Today

- Modeling with delta-function forcing
- Convolution
- Transfer functions

Delta-function forcing (6.5)

- Water with c_{in} = 2 g/L of sugar enters a tank at a rate of r = 1 L/min. The initially sugar-free tank holds V = 5 L and the contents are well-mixed. Water drains from the tank at a rate r. At t_{cube} = 3 min, a sugar cube of mass m_{cube} = 3 g is dropped into the tank.

  - Write down an ODE for the mass of sugar in the tank as a function of time.

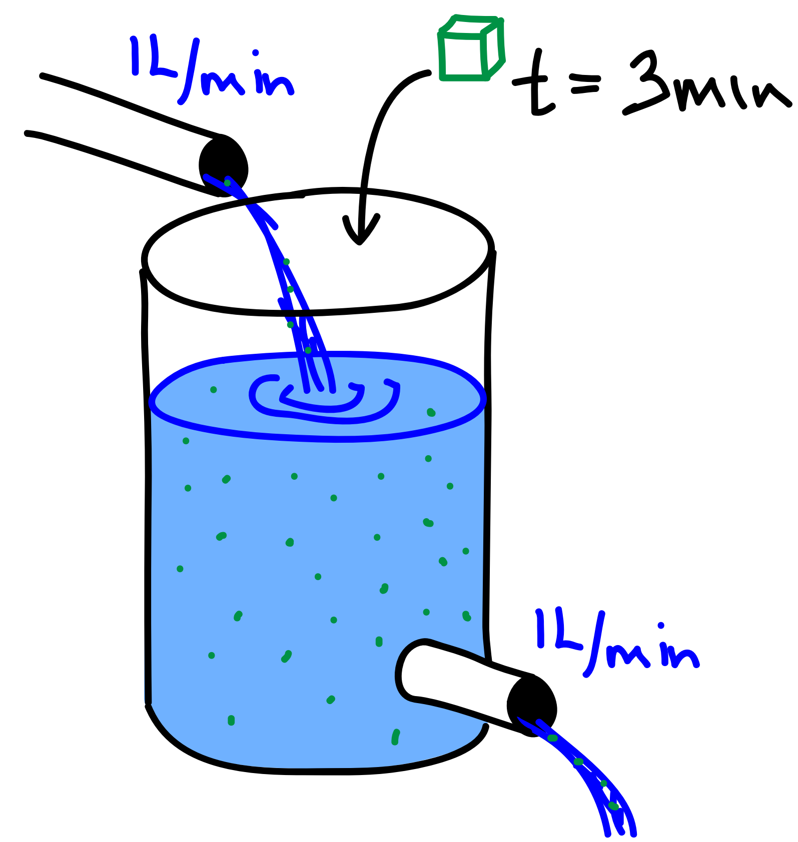
$$m' = rc_{in} - \frac{r}{V}m + m_{cube}\delta(t - t_{cube})$$

$$m' = 2 - \frac{1}{5}m + 3\ \delta(t-3)$$

  - Solve the ODE.

$$m(t) = 10(1 - e^{-t/5}) + 3u_3(t)e^{-(t-3)/5}$$

  - Sketch the solution to the ODE. How would it differ if t_{cube}=10 min?



