Today

- Complex number review and Euler's formula
- Complex roots to the characteristic equations
- Repeated roots to the characteristic equations

Tutorial poll

(A) Post worksheet online on Friday, print and hand in during tutorial.

(B) Post worksheet online on Friday, get from TA and hand in during tutorial.

(C) Hand out worksheet during tutorial, hand in during Tuesday class.

(D) As currently done.

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- For α+βi, we call α the real part and β the imaginary part.

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$$(a + bi) + (c + di) = a + c + (b + d)i$$

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$$(a+bi)+(c+di) = \underbrace{a+c}+\underbrace{(b+d)}i$$

- Multiplying two complex numbers:

$$(a+bi)(c+di) = ac-bd+(ad+bc)i$$

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  - What is the inverse of c+di?

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- What is the inverse of c+di written in the usual complex form p+qi?

(A) $c - di$

(B) $\dfrac{c + di}{c^2 + d^2}$

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- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

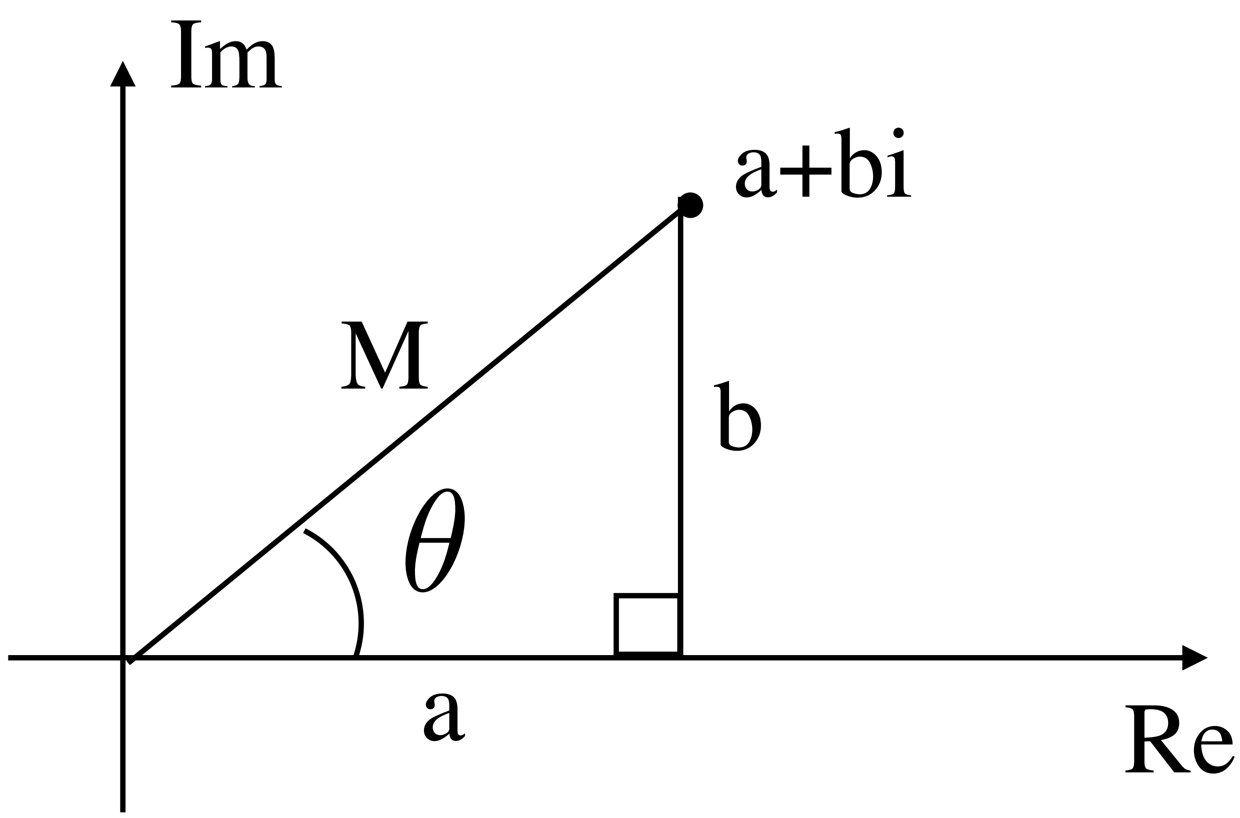
$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



