Today

- Review of solutions to the Diffusion Equation with various BCs.
- The Wave Equation.
- Separation of variables.

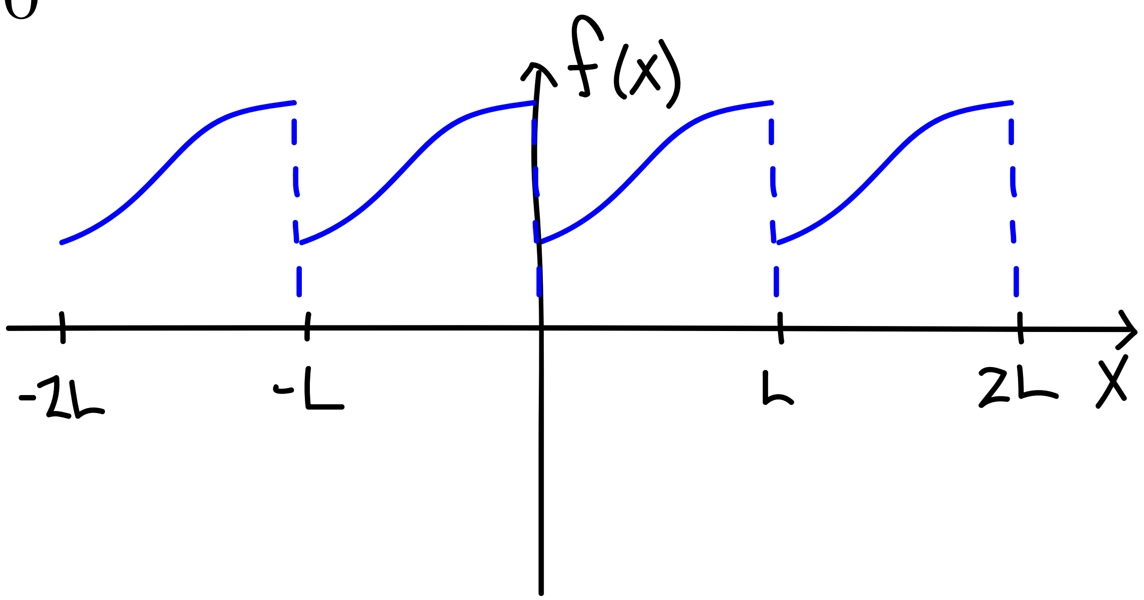
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend f(x) to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- All coefficients will be non-zero. Not particularly useful for solving the BCs.

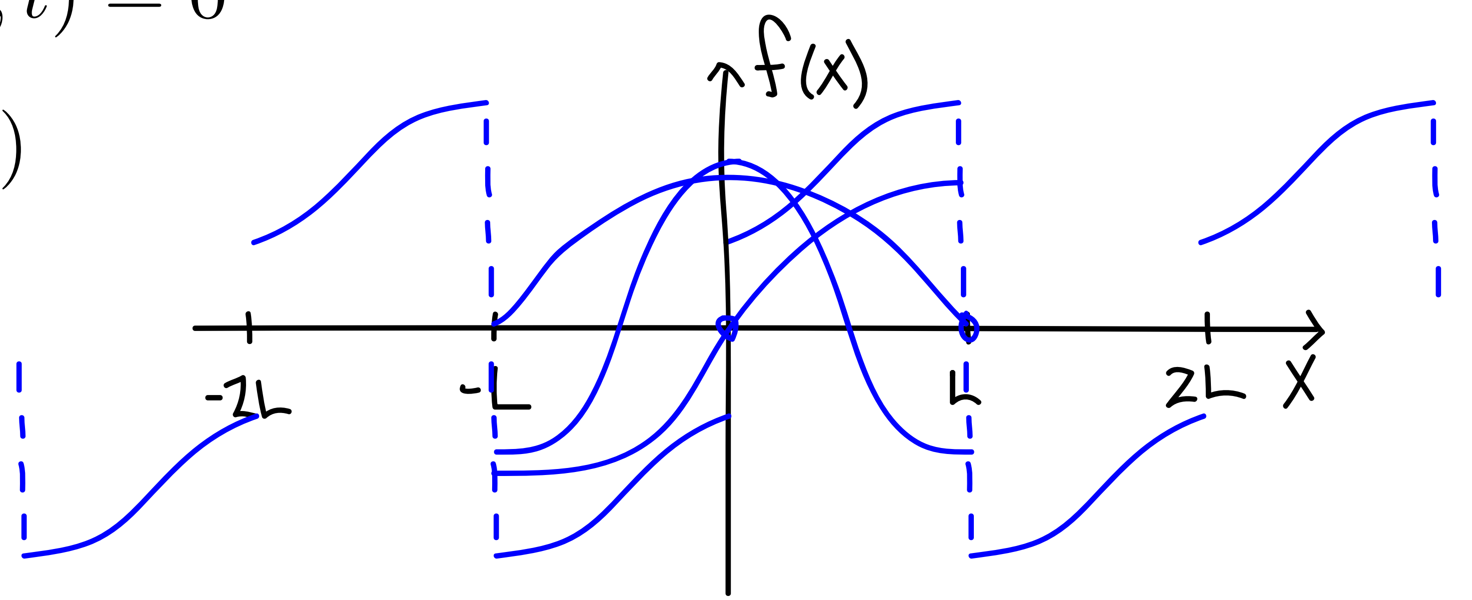
Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend to -L as an odd function and then to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- Cosine coefficients will be zero because f(x) is odd about x=0 and cosine is even. Useful for solving the Diffusion equation with Dirichlet BCs.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$u_t = D u_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x \right) \sin\frac{n\pi x}{L}\, dx$$

