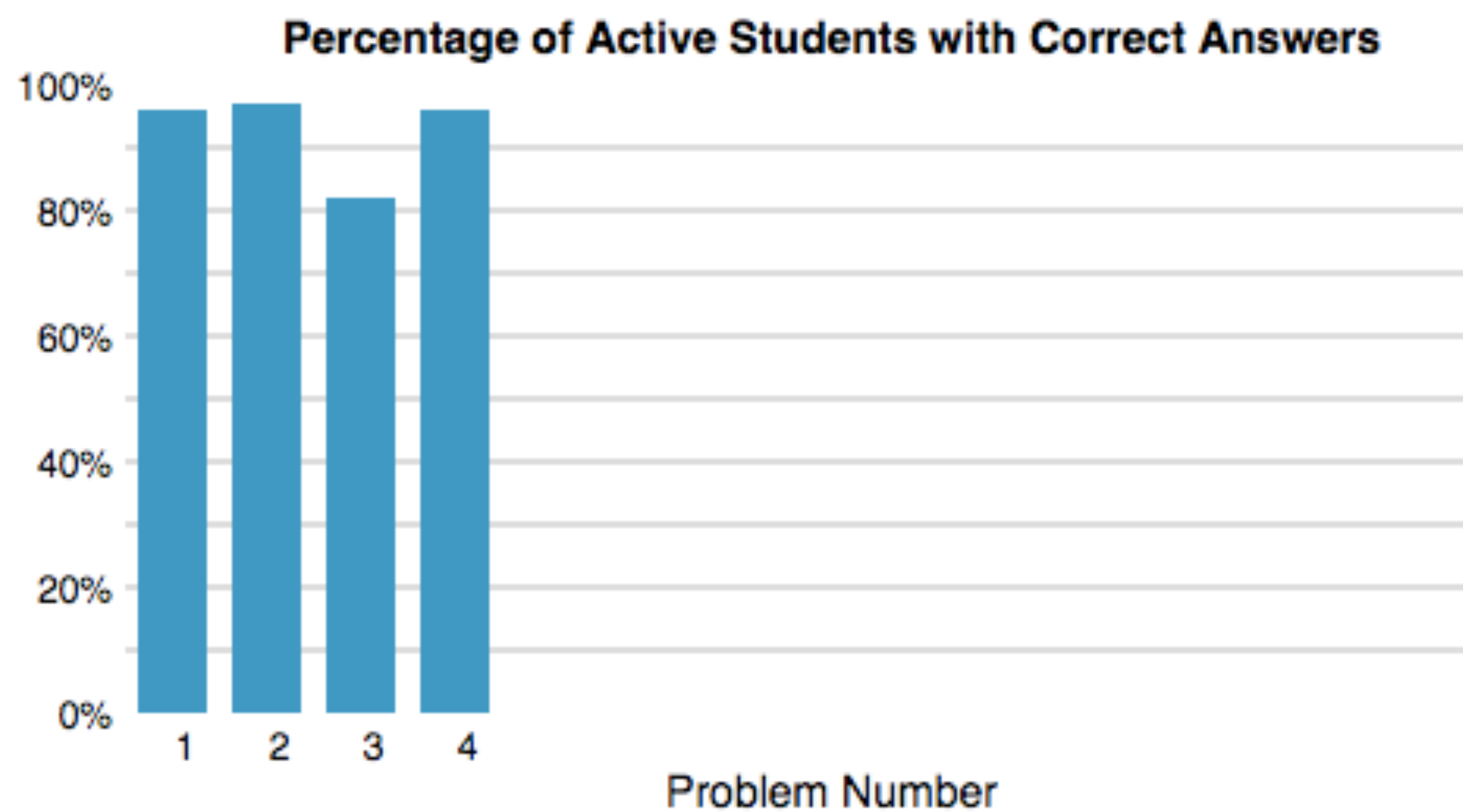
Today

- Comment on pre-lecture problems
- Finish up with integrating factors
- The structure of solutions
- Separable equations

Pre-lecture assignment comments



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Consider the initial value problem

$$\frac{dy}{dt} - 6y = 5e^{2t}, \quad y(0) = A.$$

a. Find the solution.

$y =$ ______

b. For what values of A does the above solution tend to ∞, 0 or $-\infty$ as $t \to \infty$?