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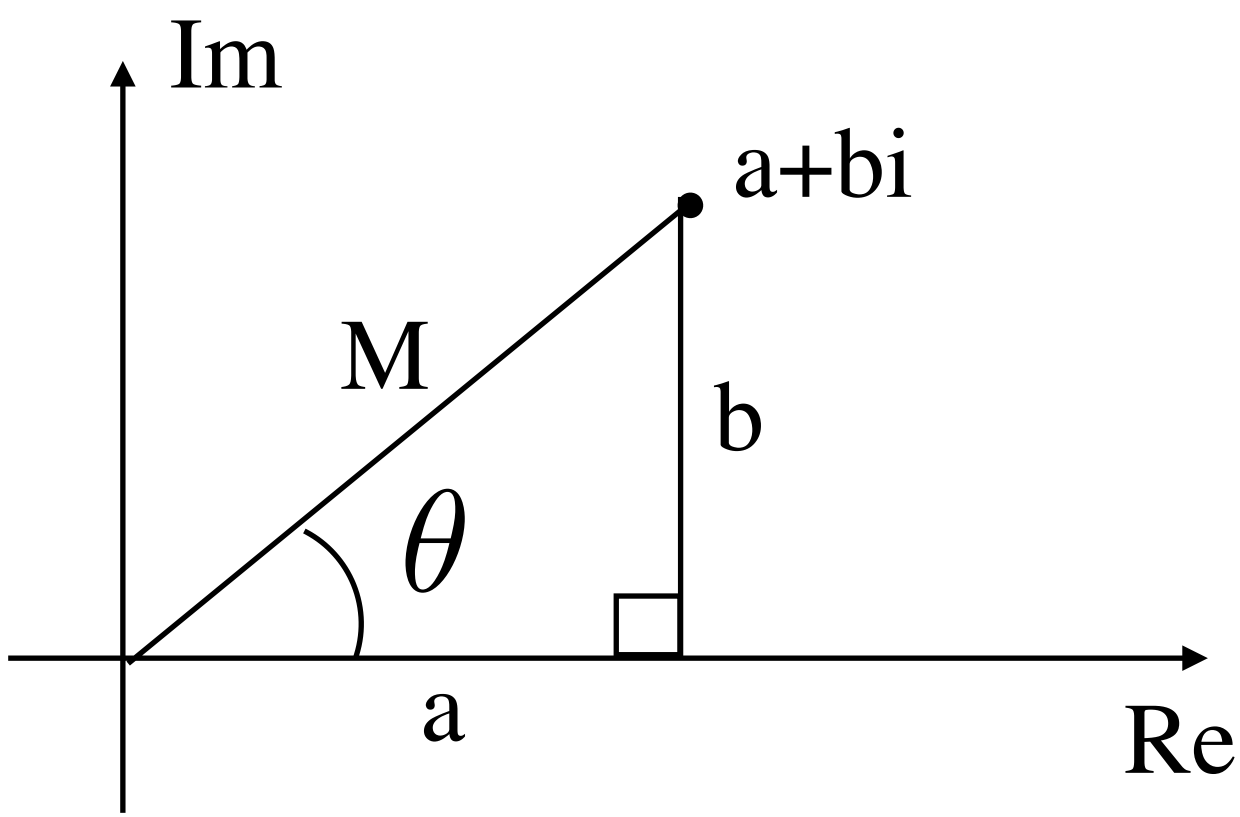
$$a = M \cos \theta$$