- Adding the linear function to the usual solution to the Dirichlet problem ensures that the BCs are satisfied without changing the fact that it satisfies the PDE.

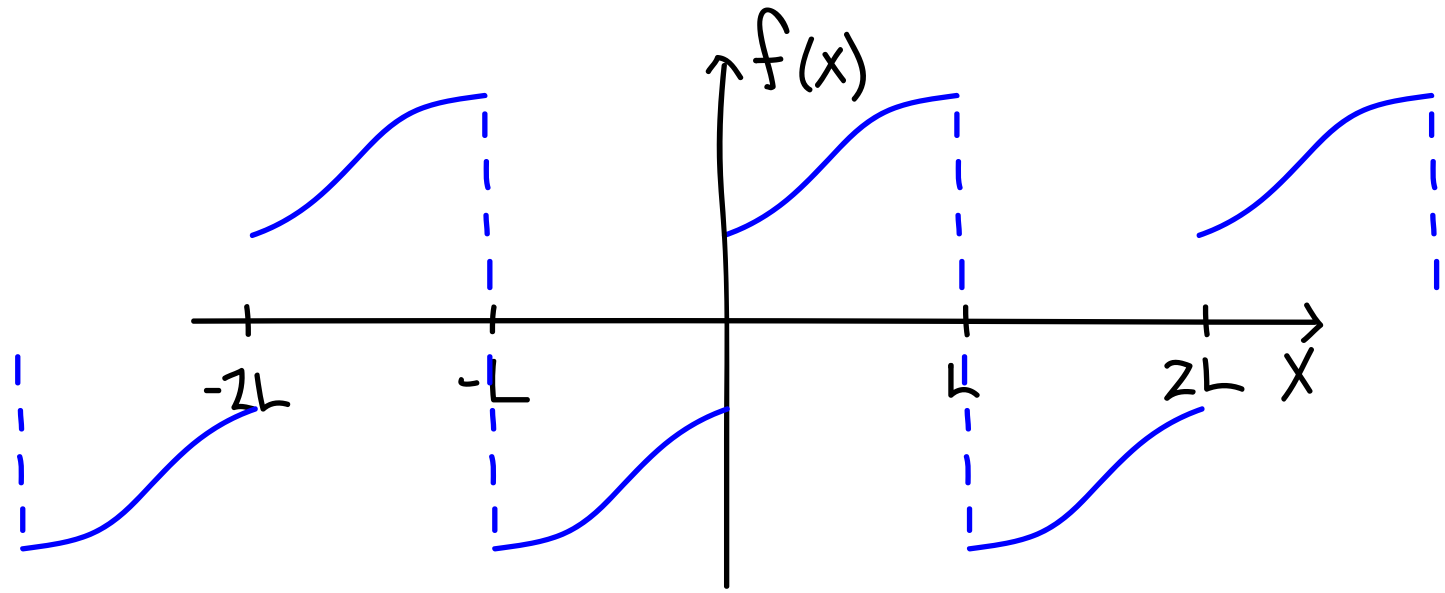
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$u(x,0) = f(x)$$