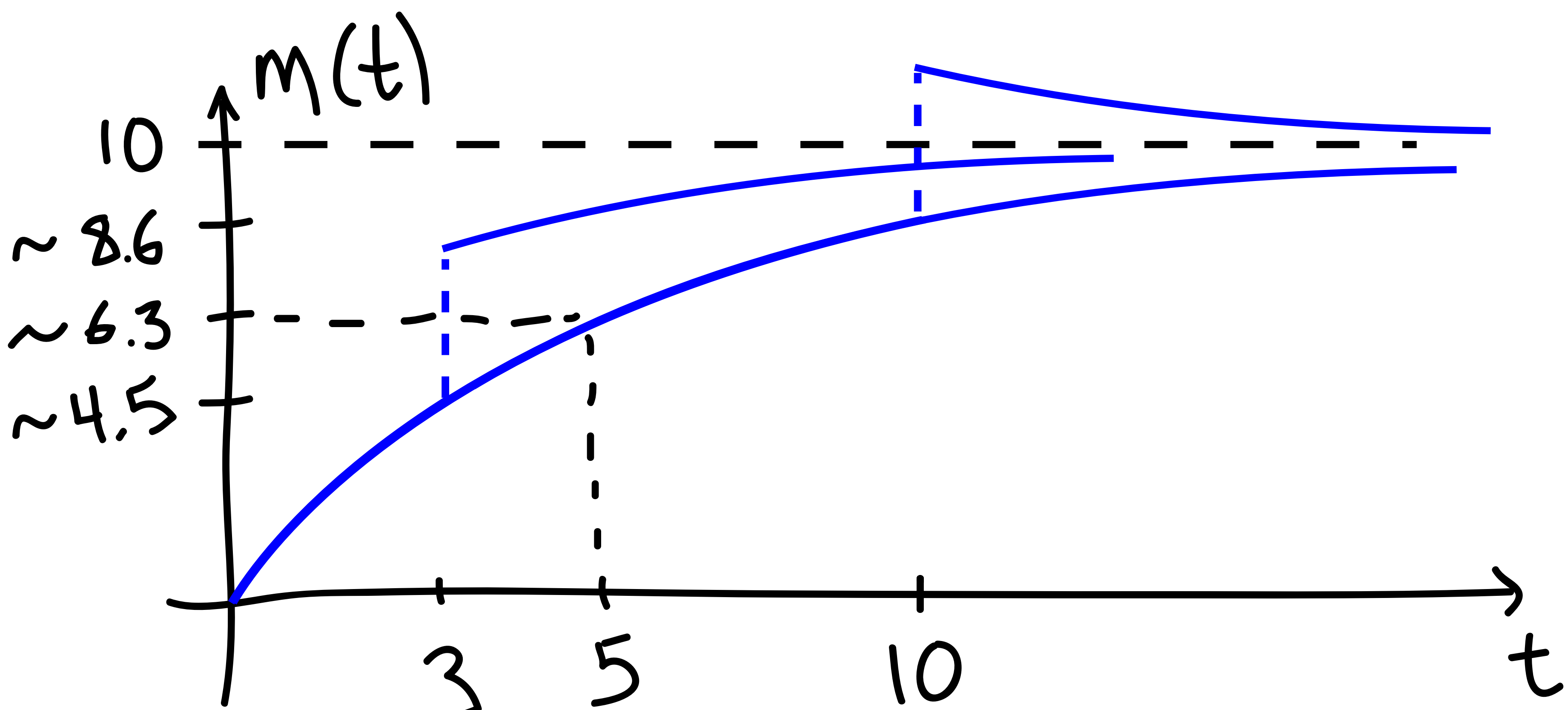
Delta-function forcing (6.5)

- Sketch the solution to the ODE.

$$m(t) = 10(1 - e^{-t/5}) + 3u_3(t)e^{-(t-3)/5}$$

$$= \begin{cases} 10(1 - e^{-t/5}) & \text{for } t < 3, \\ 10 - (10 - 3e^{3/5})e^{-t/5} & \text{for } t \geq 3. \end{cases}$$

- How would it differ if t_{cube}=10 min?



https://www.desmos.com/calculator/eizjtgf3jc

Delta-function forcing (6.5)