$$b = M \sin \theta$$



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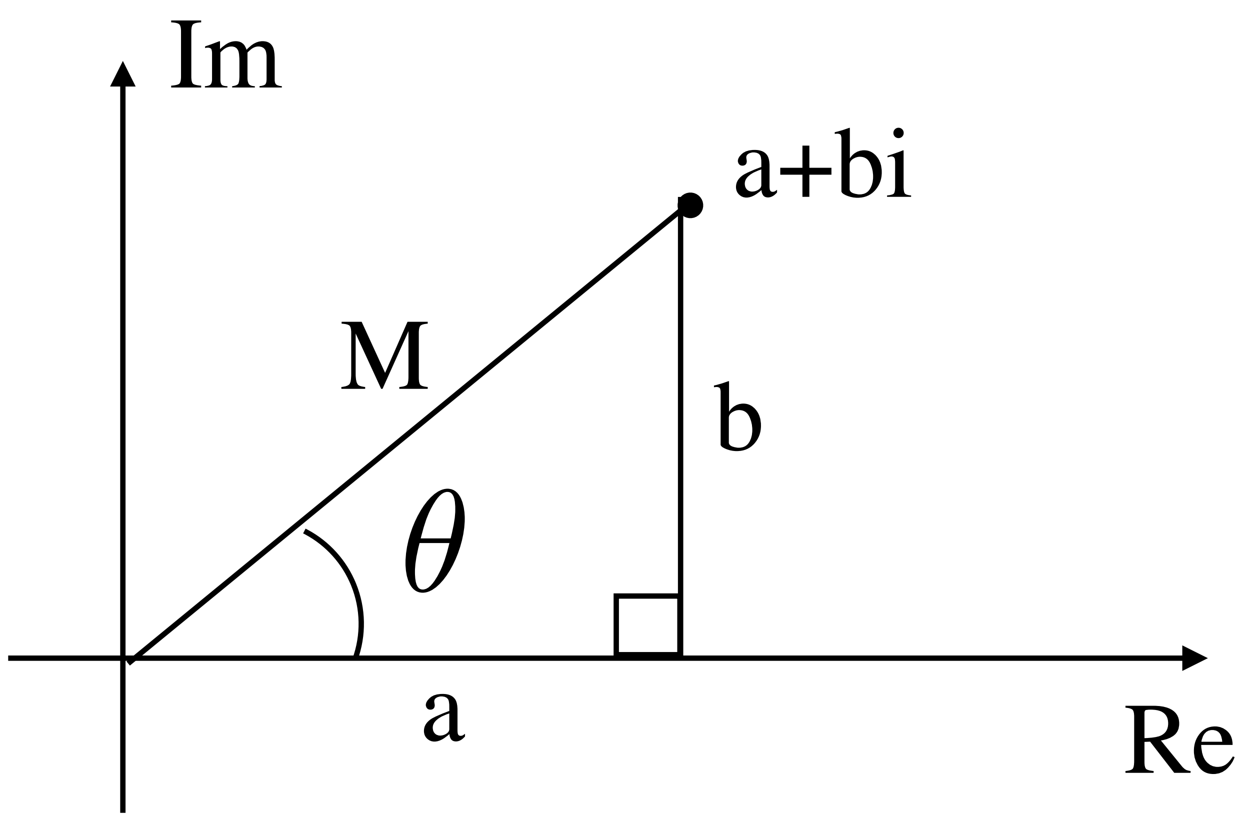

