Today

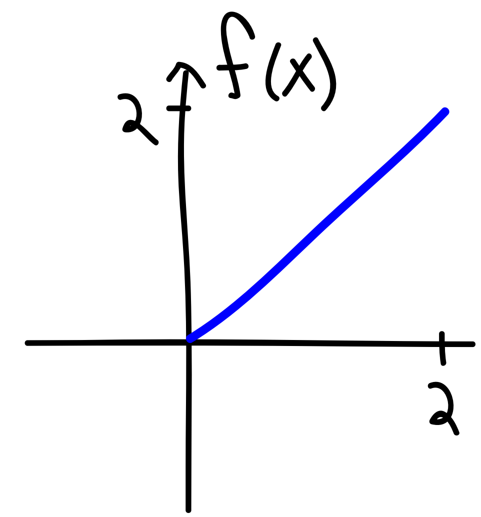
- Diffusion equation -
    - More examples: Dirichlet, Neumann BCs
    - Non homogeneous BCs

The Diffusion Equation

Solve the equation $\dfrac{dc}{dt} = D\dfrac{d^2c}{dx^2}$

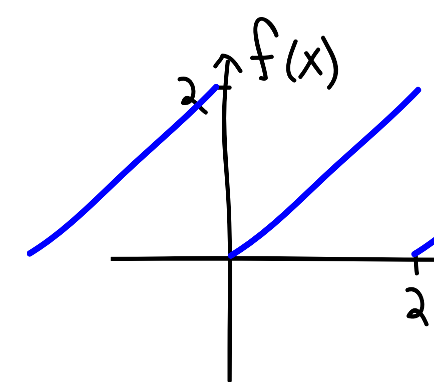
subject to boundary conditions $c(0,t) = 0,\ c(2,t) = 0$

and initial condition $c(x,0) = x$ defined on [0,2].

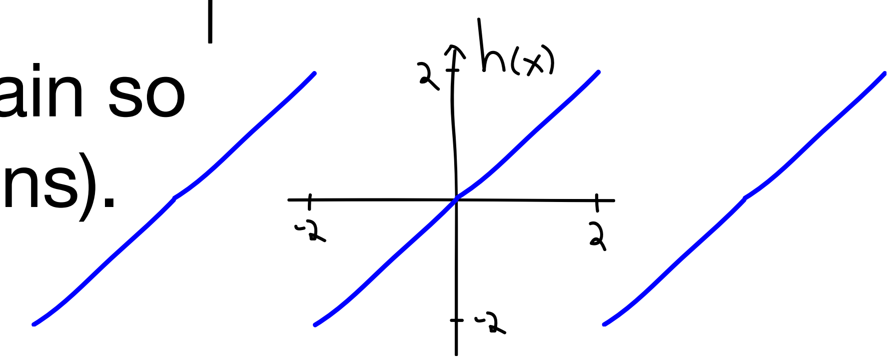


How do we solve this?

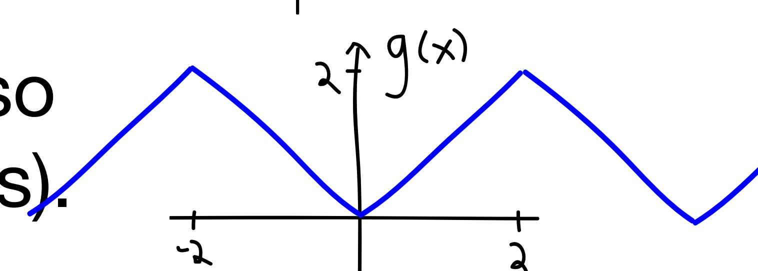
(A) Extend IC so it's periodic (P=2), then find FS.



(B) Extend IC so it's odd on [-2,2], then extend again so it's periodic (P=4), finally find FS (all sin functions).



(C) Extend IC so it's even on [-2,2], then extend again so it's periodic (P=4), finally find FS (const. and cosines).



