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$$x_1'' = x_1' + x_2' = x_1' + 4x_1 + x_2$$

$$x_2 = x_1' - x_1$$

$$x_1'' = x_1' + 4x_1 + x_1' - x_1$$

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    - Other cases (not enough e-vectors or complex e-values) next class.