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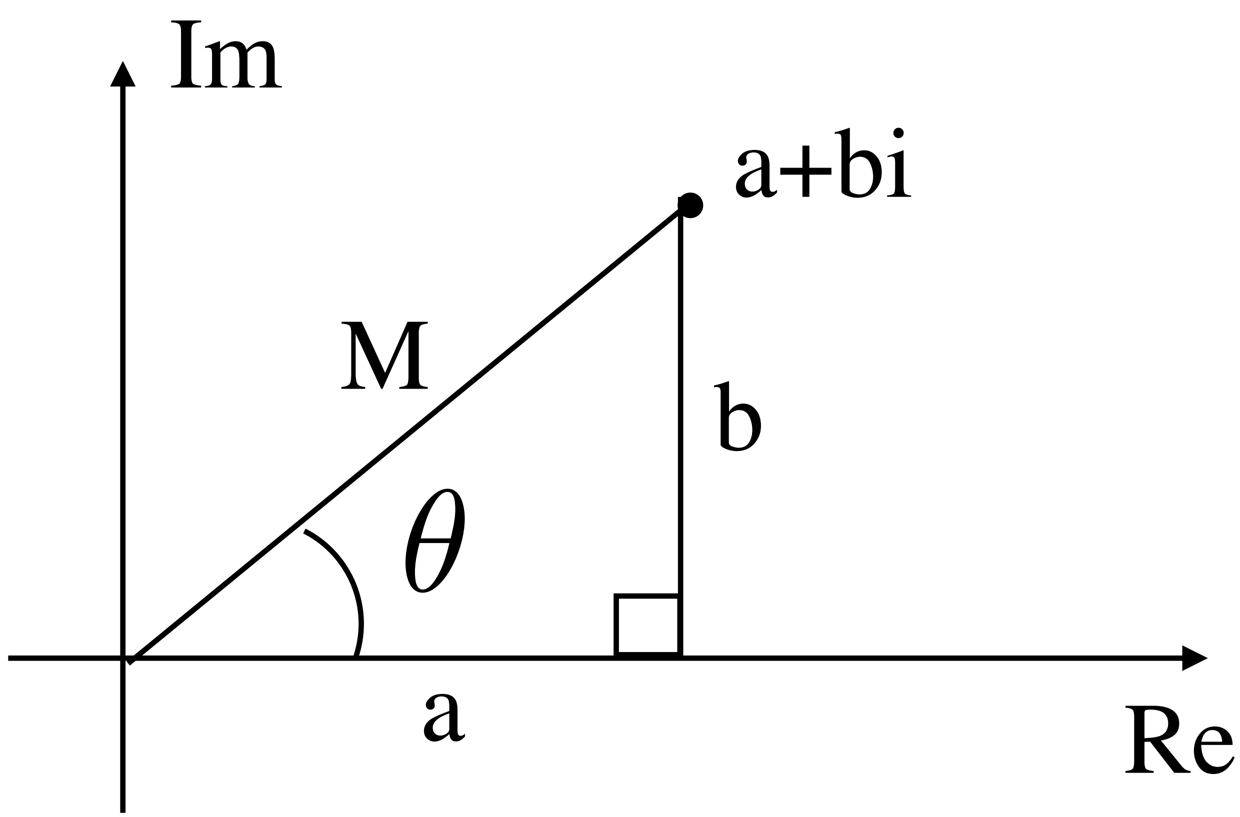

