Today

- Solving ODEs using Laplace transforms
- The Heaviside and associated step and ramp functions
- ODE with a ramped forcing function

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13}$$

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$$Y(s) = \frac{s+6}{s^2+6s+13} = \frac{s+6}{s^2+6s+9+4}$$

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2. Complete the square.

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4. Fix up coefficient of the term with no s in the numerator.

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$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

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4. Fix up coefficient of the term with no s in the numerator.
5. Invert.

Solving IVPs using Laplace transforms (6.2)

- What is the transformed equation for the IVP

$$y' + 6y = e^{2t}$$
$$y(0) = 2$$

(A) $Y'(s) + 6Y(s) = \frac{1}{s+2}$

(B) $Y'(s) + 6Y(s) = \frac{1}{s-2}$

(C) $sY(s) + 2 + 6Y(s) = \frac{1}{s+2}$

(D) $sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$

(E) Explain, please.

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{(2-s)t}\, dt$$
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$$\mathcal{L}\{y'(t)\} = sY(s) - 2$$

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$$sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$$

$$Y(s) = \left(2 + \frac{1}{s-2}\right) / (s+6)$$

$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

$$\downarrow$$

$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

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$$\frac{1}{(s-2)(s+6)} = \frac{A}{s-2} + \frac{B}{s+6}$$

$$1 = A(s+6) + B(s-2)$$

$$(s=2) \qquad 1 = 8A$$

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$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

$$y(t) = 2e^{-6t} + \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s+6}\right)$$

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$$y(t) = \frac{15}{8}e^{-6t} + \frac{1}{8}e^{2t}$$

$$y_h(t) = Ce^{-6t}$$

$$C = \frac{15}{8} \qquad y_p(t) = \frac{1}{8}e^{2t}$$

Solving IVPs using Laplace transforms (6.2)

- With a forcing term, the transformed equation is

$$ay'' + by' + cy = g(t)$$

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s)$$

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$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

transform of homogeneous solution with two degrees of freedom

transform of particular solution

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  - If denominator has unique real factors, use PFD and get

$$Y_h(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

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  - If denominator has repeated real factors, use PFD and get

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$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

- Unique real factors, $Y_h(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2} \quad \rightarrow \quad y_h(t) = Ae^{r_1 t} + Be^{r_2 t}$
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$$Y_h(s) = \frac{A(s-\alpha) + A\alpha}{(s-\alpha)^2+\beta^2} + \frac{B}{(s-\alpha)^2+\beta^2}$$

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$$Y_h(s) = \frac{A(s-\alpha) + A\alpha}{(s-\alpha)^2+\beta^2} + \frac{B}{(s-\alpha)^2+\beta^2}$$

$$Y_h(s) = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B + A\alpha}{(s-\alpha)^2+\beta^2}$$

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$$Y_h(s) = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B + A\alpha}{(s-\alpha)^2+\beta^2}$$

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Solving IVPs using Laplace transforms (6.2)

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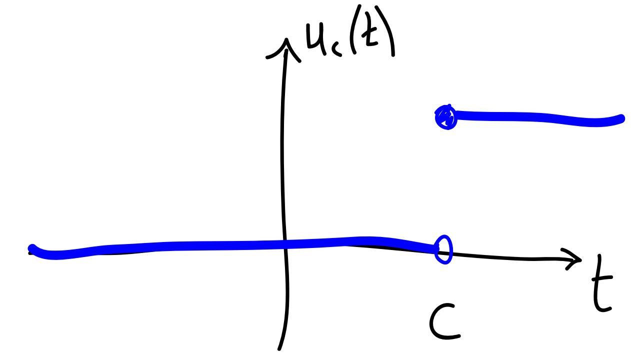
Laplace transforms (so far)

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Step function forcing (6.3, 6.4)

- We define the Heaviside function $u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$



- In WW, $u_c(t) = u(t-c) = h(t-a)$

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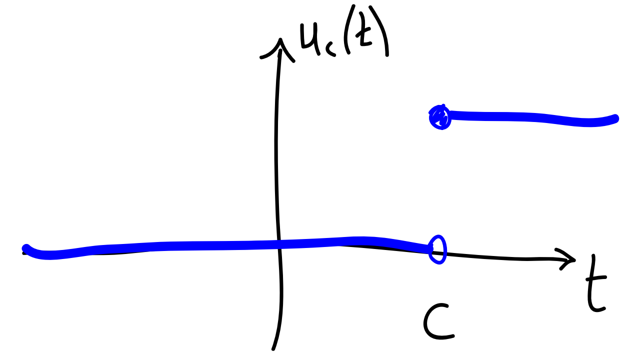

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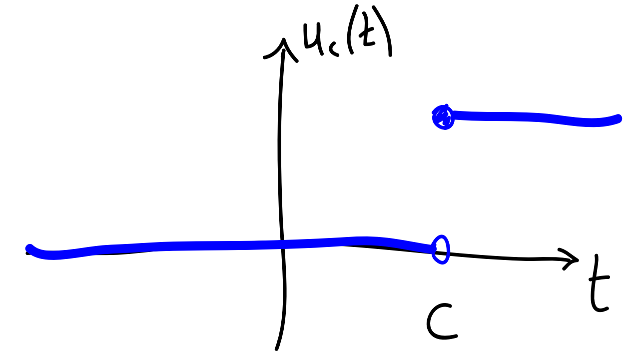

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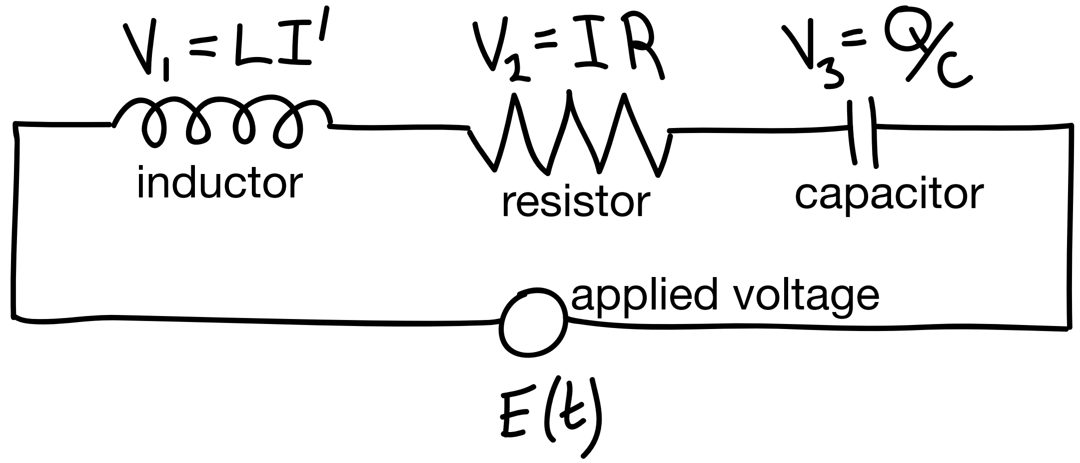