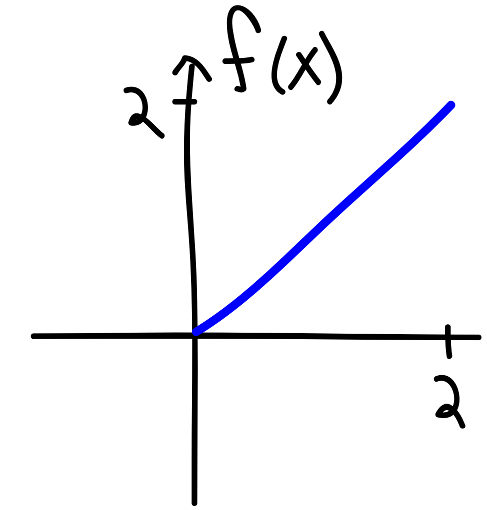
Note: the IC does not satisfy the BC at x=L in this case - that's ok.

The Diffusion Equation

Solve the equation $\dfrac{dc}{dt} = D\dfrac{d^2c}{dx^2}$

subject to boundary conditions $\dfrac{dc}{dx}(0,t) = \dfrac{dc}{dx}(L,t) = 0$

and initial condition $c(x,0) = x$ defined on [0,2].



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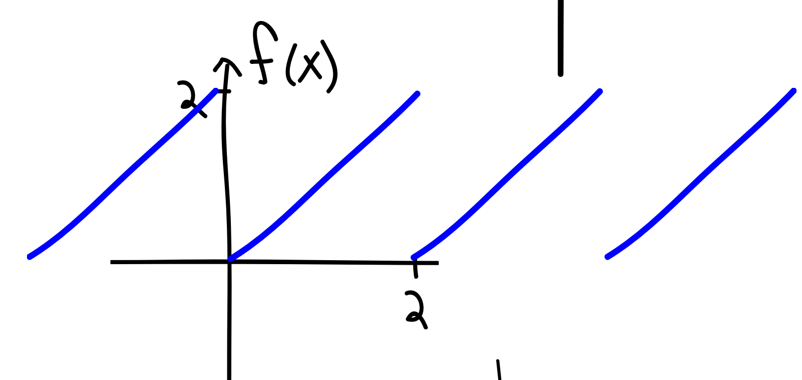

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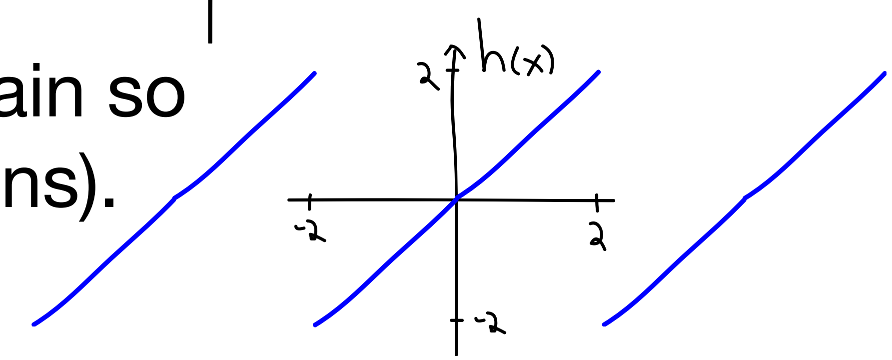

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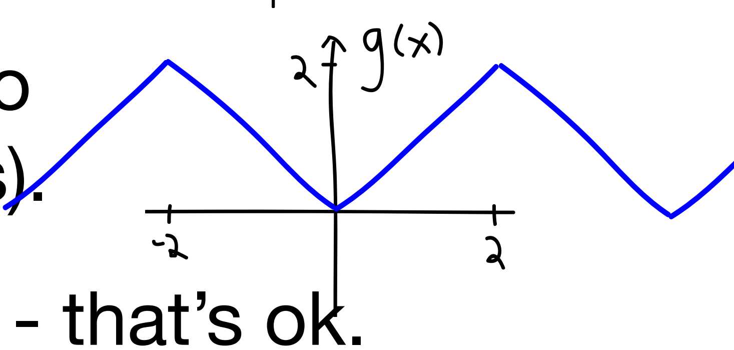


Note: the IC does not satisfy the BC at x=L in this case - that's ok.