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$$y' - 6y = 5e^{2t}$$

$$(e^{-6t}y)' = 5e^{-4t}$$

$$e^{-6t}y = -\frac{5}{4}e^{-4t} + C$$

$$y(t) = -\frac{5}{4}e^{2t} + Ce^{6t}$$

$$y(0) = -\frac{5}{4} + C = A$$

$$C = A + \frac{5}{4}$$

$$y(t) = -\frac{5}{4}e^{2t} + \left(A + \frac{5}{4}\right)e^{6t}$$

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Desmos demo: https://www.desmos.com/calculator/ne9u9c2q3b

Method of integrating factors

- Given that $\frac{d}{dt}\left(t^2 y(t)\right) = t^2 \frac{dy}{dt} + 2ty$

- if you're given the equation $t^2 \frac{dy}{dt} + 2ty = 0$

- you can rewrite is as $\frac{d}{dt}\left(t^2 y(t)\right) = 0$

arbitrary constant that appeared at an integration step

- so the solution is $t^2 y(t) = C$ or equivalently $y(t) = \frac{C}{t^2}$.

Method of integrating factors

- Solve the equation $t^2 \dfrac{dy}{dt} + 2ty(t) = \sin(t)$ (not brute force checking).

(A) $y(t) = -\cos(t) + C$

(B) $y(t) = \dfrac{C - \cos(t)}{t^2}$

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(D) $y(t) = -\dfrac{1}{t^2}\cos(t)$

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(D) $y(t) = -\dfrac{1}{t^2}\cos(t)$ ← a particular solution

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- An Initial Value Problem (IVP) is a ODE together with an IC.

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Technical definition of integrating factor

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$$y' + p(t)y = q(t)$$

- The function that, when multiplied through, make the LHS a perfect product rule is called the integrating factor.

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The structure of solutions

- When the equation is of the form (called homogeneous)

$$\frac{dy}{dt} + p(t)y = 0$$

- the solution is

$$y(t) = C\mu(t)^{-1}$$

- where

$$\mu(t) = \exp\left(\int p(t)dt\right)$$

- is the integrating factor.

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- Think about this expression as $y(t) = y_p(t) + y_h(t)$

- Directly analogous to solving the vector equations $A\overline{x} = 0$ and $A\overline{x} = \overline{b}$.

Examples

- Find the general solution to

$$t\frac{dy}{dt} + 2y = 4t^2$$

- and plot a few of the integral curves.