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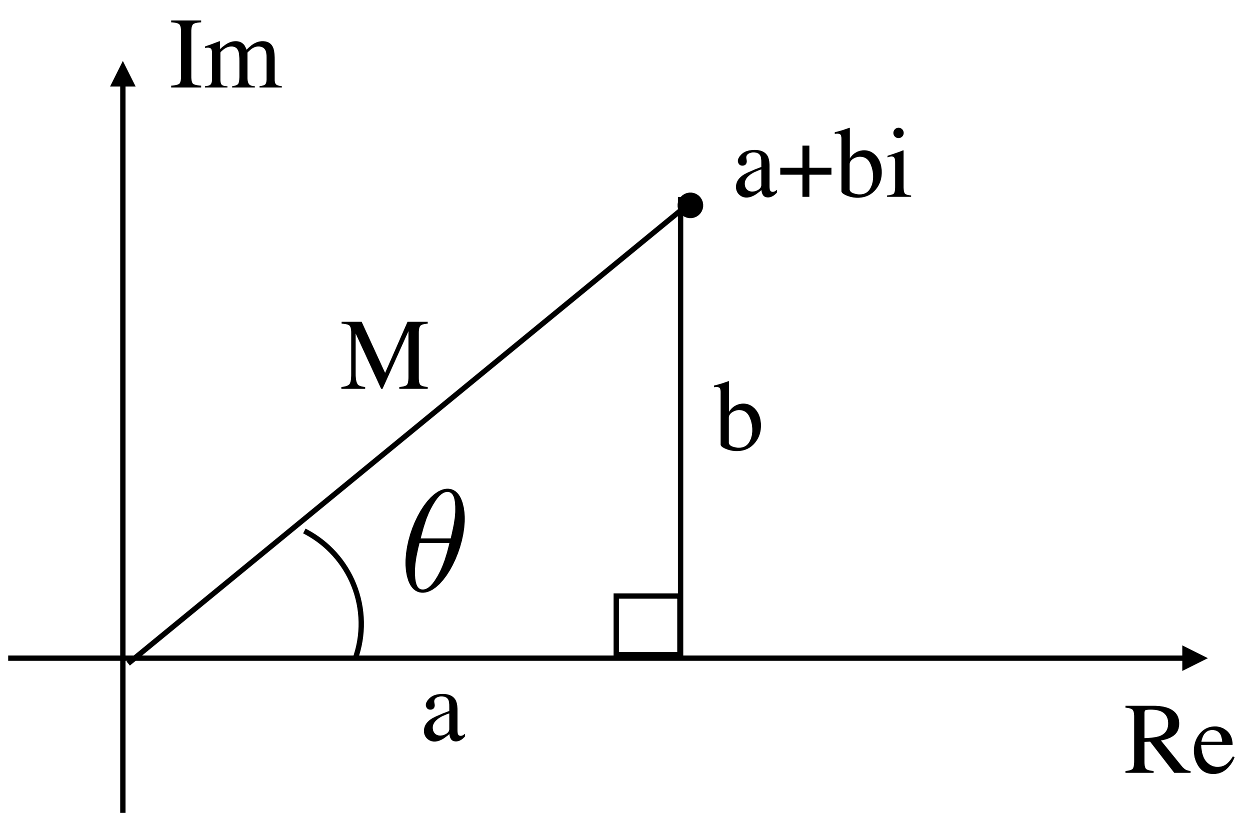

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θ is sometimes called the argument or phase of $a + bi$.

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  - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

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- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

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$$\cos\theta + i\sin\theta$$

$$= e^{i\theta}$$

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Euler's formula:

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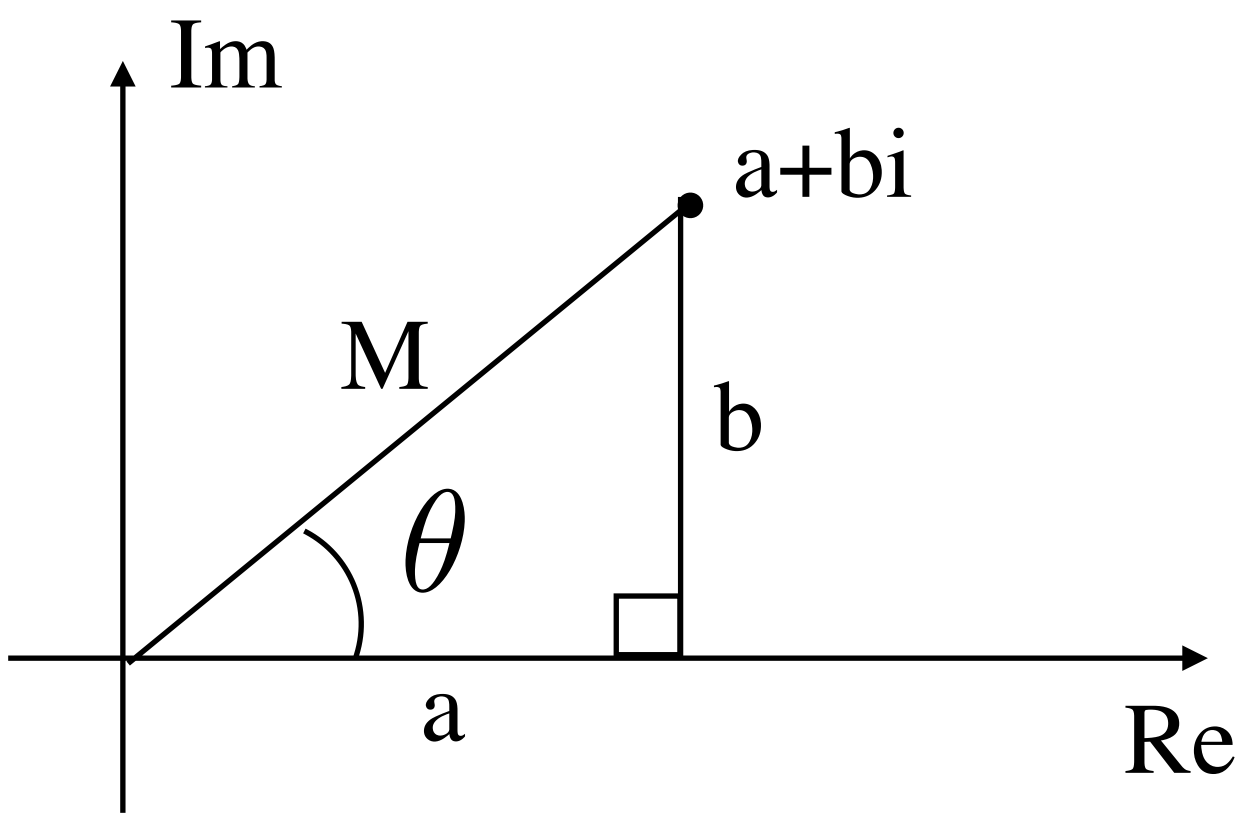
  - e.g. $a + bi$