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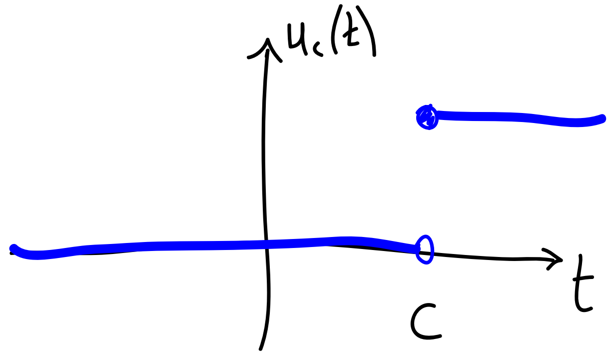

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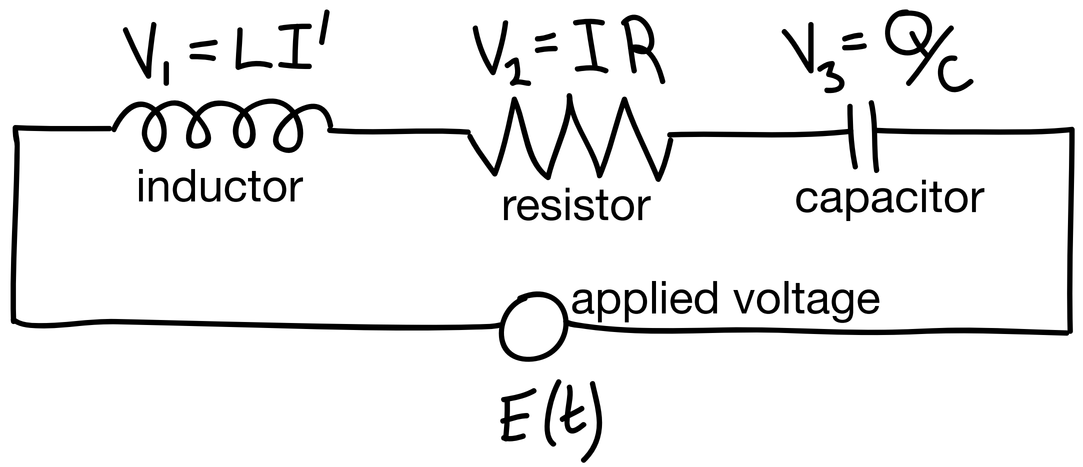

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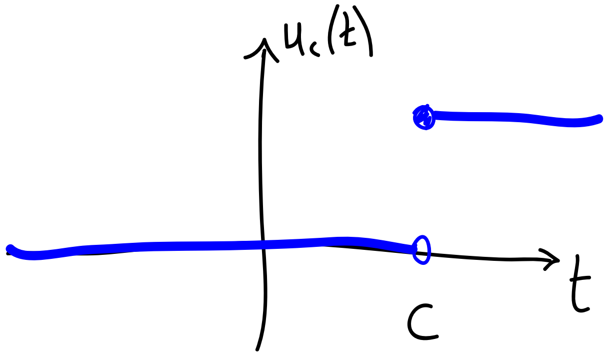

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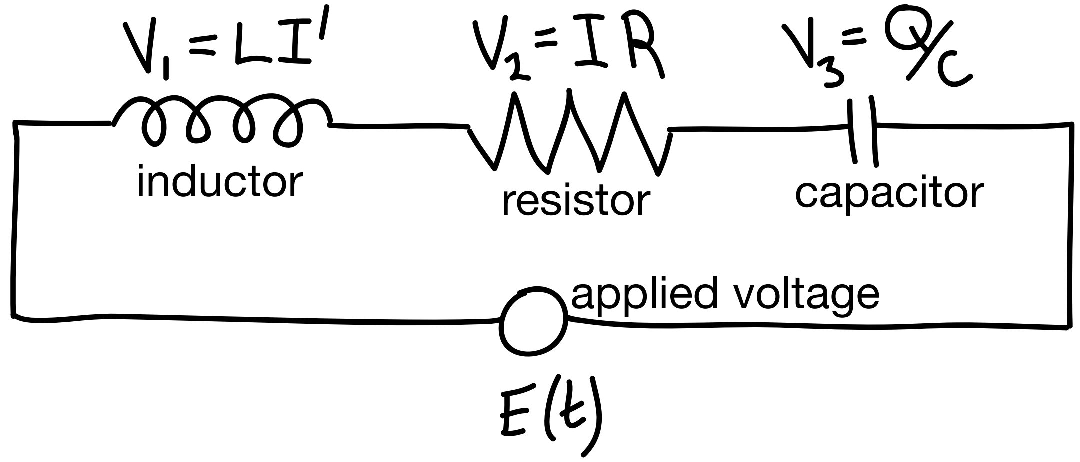

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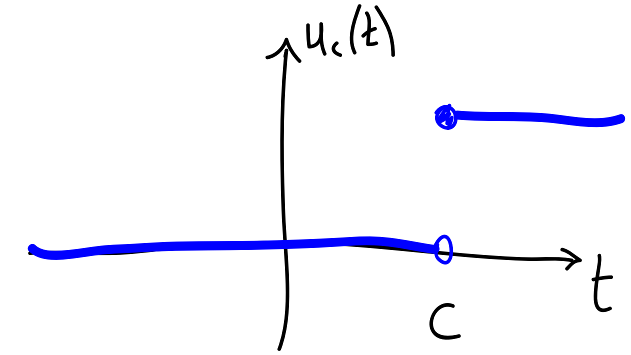