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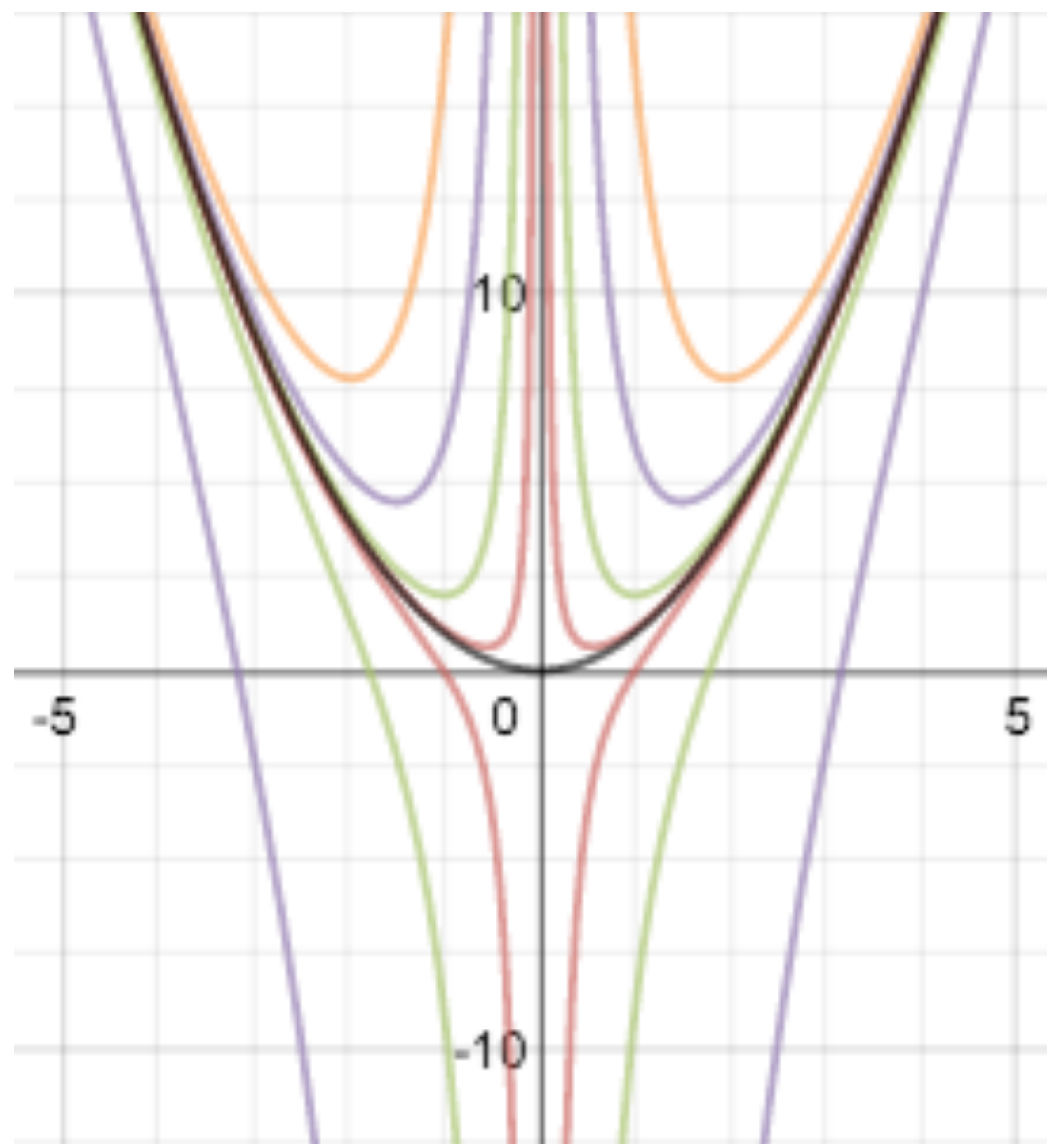

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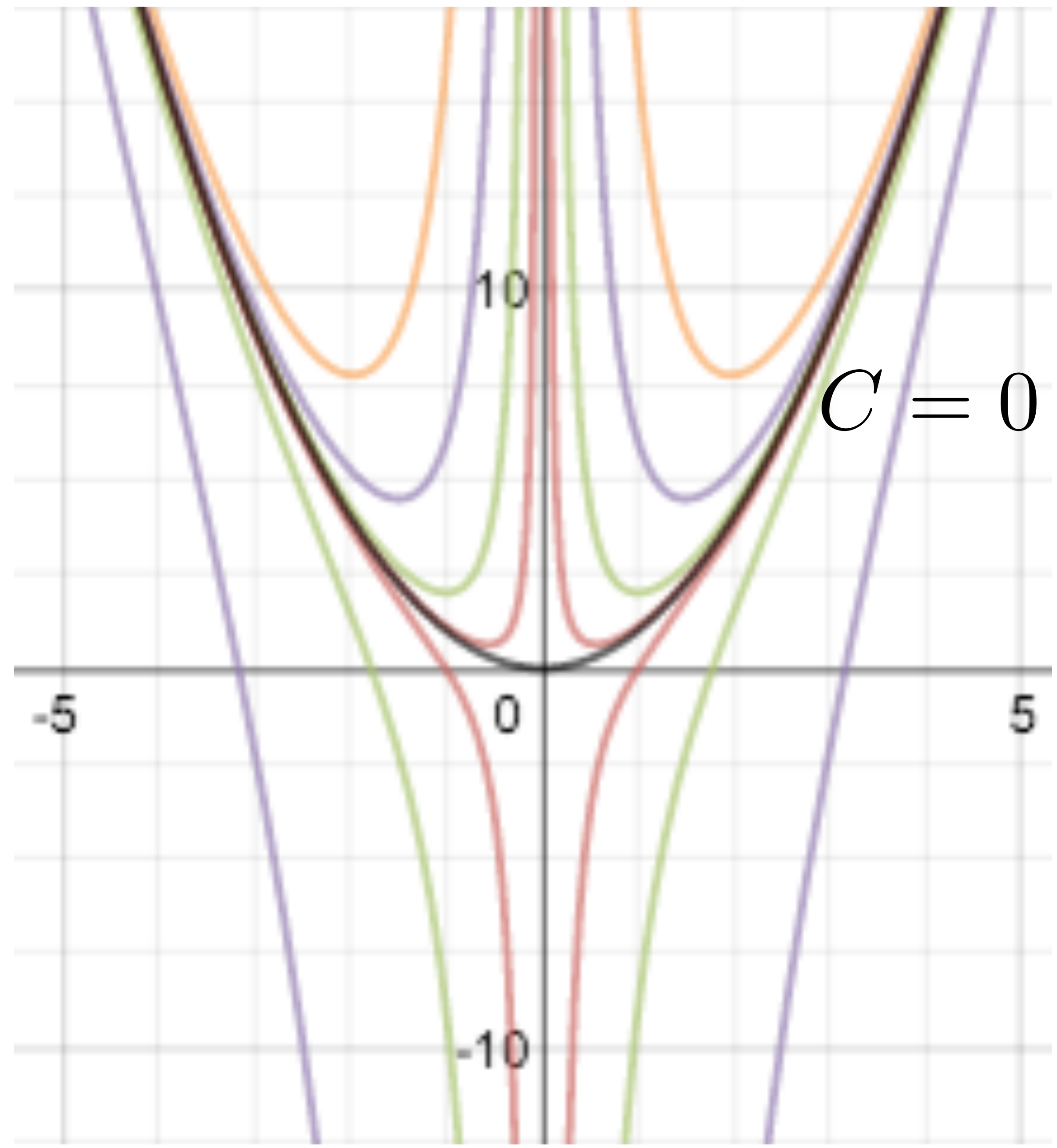

