Find the solution to the equation

$$u_t = 4u_{xx}$$

with IC $u(x,0) = 6x$

and BCs $u_x(0,t) = 4$

$u_x(1,t) = 4$

$u_{ss}(x) = ax + b$

$\frac{\partial u_{ss}}{\partial x}(0) = a = 4 = \frac{\partial u_{ss}}{\partial x}(5)$

To determine b, note that

$J_0 = -Du_x(0,t) = -16 = -Du_x(1,t) = J_1$

so that the total amount of mass (assuming $u(x,t)$ is mass/unit length) inside the domain $[0,1]$ is constant in time. That is,

$$\int_0^1 u(x,t)\,dx = \text{Total mass} = \text{constant.}$$

So $\underbrace{\text{initial mass}}$ = $\underbrace{\text{mass at steady state}}$

$$\int_0^1 u(x,0)\,dx = \int_0^1 6x\,dx = \int_0^1 (4x+b)\,dx$$

$$3x^2\Big|_0^1 = 2x^2 + bx\Big|_0^1$$

$$3 = 2 + b \longrightarrow b = 1$$

$u_{ss}(x) = 4x+1$

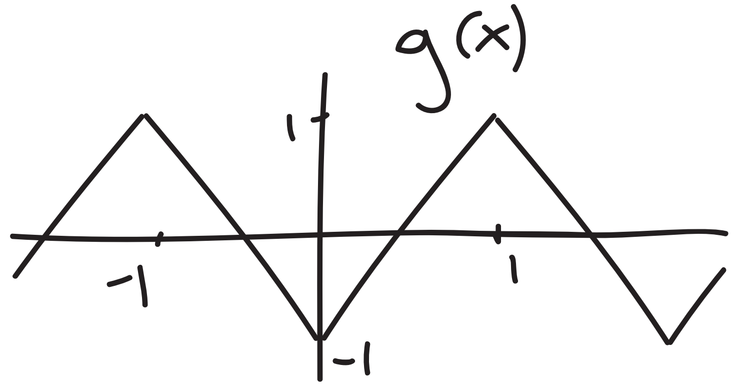
$$u(x,t) = 4x+1+\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n e^{-4n^2\pi^2 t} \cos n\pi x$$

$$u(x,0) = 4x+1+\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x = 6x$$

Choose a_n so that

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x = 2x-1 \text{ on } (0,1).$$

Extend $2x-1$ as an even function about $x=0$. Then extend that function as a periodic function with period 2.



$$a_0 = \int_{-1}^{1} g(x)\,dx = 2\int_0^1 (2x-1)\,dx = 0$$

Note: This is zero precisely because we subtracted the s.s.

$$a_n = \int_{-1}^{1} g(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 (2x-1)\cos n\pi x\,dx$$

$$= \frac{4}{n^2\pi^2}\left((-1)^n-1\right)$$

This is zero for n even because $2x-1$ happens to be odd about $x=\frac{1}{2}$.

$$u(x,t) = 4x+1+\sum_{n=1}^{\infty} a_n e^{-4n^2\pi^2 t} \cos n\pi x$$

where $a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$.