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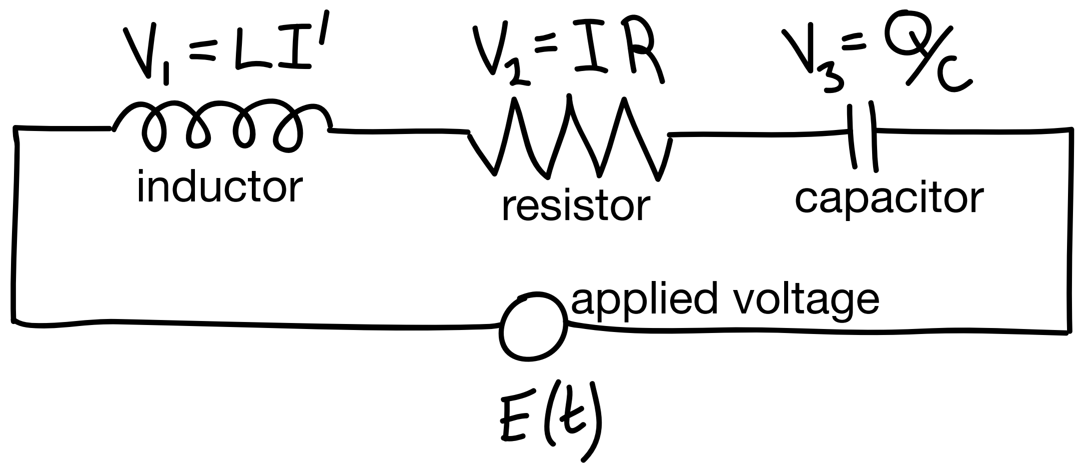

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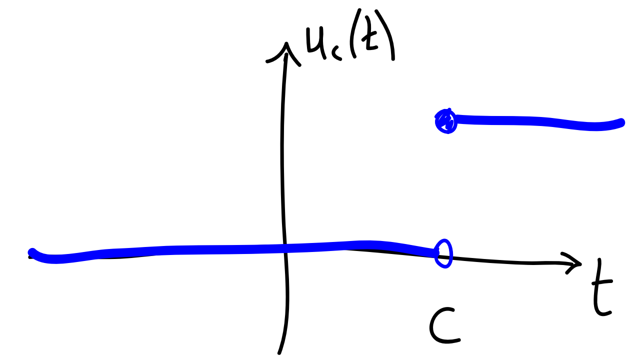

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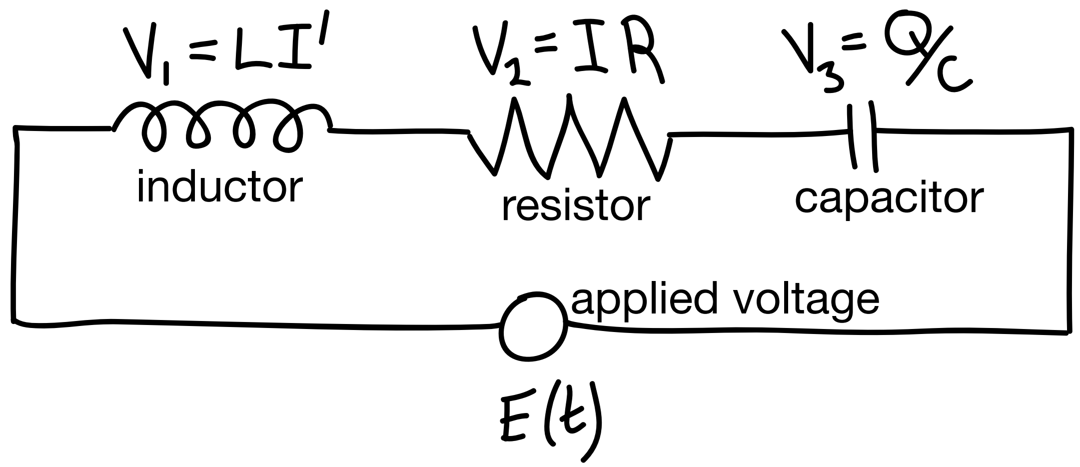

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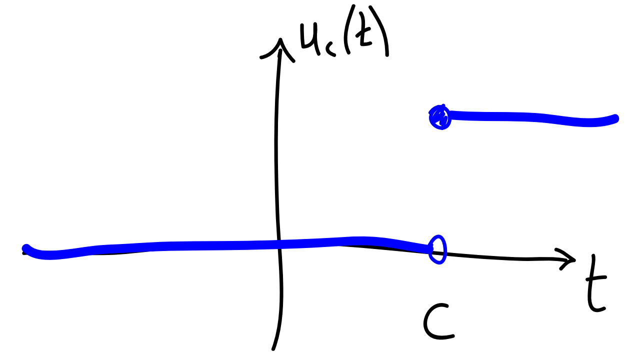

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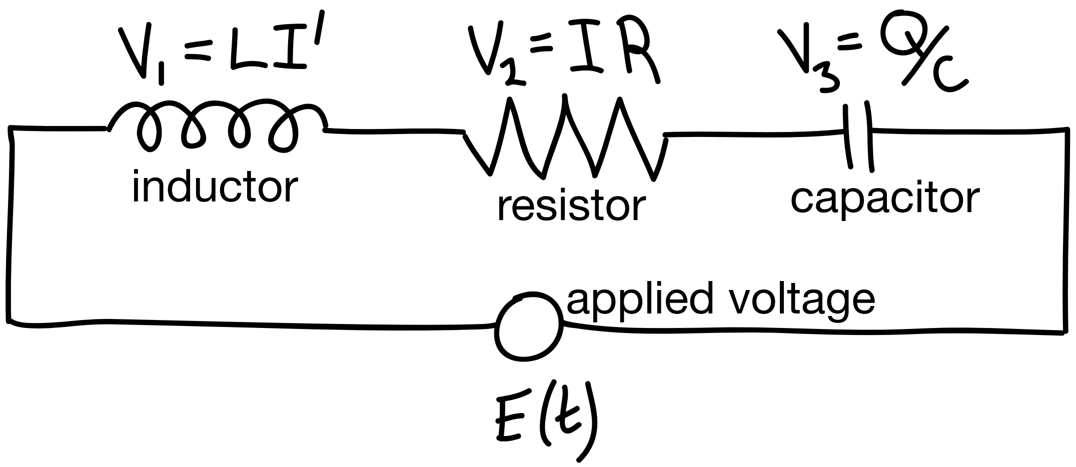