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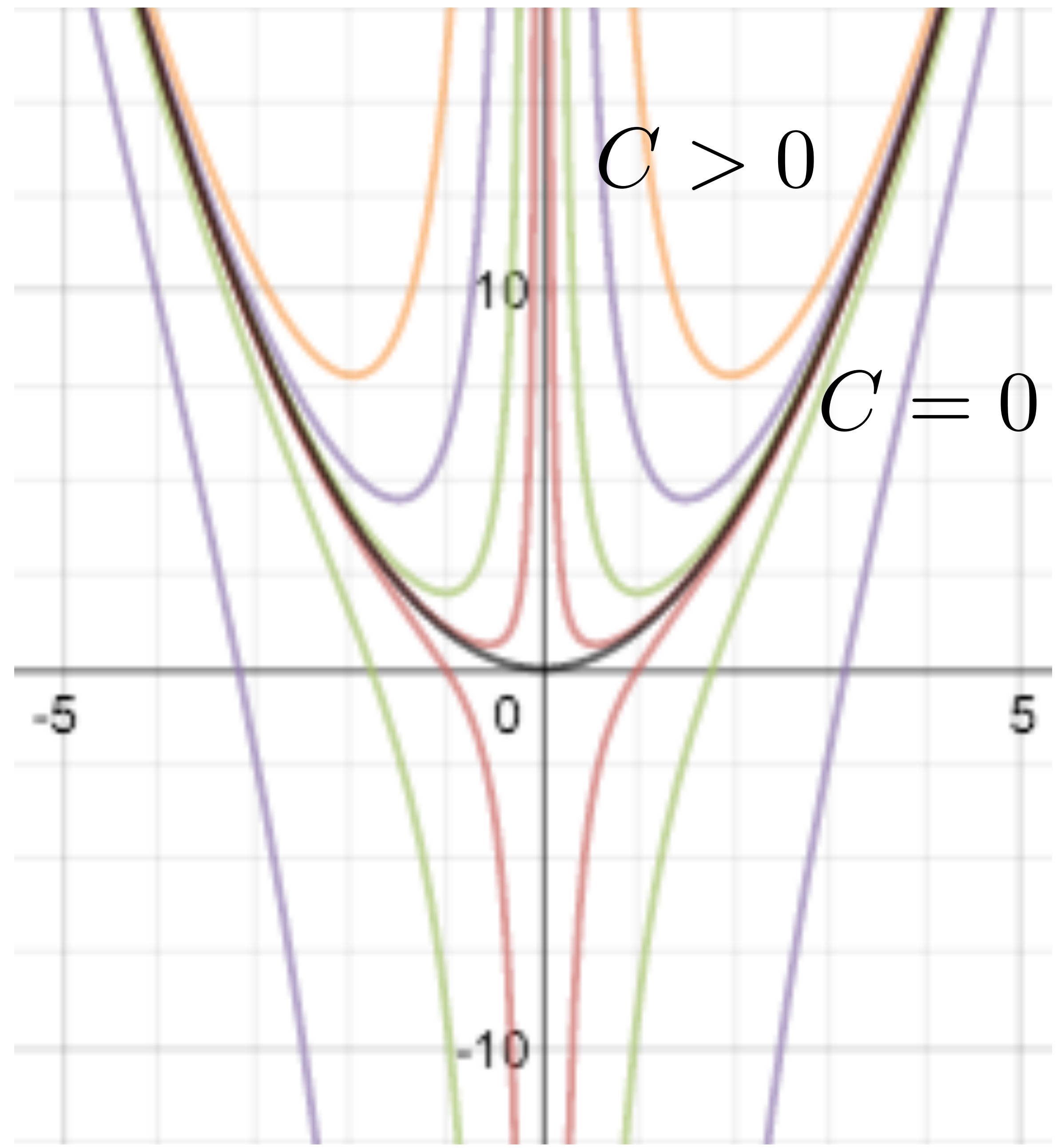

