Today

- Homework
  - one WW problem to appear today
  - more TBA from the textbook to be handed in at the start of the tutorial Monday April 7.
- Tutorial on Monday - worksheet instead of quiz.
- Orthogonality of sine and cosine functions
- Fourier series approximations to functions
- Using Fourier series to solve the Diffusion Equation

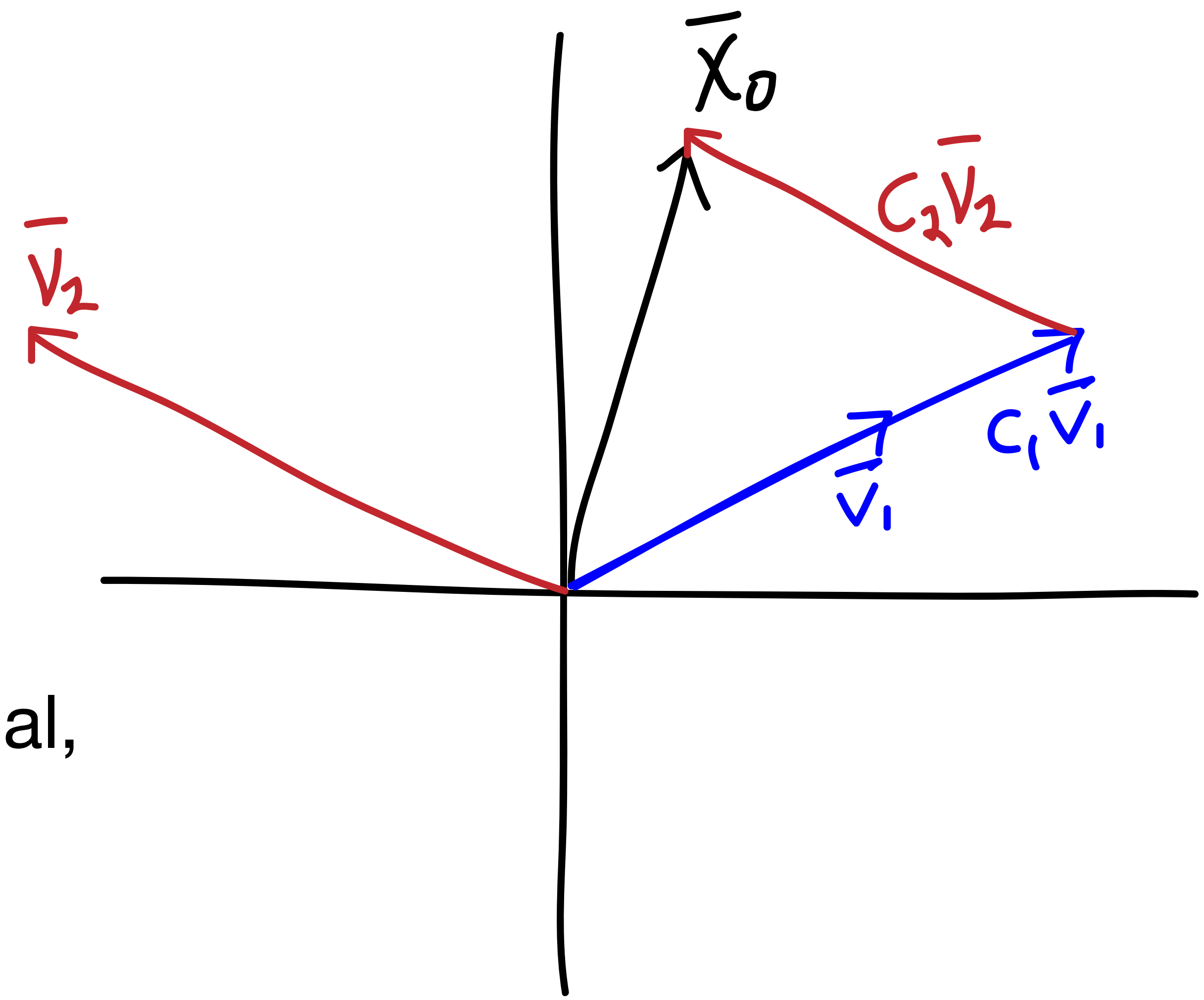
Solving initial conditions using linear algebra