...with nonhomogeneous boundary conditions

$$u_t = Du_{xx}$$

$$u(0,t) = 0$$
$$u(2,t) = 4$$ ⟶ Nonhomogeneous BCs

$$u(x,0) = x^2$$

Still use sin(nπx/L) but need to get end(s) away from zero!

What is steady state? $u_{ss}(x) = 2x$

Ultimately, we want $u(x,t) = 2x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$

What function do we use to calculate the Fourier series $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$?

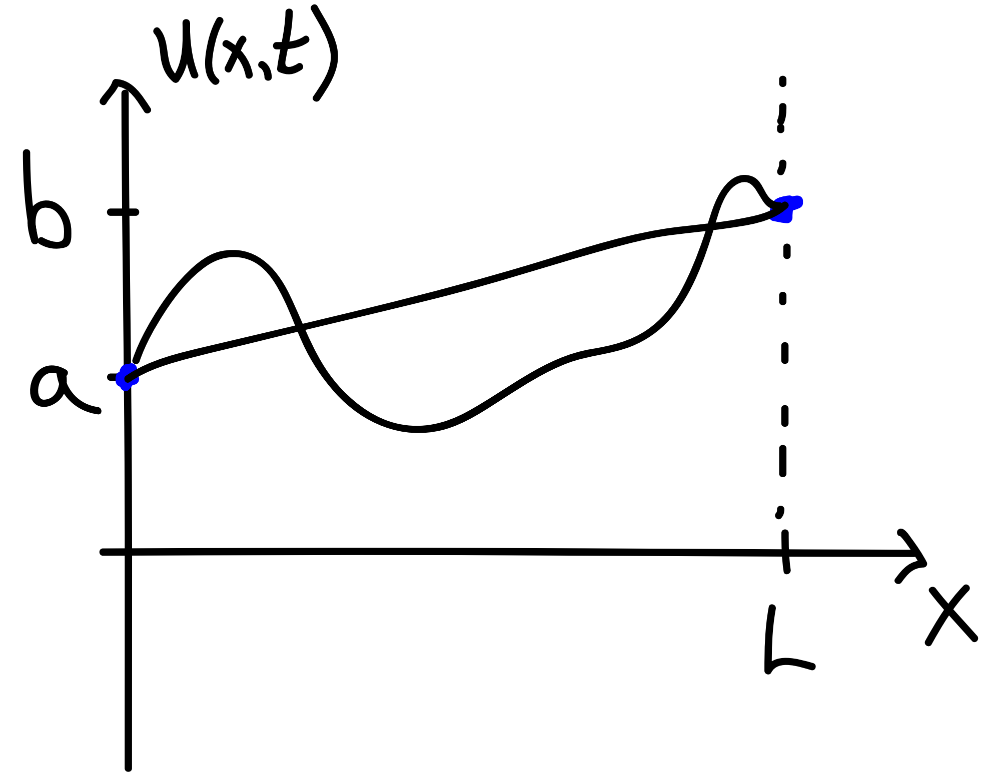
(A) x^2 (B) $x^2 - 2$ (C) $x^2 - 2x$ ★ (D) $x^2 + 2x$

...with nonhomogeneous boundary conditions

- Solve the Diffusion Equation with nonhomogeneous BCs:

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

- Recall - rate of change is proportional to concavity so bumps get ironed out.



$$v(x,t) = u(x,t) - \left(a + \frac{b-a}{L}x\right)$$

$$\left.\begin{aligned} v_t &= u_t \\ v_{xx} &= u_{xx} \end{aligned}\right\} \Rightarrow v_t = Dv_{xx}$$

$$v(0,t) = u(0,t) - a = 0$$
$$v(L,t) = u(L,t) - b = 0$$
$$v(x,0) = u(x,0) - \left(a + \frac{b-a}{L}x\right)$$

- v(x,t) satisfies the Diffusion Eq with homogeneous Dirichlet BCs and a new IC.