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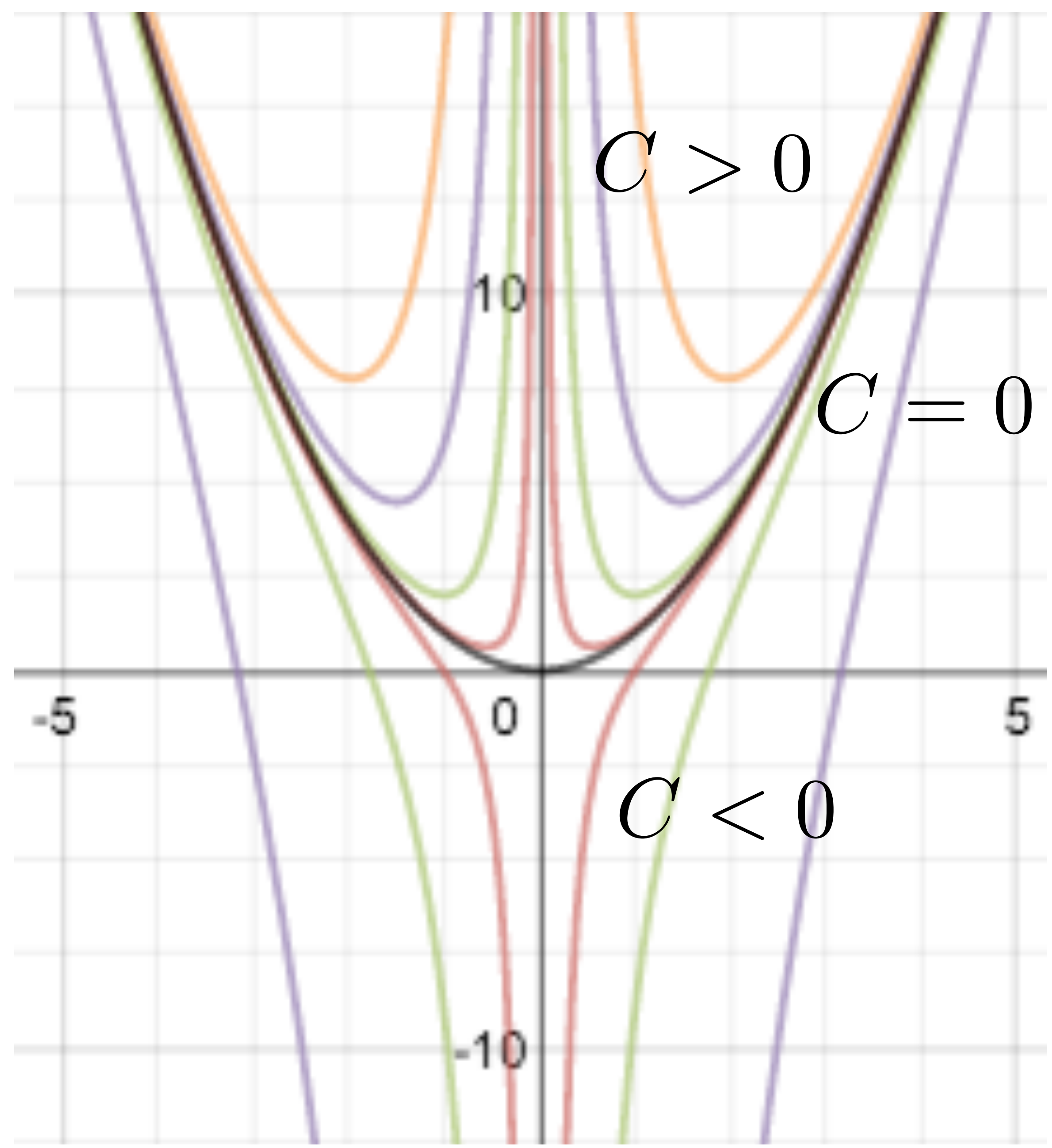