Odd symmetry about x=0

Odd symmetry about x=L

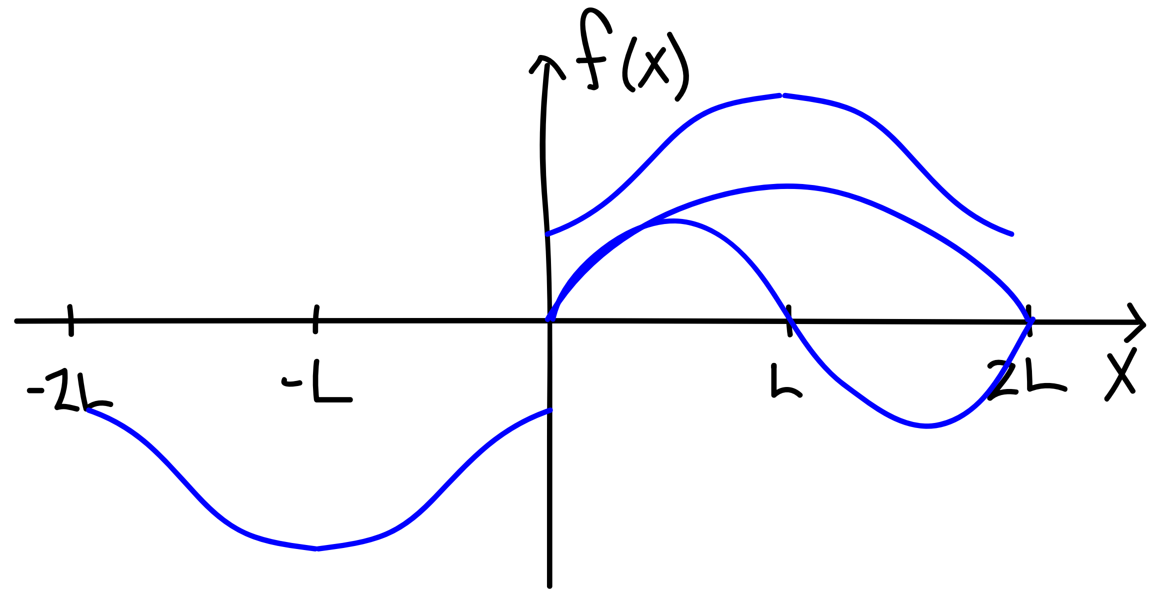
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$u(x,0) = f(x)$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2L}$$

$$b_n = \frac{1}{2L}\int_{-2L}^{2L} f(x)\sin\frac{n\pi x}{2L}\,dx$$

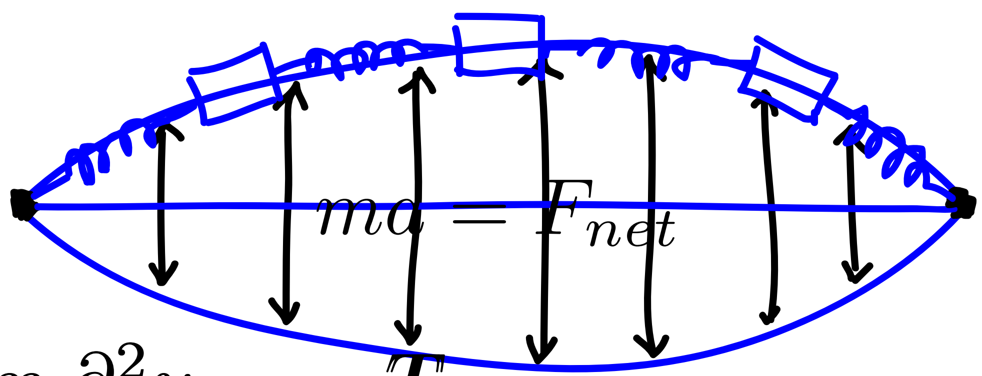
$$b_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx & \text{for } n \text{ odd.} \end{cases}$$

$$= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{2L}\,dx = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{2L}\,dx$$

(for n odd)

The Wave Equation

- Motion of a string



$$ma = F_{net}$$

m = mass of string,

L = length,

ρ = m/L = “density”,

T = tension in string,

u_n = vertical displacement of n^{th} mass.

