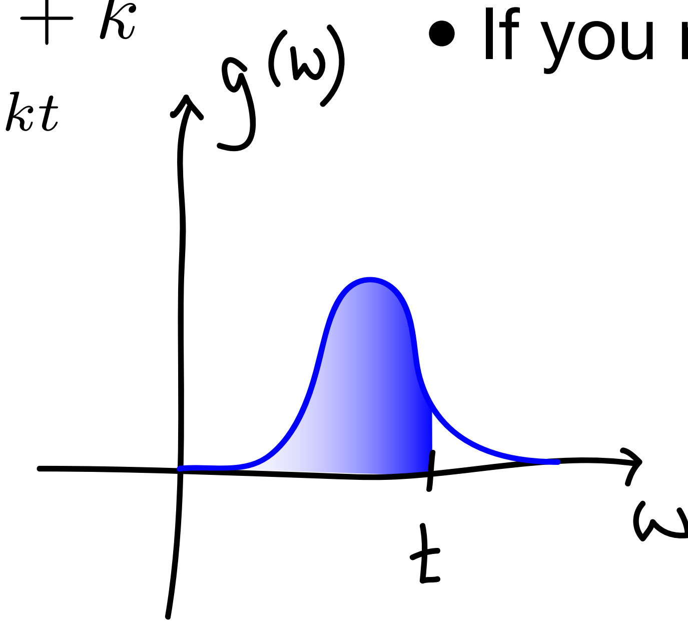

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