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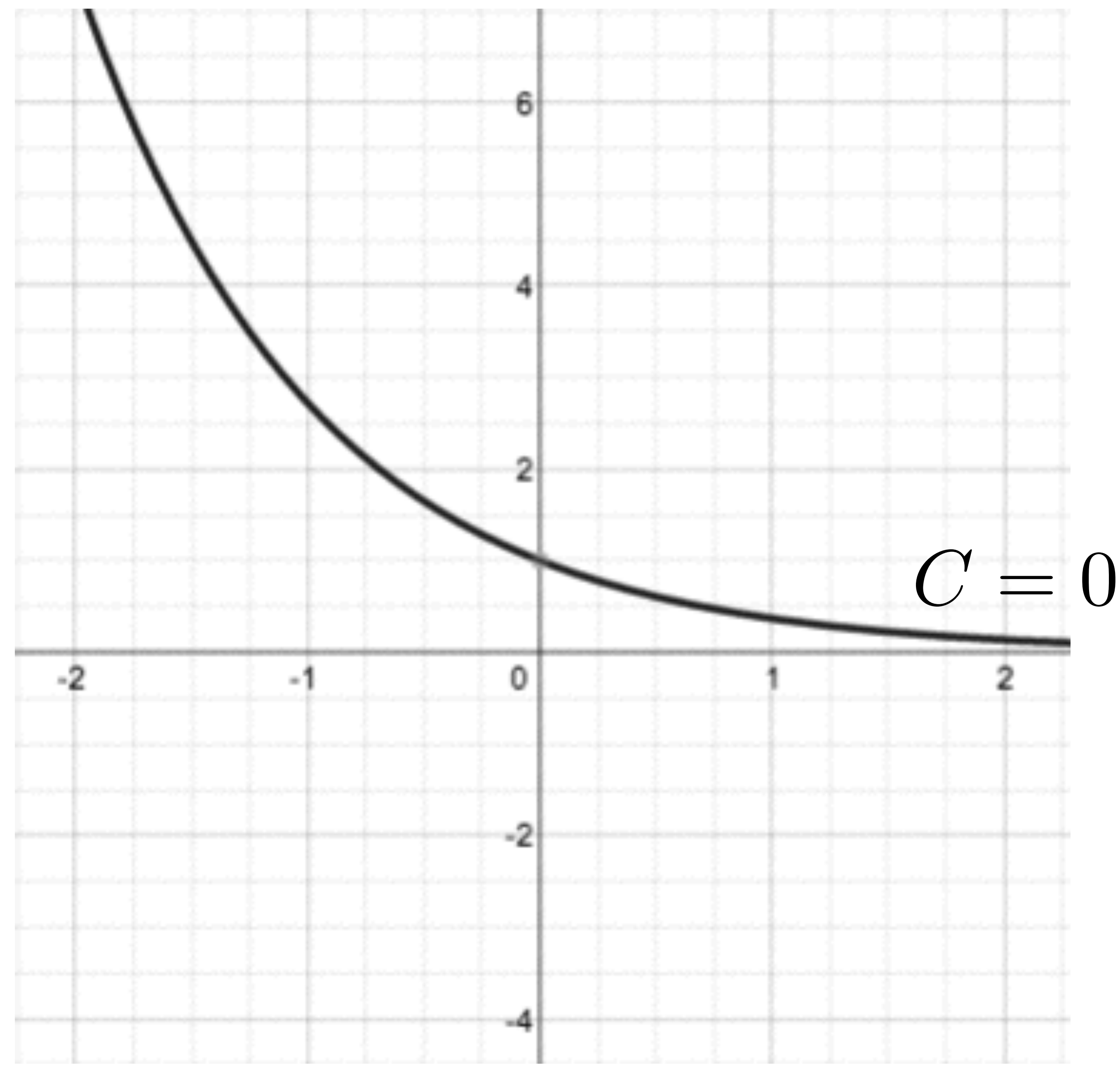