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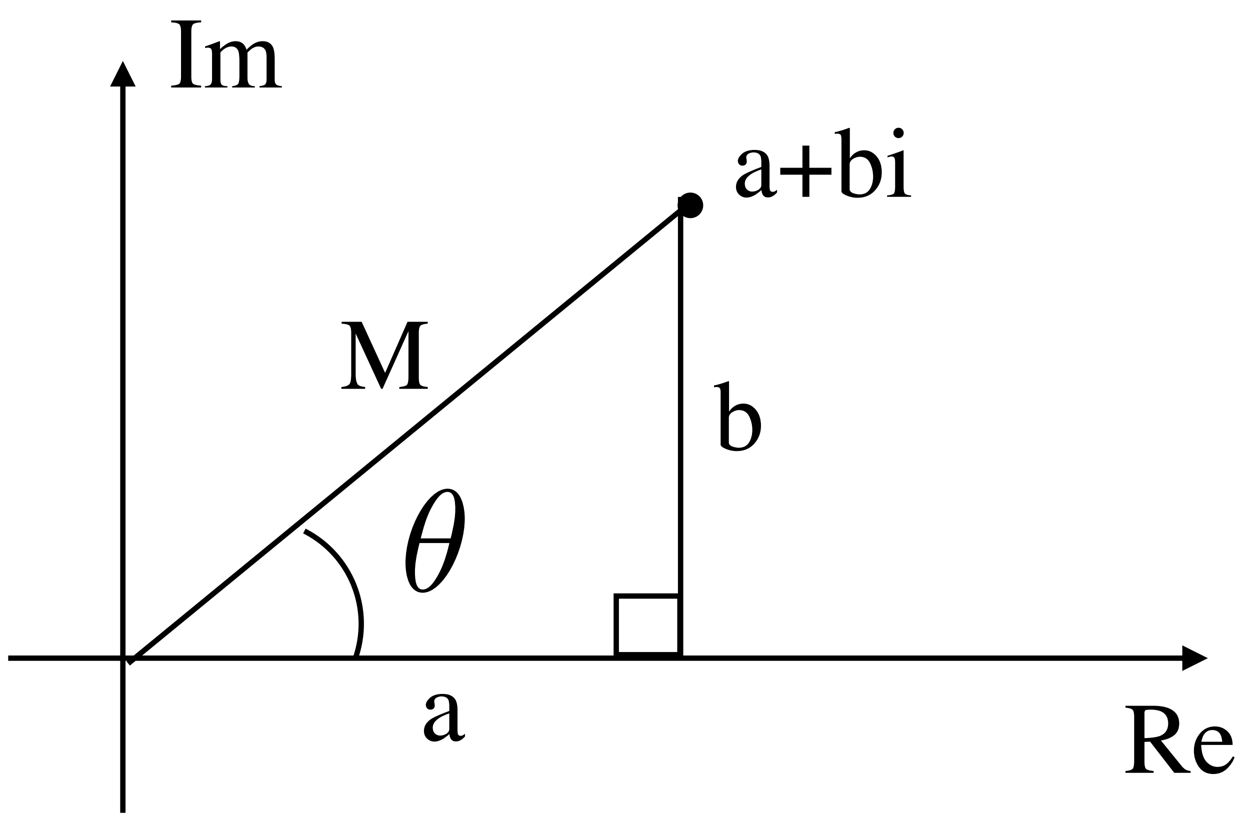