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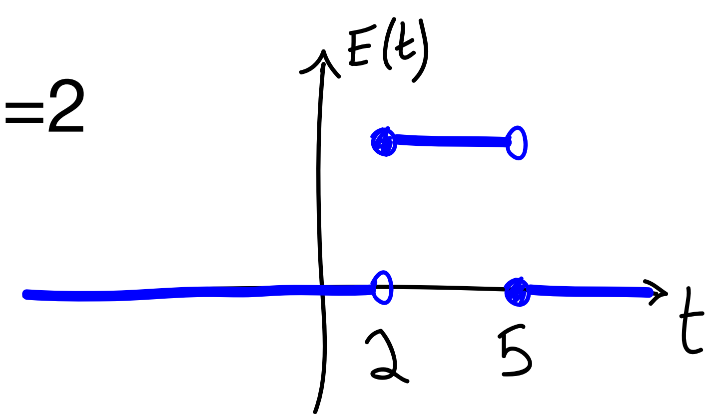
- For example, turn E on at t=2 and off again at t=5:



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- Use the Heaviside function to rewrite $g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$

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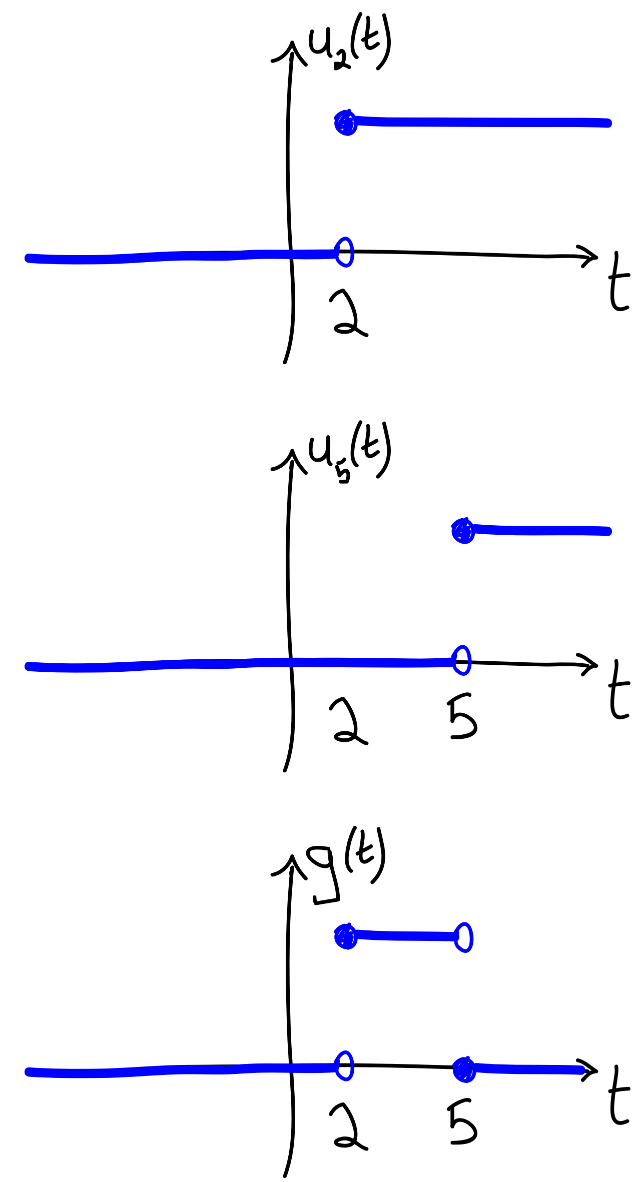

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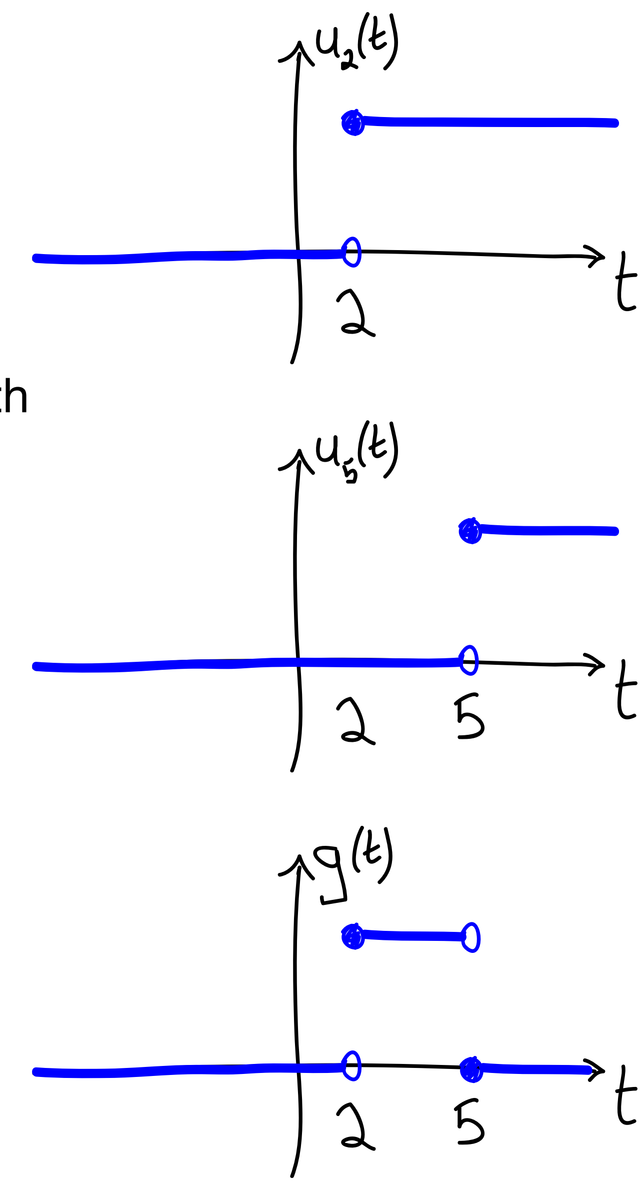