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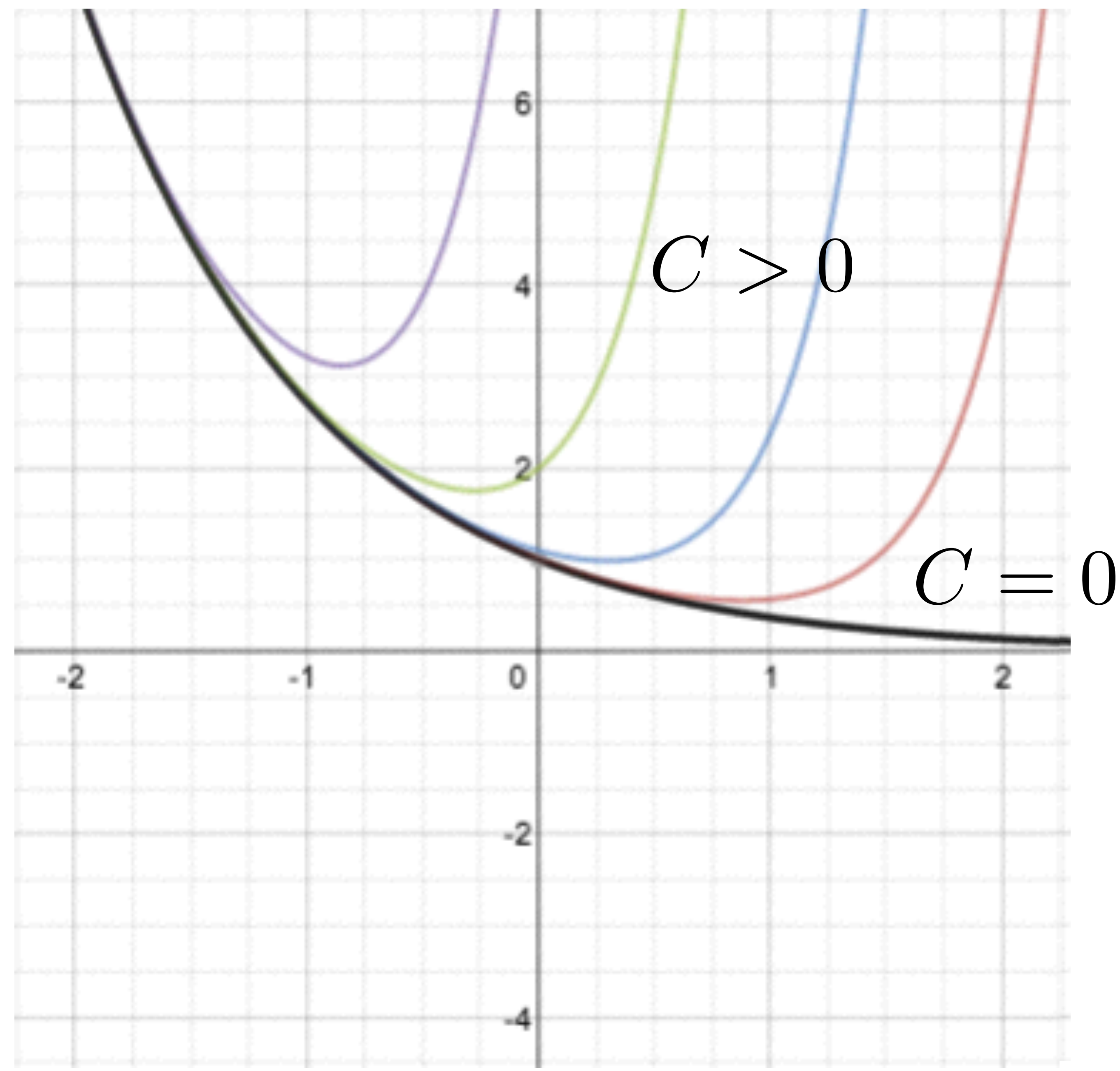