$$\frac{m}{3}\frac{\partial^2 u_2}{\partial t^2} = \frac{T}{\Delta x}\left((u_3 - u_2) - (u_2 - u_1)\right)$$

$$\frac{m}{n}\frac{\partial^2 u_k}{\partial t^2} = \frac{T}{\Delta x}\left((u_{k+1} - u_k) - (u_k - u_{k-1})\right)$$

(not obvious - requires more detail)

$$n = L/\Delta x - 1 \approx L/\Delta x$$

$$\frac{m\Delta x}{L}\frac{\partial^2 u_k}{\partial t^2} = \frac{T}{\Delta x}\left((u_{k+1} - u_k) - (u_k - u_{k-1})\right)$$

$$\frac{\partial^2 u_k}{\partial t^2} = \frac{T}{\rho \Delta x^2}\left((u_{k+1} - u_k) - (u_k - u_{k-1})\right) \qquad k = 1, 2, 3, \ldots, n$$

$$\frac{\partial^2 u_k}{\partial t^2} = \frac{T}{\rho}\frac{\frac{u_{k+1}-u_k}{\Delta x} - \frac{u_k - u_{k-1}}{\Delta x}}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

BCs:

$$u(0,t) = 0$$

$$u(L,t) = 0$$

ICs: $u(x,0) = f(x), \ \frac{\partial}{\partial t}u(x,0) = g(x)$

The Wave Equation

- Other physical systems described by the wave equation:
- Sound waves - u(x,t) is the air pressure, c is speed of sound.
- Water waves - u(x,t) is water height, c is wave speed.
- Electromagnetic waves - u(x,t) is field intensity, c is speed of light ...

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Separation of variables

- There is a way to calculate eigenvalues as we did for the Diffusion Equation but that requires rewriting it as a first order system:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

find evalues/vectors of this operator

- Easier approach - separation of variables...
- Inspired by the fundamental solutions to the Diffusion Equation, assume

$$u_n(x,t) = T(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial t^2} = T''(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial x^2} = T(t)v''(x)$$

$$T''(t)v(x) = c^2 T(t)v''(x)$$

$$u_n(x,t) = e^{\lambda_n t} \sin \frac{n\pi x}{L}$$

$$T(t) = e^{\lambda t}$$

Expect oscillations so

$$T(t) = e^{i\beta t}$$

(or sines and cosines)

Separation of variables

- Inspired by the fundamental solutions to the Diffusion Equation, assume

$$u_n(x,t) = T(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial t^2} = T''(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial x^2} = T(t)v''(x)$$

$$T''(t)v(x) = c^2 T(t)v''(x)$$

Separation of variables

- Inspired by the fundamental solutions to the Diffusion Equation, assume

$$u_n(x,t) = T(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial t^2} = T''(t)v(x)$$

$$\frac{\partial^2 u_n}{\partial x^2} = T(t)v''(x)$$

$$T''(t)v(x) = c^2 T(t)v''(x)$$

$$\frac{T''(t)}{T(t)} = c^2\frac{v''(x)}{v(x)} = \alpha$$

$$T''(t) = \alpha T(t)$$

$$v''(x) = \frac{\alpha}{c^2}v(x)$$

$$\alpha > 0 \Rightarrow \quad T(t) = e^{\pm\sqrt{\alpha}t}$$

$$v(x) = e^{\pm\frac{\sqrt{\alpha}}{c}x}$$

- no oscillations in time,
- can't satisfy BCs! (v(0)=v(L)=0)

$\alpha = 0$ doesn't work either (check).

$$\alpha < 0 \Rightarrow$$

$$T(t) = A_1\cos\sqrt{-\alpha}t + A_2\sin\sqrt{-\alpha}t$$

$$v(x) = \cancel{B_1\cos\frac{\sqrt{-\alpha}}{c}x} + B_2\sin\frac{\sqrt{-\alpha}}{c}x$$

$$u(0,t) = 0$$

$$u(L,t) = 0 \quad \Rightarrow \alpha = -(n\pi c/L)^2$$

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

Separation of variables

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial}{\partial t}u(x,0) = g(x)$$

$$u_n(x,t) = \left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

- Pull, hold and let go of a guitar string:

$$\frac{\partial}{\partial t}u(x,0) = 0 \quad \Rightarrow B_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

https://www.desmos.com/calculator/dmfijt0e6q