- To solve vector ODEs with ICs, we had to express the initial vector as a linear combination of the eigenvectors:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x_0}$$

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}$$

$$\mathbf{x}(0) = c_1\mathbf{v_1} + c_2\mathbf{v_2} = \mathbf{x}_0$$



- If **v₁** and **v₂** are independent, then c_1 and c_2 can always be found.

- Even better, if **v₁** and **v₂** are orthogonal, then $\mathbf{v_1} \circ \mathbf{v_2} = 0$ and

$$\mathbf{x_0} \circ \mathbf{v_1} = c_1\mathbf{v_1} \circ \mathbf{v_1} + c_2\mathbf{v_2} \circ \mathbf{v_1}$$

$$c_1 = \frac{\mathbf{x_0} \circ \mathbf{v_1}}{\mathbf{v_1} \circ \mathbf{v_1}} \qquad \mathbf{v_1} \circ \mathbf{v_1} = ||\mathbf{v_1}||^2 \qquad c_2 = \frac{\mathbf{x_0} \circ \mathbf{v_2}}{\mathbf{v_2} \circ \mathbf{v_2}}$$

Solving initial conditions using linear algebra

- For the Diffusion Equation, we found that to solve the problem

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2} \qquad \begin{array}{l} c(L,t) = 0 \\ c(0,t) = 0 \end{array} \qquad c(x,0) = f(x)$$

we have to add up eigenfunctions

$$c(x,t) = b_1 e^{\lambda_1 t}\sin(\omega_1 x) + b_2 e^{\lambda_2 t}\sin(\omega_2 x) + b_3 e^{\lambda_3 t}\sin(\omega_3 x) + \cdots$$

and then figure out values for the b_n by imposing the initial condition

$$c(x,0) = b_1\sin(\omega_1 x) + b_2\sin(\omega_2 x) + b_3\sin(\omega_3 x) + \cdots = f(x)$$

- Generalize inner product to functions:

$$g(x)\circ h(x) = \int_{-L}^{L} g(x)h(x)\ dx$$

Solving initial conditions using linear algebra

- Define $v_0(x) = 1$ $v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ $n = (0,)1, 2, 3, \dots$