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Recall: $\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty e^{-st}(f(t) + g(t))\ dt$

$$= \int_0^\infty e^{-st} f(t)\ dt + \int_0^\infty e^{-st} g(t)\ dt$$

$$= \quad \mathcal{L}\{f(t)\} \quad + \quad \mathcal{L}\{g(t)\}$$

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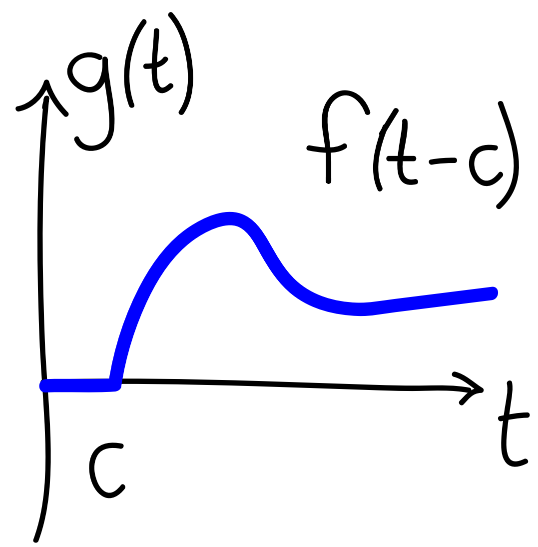

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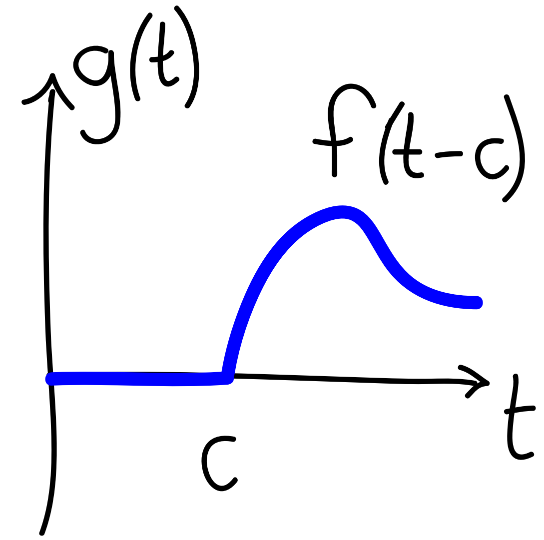

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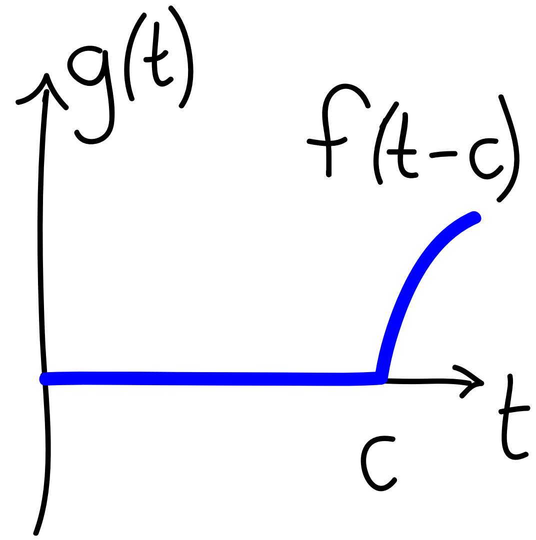

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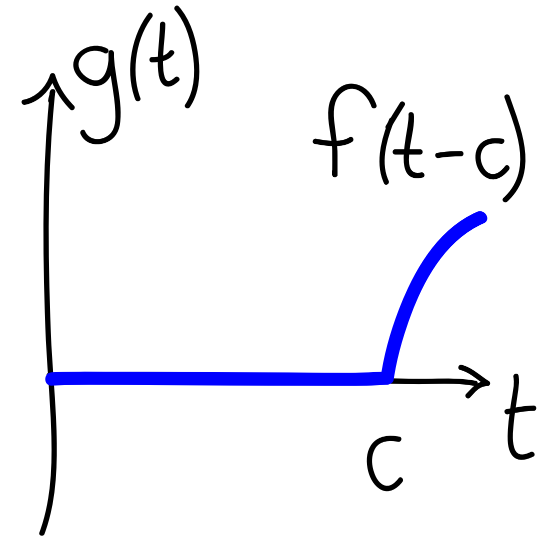

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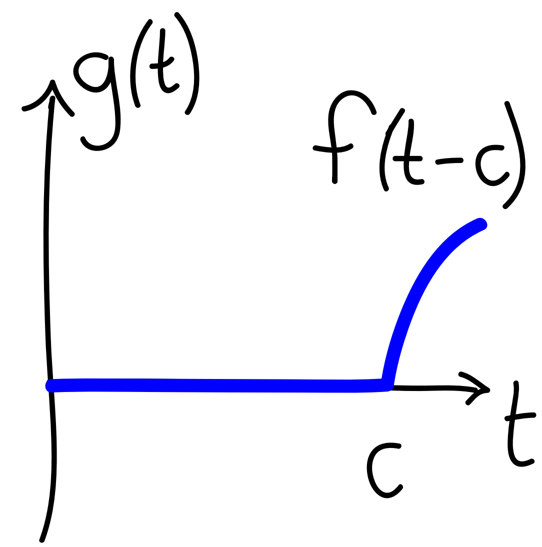

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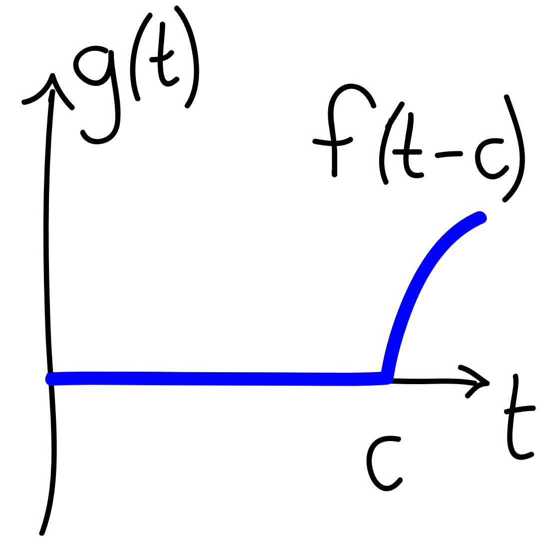