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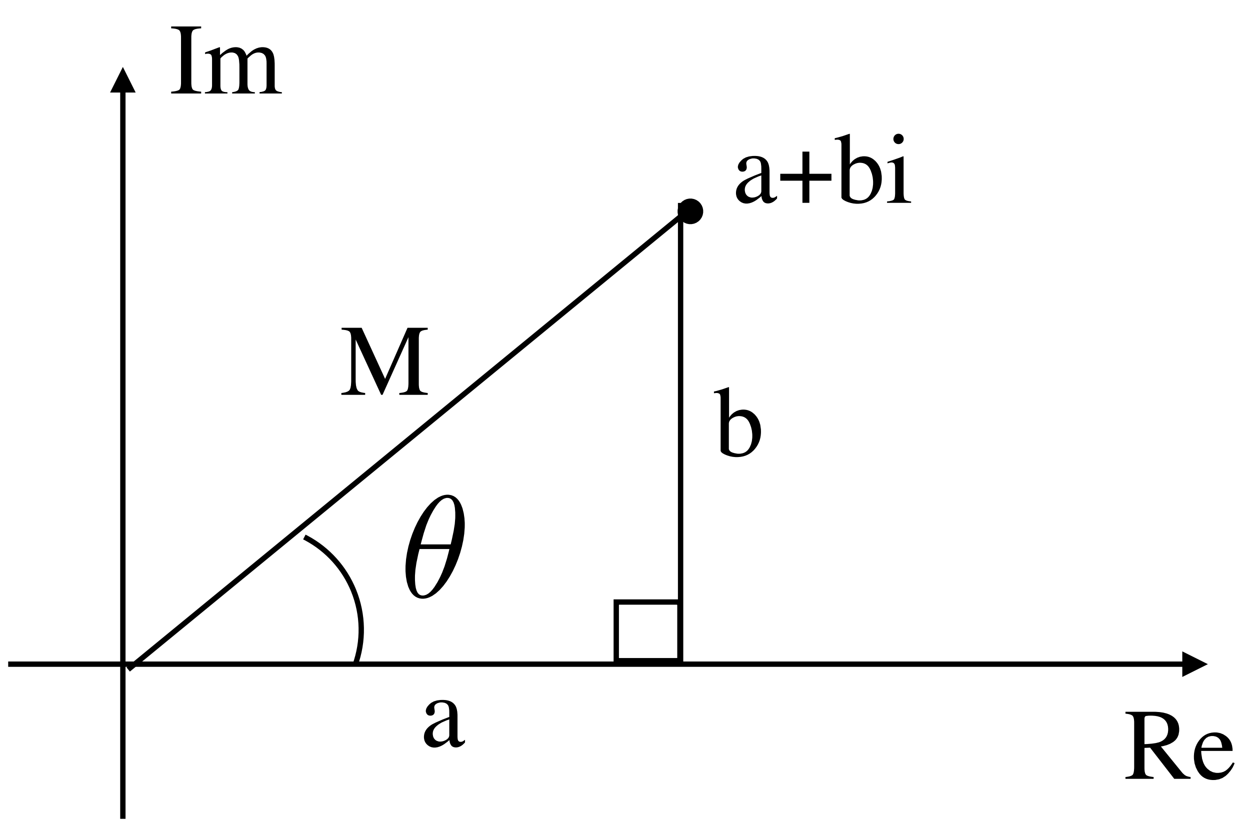

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(Polar form makes multiplication much cleaner)