$\dots$

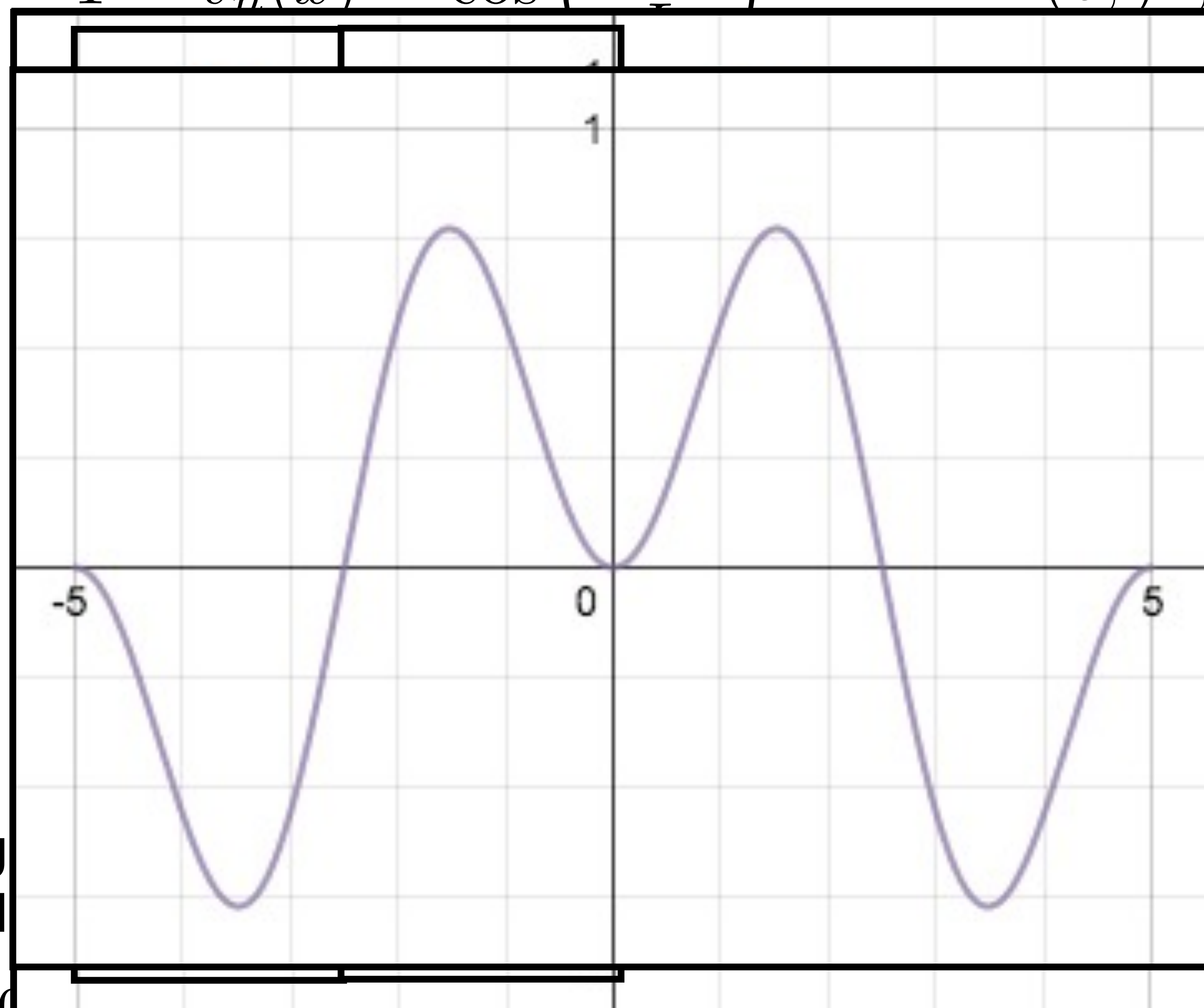
$v_0 \circ v_n =$ $(m \neq n)$

(A) 0

(B) π

(C) π/2

(D) nπ/2



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Integral of a trig
over one period

$v_0 \circ w_n = 0$

$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\ dx$$

Solving initial conditions using linear algebra

- The only inner products of eigenfunctions that aren't zero:

$$v_0 \circ v_0 = \int_{-L}^{L} 1 \cdot 1 \; dx = 2L$$

$$v_n \circ v_n = \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) \; dx = L$$

$$w_n \circ w_n = \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) \; dx = L$$

- We use this orthogonality property (as with vectors) to find the coefficients in the eigenvector sum

$$c(x,0) = b_1 \sin(\omega_1 x) + b_2 \sin(\omega_2 x) + b_3 \sin(\omega_3 x) + \cdots = f(x)$$

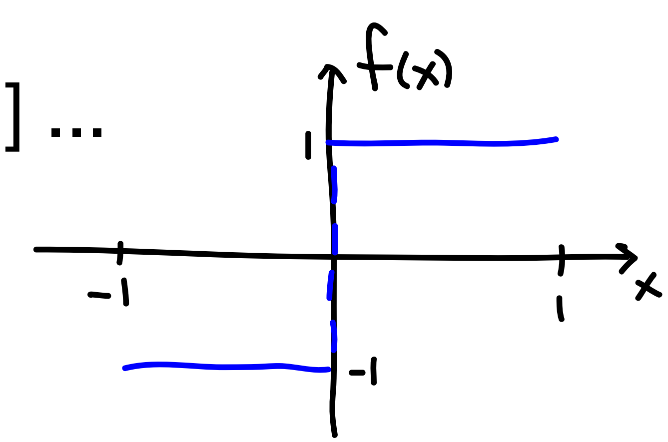
$$b_n = \frac{f(x) \circ v_n(x)}{v_n(x) \circ v_n(x)} = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \; dx$$