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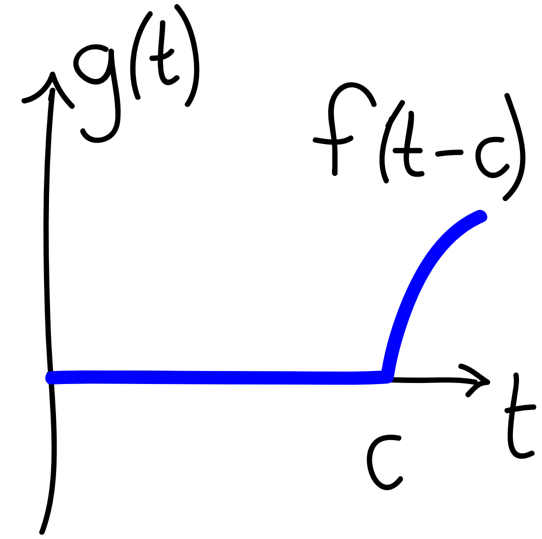

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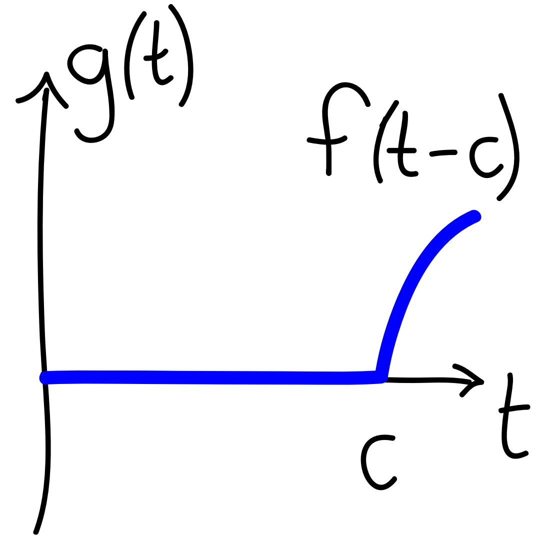

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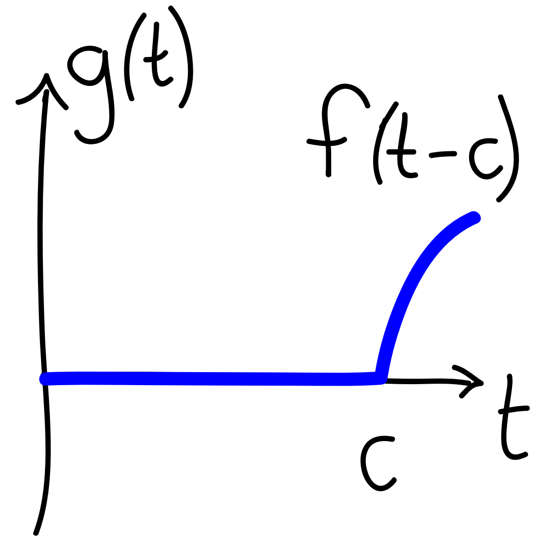

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$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

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- So we just need h(t) and we're done.

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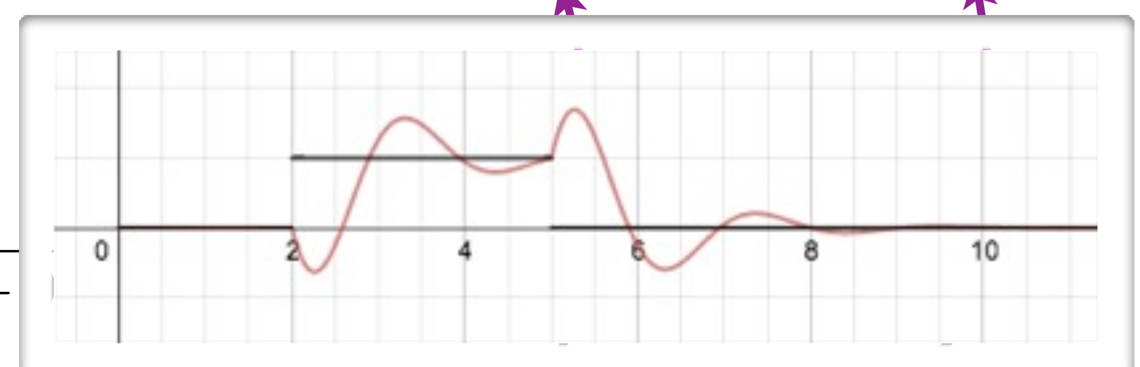

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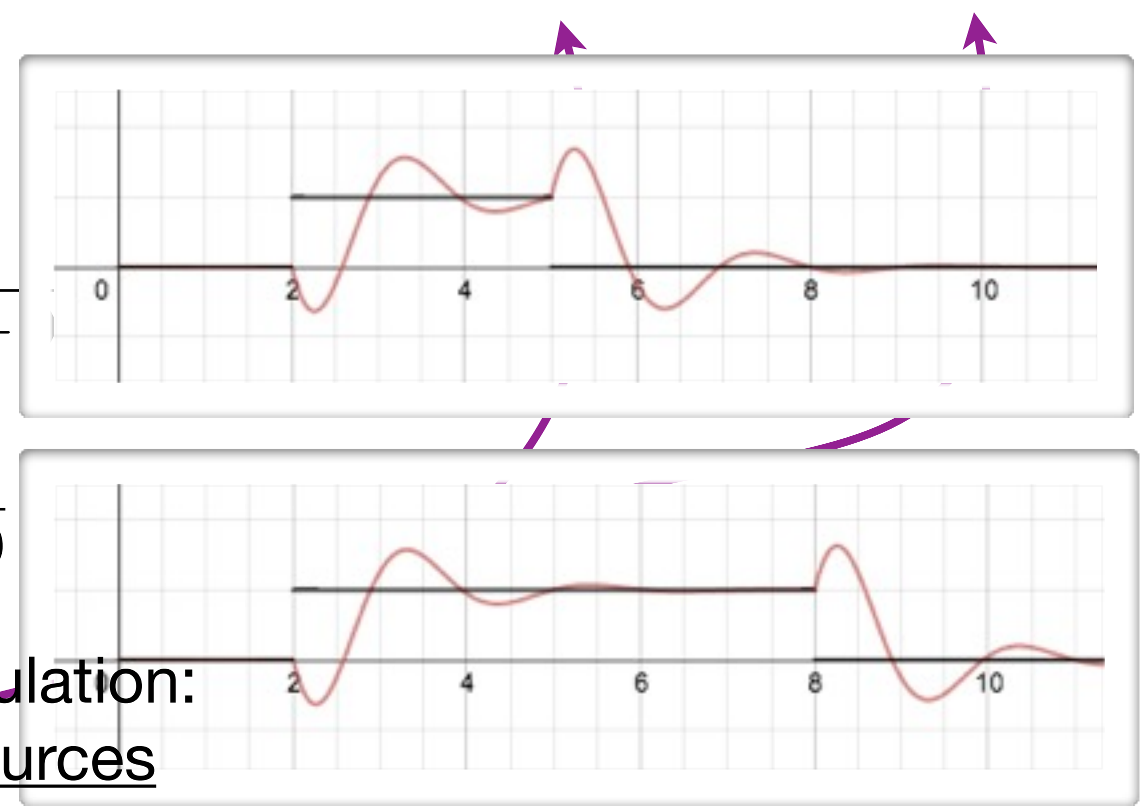