- General trick: define v=u-SS and find v as before.

https://www.desmos.com/calculator/6jp7jggsf9

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2}$

(B) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$

(C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2} + 9 - 2x$

(D) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2} + 9 - 2x$

where $b_n = ?$

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $b_n = \displaystyle\int_0^2 \sin\frac{3\pi x}{2}\cos\frac{n\pi x}{2}\;dx$

(B) $b_n = \displaystyle\int_0^2 \sin\frac{3\pi x}{2}\sin\frac{n\pi x}{2}\;dx$

★ (C) $b_n = \displaystyle\int_0^2 \left(\sin\frac{3\pi x}{2} - 9 + 2x\right)\sin\frac{n\pi x}{2}\;dx$

(D) $b_n = \displaystyle\int_0^2 \left(\sin\frac{3\pi x}{2} + 9 - 2x\right)\sin\frac{n\pi x}{2}\;dx$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2} + 9 - 2x$$

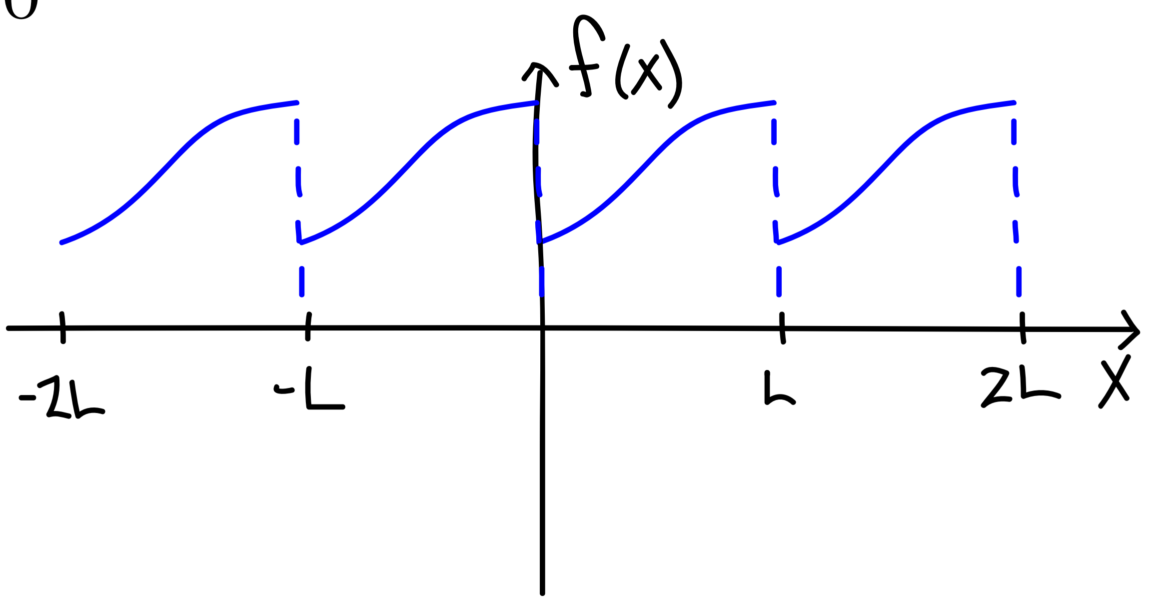
Review of solutions to the Diffusion Equation

$u_t = D u_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

- Extend f(x) to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

- All coefficients will be non-zero. Not particularly useful for solving the BCs.