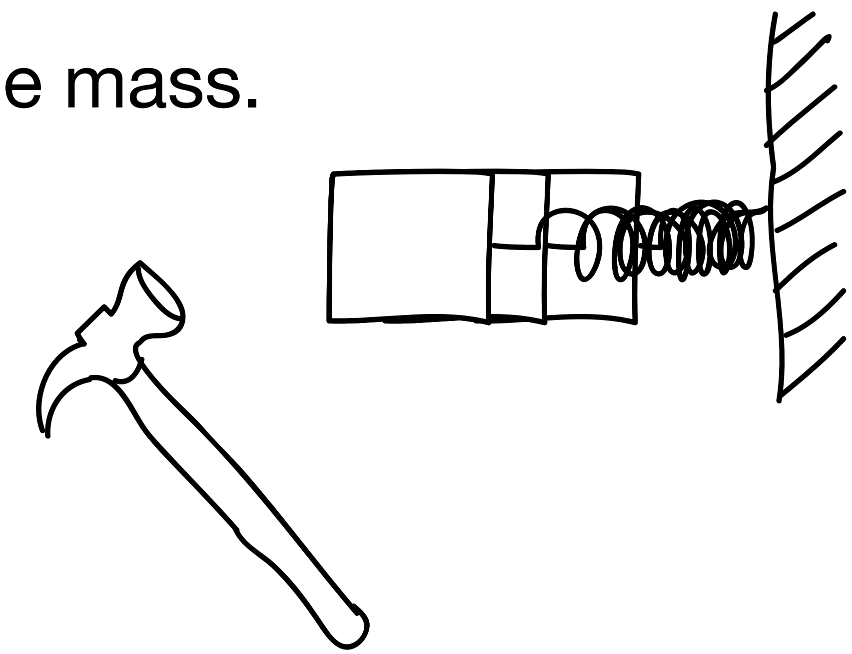
- A hammer hits a mass-spring system imparting an impulse of $I_0 = 2\ \mathrm{N\ s}$ at $t = 5\ \mathrm{s}$. The mass of the block is $m = 1\ \mathrm{kg}$. The drag coefficient is $\gamma = 2\ \mathrm{kg/s}$ and the spring constant is $k = 10\ \mathrm{kg/s}^2$. The mass is initially at $y(0) = 5\ \mathrm{m}$ with velocity $y'(0) = 0\ \mathrm{m/s}$.
  - Write down an equation for the position of the mass.

(A) $y'' + 2y' + 10y = 2\ u_0(t)$

(B) $y'' + 2y' + 10y = 2\ (u_0(t) - u_5(t))$

(C) $y'' + 2y' + 10y = 2\ u_5(t)$

(D) $y'' + 2y' + 10y = 2\ \delta(t - 5)$



$$s^2Y - 2s + 2sY - 4 + 10Y = 2e^{-5c}$$

$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10}$$

Delta-function forcing (6.5)

- Inverting Y(s)... (go through this on your own)

$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10} = \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+2}{s^2 + 2s + 10}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+2}{(s+1)^2 + 9}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{(s+1)^2 + 9}$$

$$= \frac{2e^{-5s}}{s^2 + 2s + 10} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{3}\frac{3}{(s+1)^2 + 9}$$

$$= \frac{2}{3}\frac{3e^{-5s}}{(s+1)^2 + 9} + 2\frac{s+1}{(s+1)^2 + 9} + \frac{2}{3}\frac{3}{(s+1)^2 + 9}$$

$$y(t) = \frac{2}{3}u_5(t)e^{-(t-5)}\sin(3(t-5)) + 2e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)$$

Convolution (6.6)

- We often end up with transforms to invert that are the product of two known transforms. For example,

$$Y(s) = \frac{2}{s^2(s^2+4)} = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$$

- Can we express the inverse of a product in terms of the known pieces?

$$F(s)G(s) = \mathcal{L}\{??\}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt \quad \longrightarrow \quad F(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau$$

$$G(s) = \int_0^\infty e^{-st} g(t)\ dt \quad \longrightarrow \quad G(s) = \int_0^\infty e^{-sw} g(w)\ dw$$

Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \int_0^\infty e^{-sw} g(w) \int_0^\infty e^{-s\tau} f(\tau)\ d\tau\ dw$$

(on the board)

$$= \int_0^\infty g(w) \int_0^\infty e^{-s(\tau+w)} f(\tau)\ d\tau\ dw$$

Replace τ using $u = \tau + w$.

$$= \int_0^\infty g(w) \int_w^\infty e^{-s(u)} f(u-w)\ du\ dw$$

$$= \int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

Convolution (6.6)

- What are the correct values for a, b, c and d?

$$\int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

☆ (C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

(E) Huh?

Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \boldsymbol{\int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du}$$

$$= \int_0^\infty e^{-su} \int_0^u g(w) f(u-w)\ dw\ du$$

$$= \int_0^\infty e^{-su} h(u)\ du\ = H(s)$$

The transform of a convolution is the product of the transforms.

$$h(t) = f * g(t) = \int_0^u g(w) f(t-w)\ dw$$

$$\Rightarrow H(s) = F(s)G(s)$$

where $h(u) = \int_0^u g(w) f(u-w)\ dw$

This is called the convolution of f and g. Denoted $f * g$.

Convolution (6.6)

- To invert $Y(s) = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$, we can use the fact that the inverse is the convolution of the inverses of the two pieces (instead of PFD...).

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin(2t)$$

$$f * g = g * f$$

$$\int_0^t f(t-w)g(w)\ dw = \int_0^t f(t)g(t-w)\ dw$$

$$y(t) =$$

☆ (A) $\int_0^t (t-w)\sin(2w)\ dw$

☆ (C) $\int_0^t w\sin(2(t-w))\ dw$

(B) $\int_0^t (t-w)\sin(2t)\ dw$

(D) $\int_0^t w\sin(2(w-t))\ dw$

Convolution (6.6)

- Transfer functions

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0$$

$$Y(s) = \frac{1}{as^2 + bs + c} G(s)$$

- Define the transfer function for the ODE:

$$H(s) = \frac{1}{as^2 + bs + c}$$

$$y(t) = (h * g)(t)$$

- h(t) is called the impulse response because it solves (1) when g(t)=δ(t).

$$g(t) = \delta(t)$$

$$G(s) = 1$$

$$Y(s) = \frac{1}{as^2 + bs + c}$$

$$y_{IR}(t) = h(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}$$

Convolution (6.6)

- Interpreting the transfer function in a model of memory.
- Suppose your contact list got deleted and you are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget number at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.

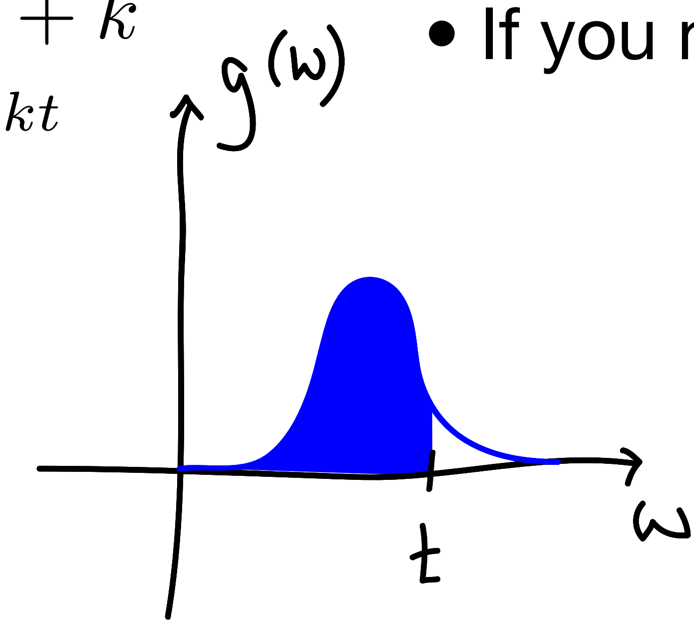
$$n' = -kn + g(t)$$

$$N(s) = \frac{G(s)}{s+k}$$

$$H(s) = \frac{1}{s+k}$$

$$h(t) = e^{-kt}$$

- If you memorize one phone number (g(t)=δ(t)), h(t) tells you what's left of that memory.
- If you memorize numbers over time,



$$n(t) = \int_0^t h(t-w)g(w)\ dw$$
$$= \int_0^t e^{-(t-w)}g(w)\ dw$$