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$$y'' + 4y = \begin{cases} 0 & \text{for } t < 5, \\ \frac{t-5}{5} & \text{for } 5 \leq t < 10, \\ 1 & \text{for } t \geq 10. \end{cases}$$

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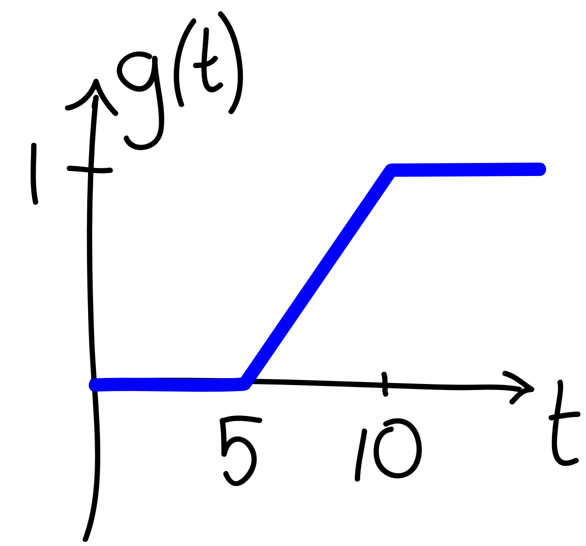

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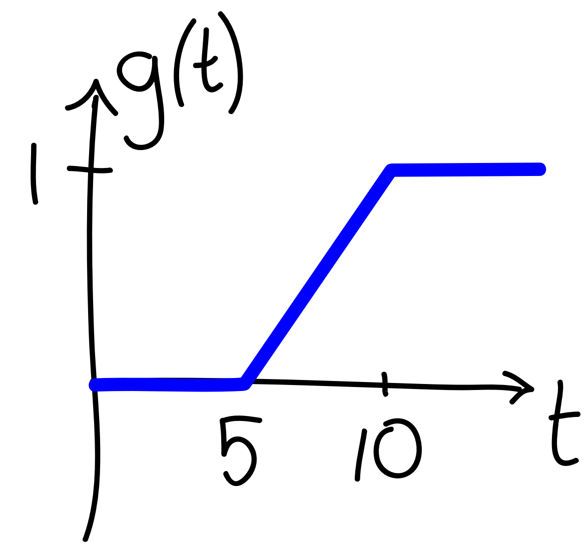

- Write g(t) is terms of $u_c(t)$:

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(B) $g(t) = u_5(t)(t-5) - u_{10}(t)(t-5)$

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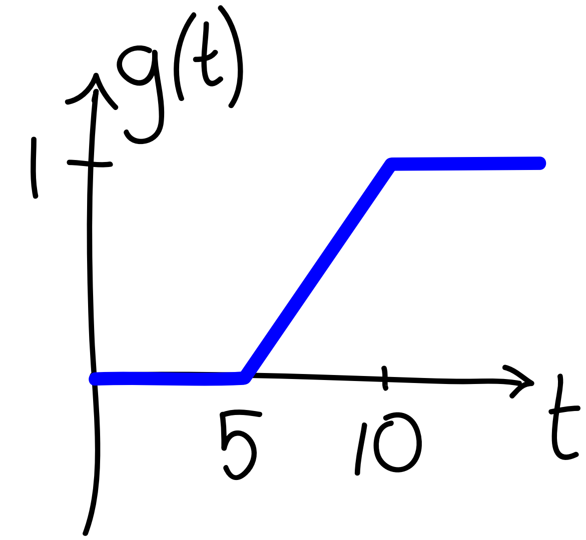

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Two methods:

1. Build from left to right, adding/subtracting what you need to make the next section:

$$g(t) = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

2. Build each section independently:

$$g(t) = \left(u_5(t) - u_{10}(t)\right)\frac{1}{5}(t-5) + u_{10}(t)\cdot 1$$

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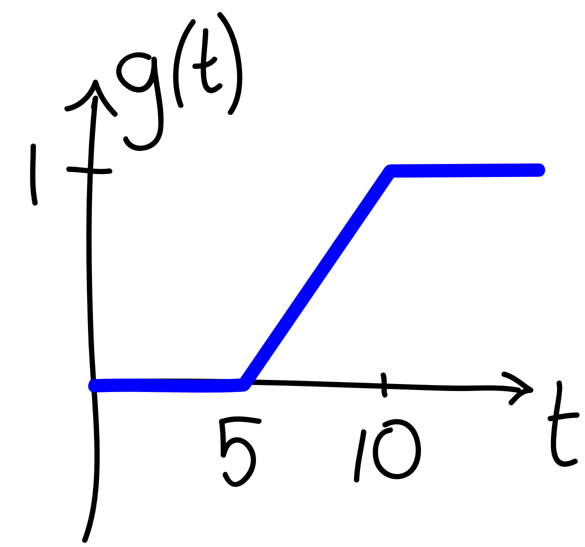

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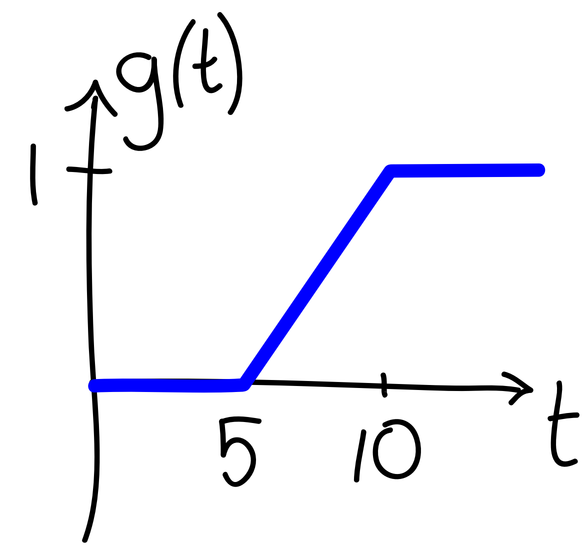