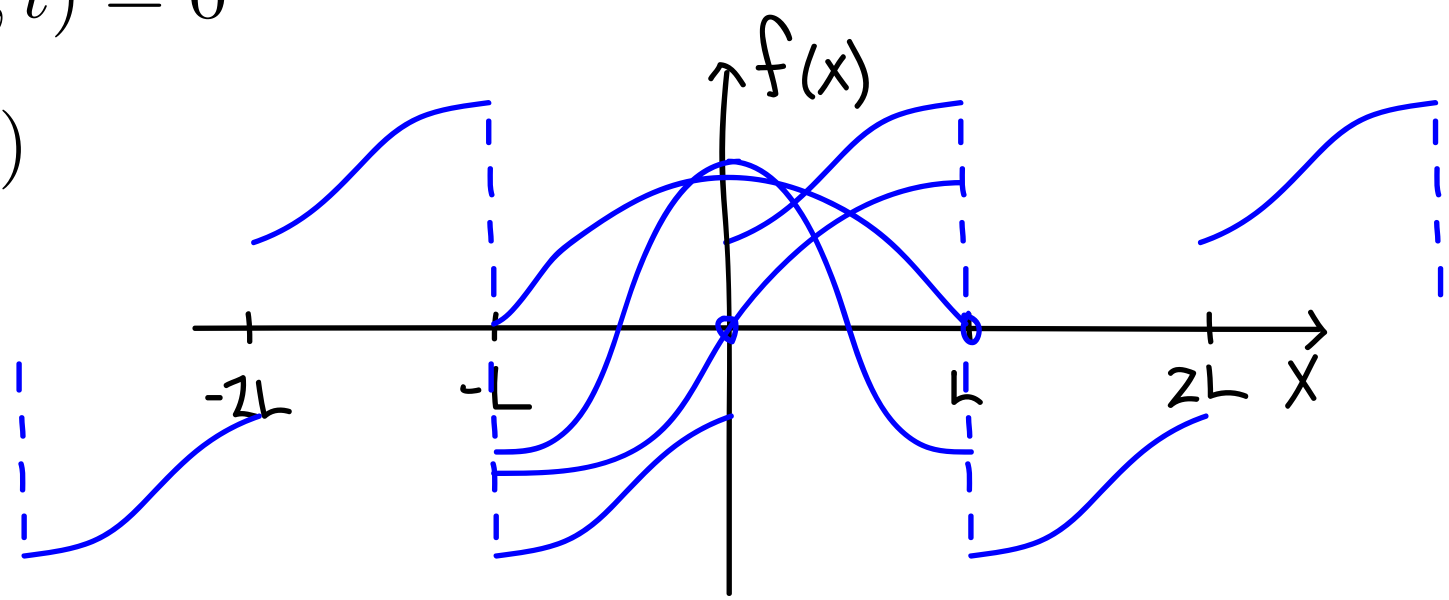
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend to -L as an odd function and then to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- Cosine coefficients will be zero because f(x) is odd about x=0 and cosine is even. Useful for solving the Diffusion equation with Dirichlet BCs.

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\cos\frac{n\pi x}{L}\,dx \qquad a_n = 0 \qquad b_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\sin\frac{n\pi x}{L}\,dx \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$u_t = Du_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\ dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x\right)\sin\frac{n\pi x}{L}\, dx$$

- Adding the linear function to the usual solution to the Dirichlet problem ensures that the BCs are satisfied without changing the fact that it satisfies the PDE.

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = a$$

$$u(x,0) = f(x)$$

$$u_{ss}(x) = ax + B$$

$$B = \frac{1}{L}\int_0^L f(x)\ dx - \frac{1}{2}aL$$

$$u(x,t) = ax + B + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L (f(x) - ax - B)\cos\frac{n\pi x}{L}\ dx$$

Review lectures

- What days are best for a review session / office hours?
- WeBWorK problems with low success.
- Exam review questions (on the wiki).
- Student requested problems.

Nonhomogeneous boundary conditions

- How would you solve this one?

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = -2$$

$$u(x,0) = \cos\frac{3\pi x}{2}$$