Fourier series

- Taking a step back from PDEs, let's define what a Fourier series is.
- Define a function $f_{FS}(x)$ on the interval [-L,L] by choosing coefficients A_0, a_n and b_n and setting

$$f_{FS}(x) = A_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

- This is called a Fourier series. It may or may not converge for different values of x, depending on the choice of coefficients.
- Given any function f(x) on [-L,L], can it be represented by some $f_{FS}(x)$?
- Let's check for $f(x) = 2u_0(x)-1$ on the interval [-1,1] ...

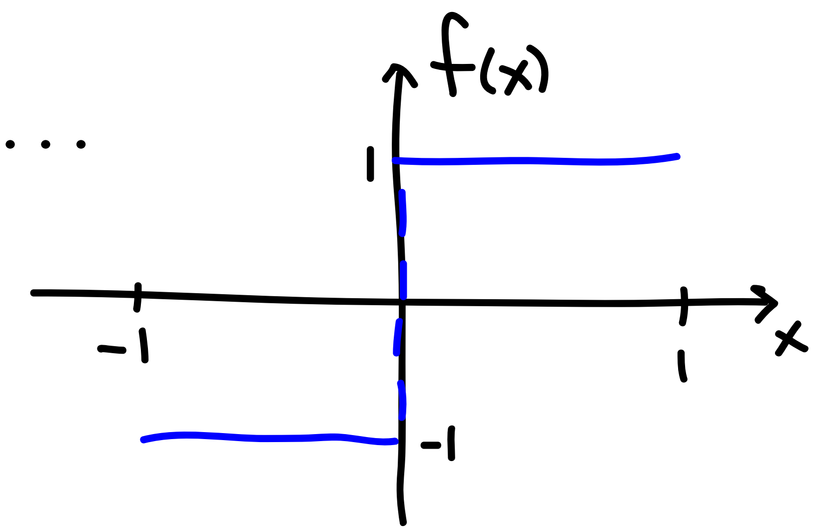


Fourier series

- Find the Fourier series for f(x) = $2u_0(x)-1$ on the interval [-1,1].

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$



- Our hope is that $f(x) = f_{FS}(x)$ so we calculate coefficients as if they were equal:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

- To simplify formulas, usually define

$$a_0 = 2A_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$$

Fourier series

- Calculate the coefficients.

$$a_0 = \int_{-1}^{1} f(x)\ dx$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\ dx$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\ dx$$

$$f_{FS}(x) = \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{L}\right) + a_2\cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1\sin\left(\frac{\pi x}{L}\right) + b_2\sin\left(\frac{2\pi x}{L}\right) + \cdots$$

$a_0 =$

(A) 0

(B) $\frac{1}{\pi}$

(C) undefined

(D) $\frac{1}{n\pi}$

$b_n =$

(B) $\frac{2}{n\pi}$

(C) $\frac{4(-1)^n}{n\pi}$

(D) $\frac{2(1-(-1)^n)}{n\pi}$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

$$f_{FS}(x) = \frac{4}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi x}{L}\right) + \cdots$$

https://www.desmos.com/calculator/tlvtikmi0y

Does f(x) = f_{FS}(x) for all x?

Problems at jumps! x=-1, 0, 1

Fourier series

- **Theorem** Suppose f anf f' are piecewise continuous on [-L,L] and periodic beyond that interval. Then $f(x) = f_{FS}(x)$ at all points at which f is continuous. Furthermore, at points of discontinuity, $f_{FS}(x)$ takes the value of the midpoint of the jump. That is,

$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

Examples