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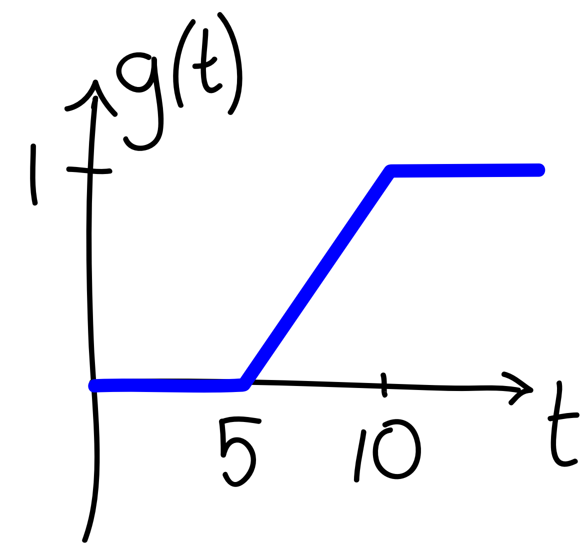

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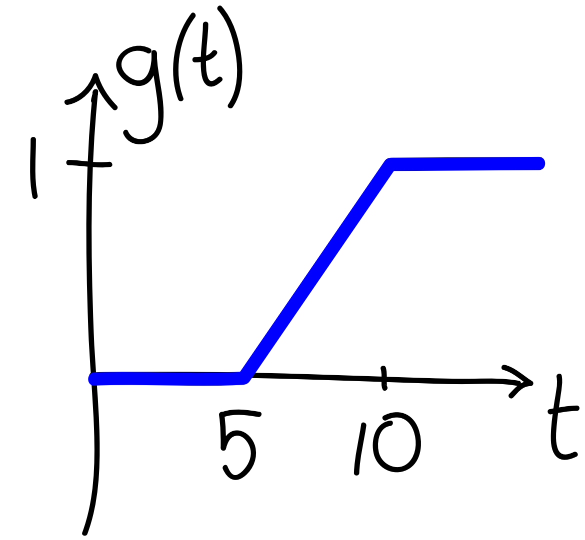

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Step function forcing (6.3, 6.4)

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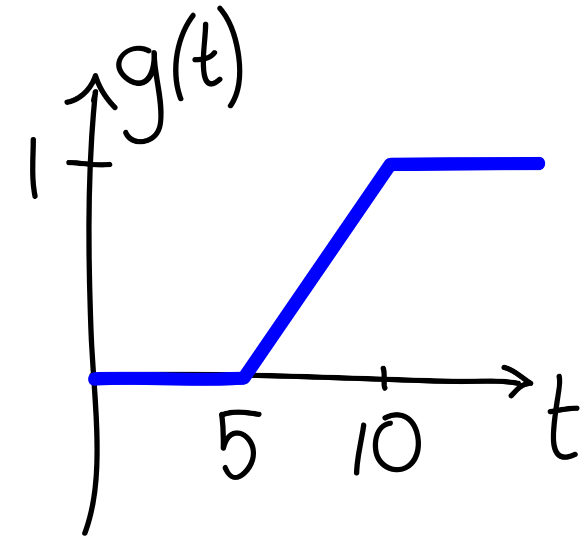

$$y'' + 4y = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

$$y(0) = 0,\ y'(0) = 0.$$

$$s^2Y + 4Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$Y(s) = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)} \quad = \frac{1}{5}(e^{-5s} - e^{-10s})H(s)$$

$$y(t) = \frac{1}{5}[u_5(t)h(t-5) - u_{10}(t)h(t-10)]$$

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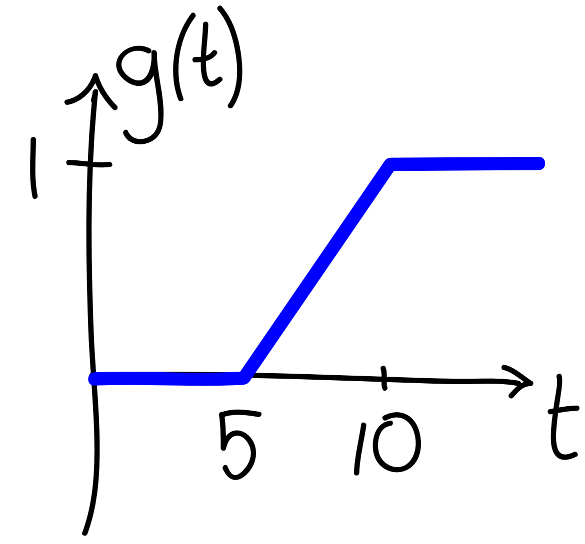

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Find h(t) given that $H(s) = \dfrac{1}{s^2(s^2+4)}$.

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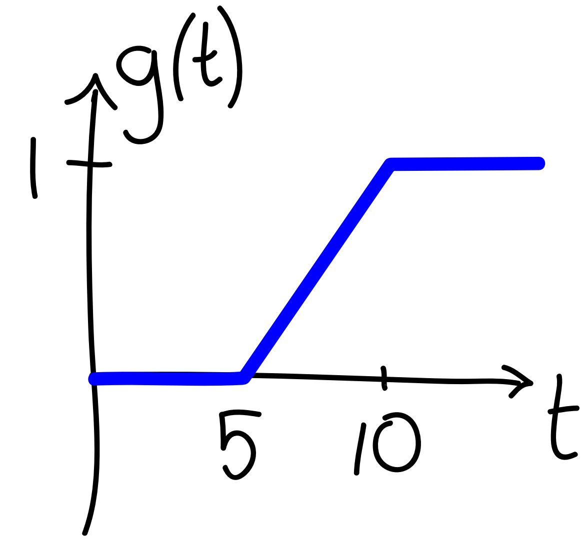

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Find h(t) given that $H(s) = \dfrac{1}{s^2(s^2+4)}$.

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$