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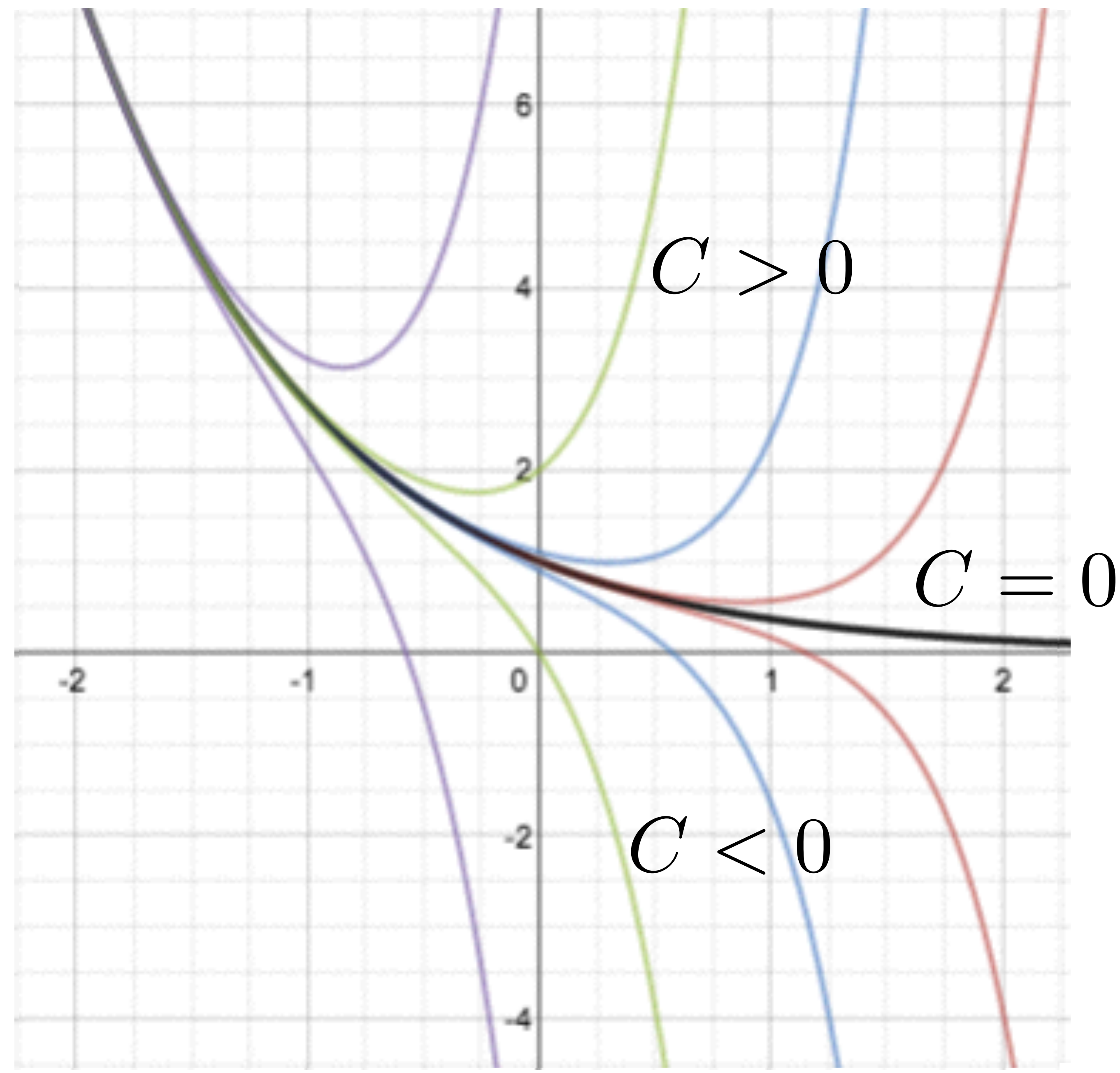

Limits at infinity

- If y(t) is a particular solution to

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- depending on C, how many different results are possible for

$$\lim_{t\to\infty} y(t) \text{ ?}$$

(A) 0

(B) 1

(C) 2

(D) 3

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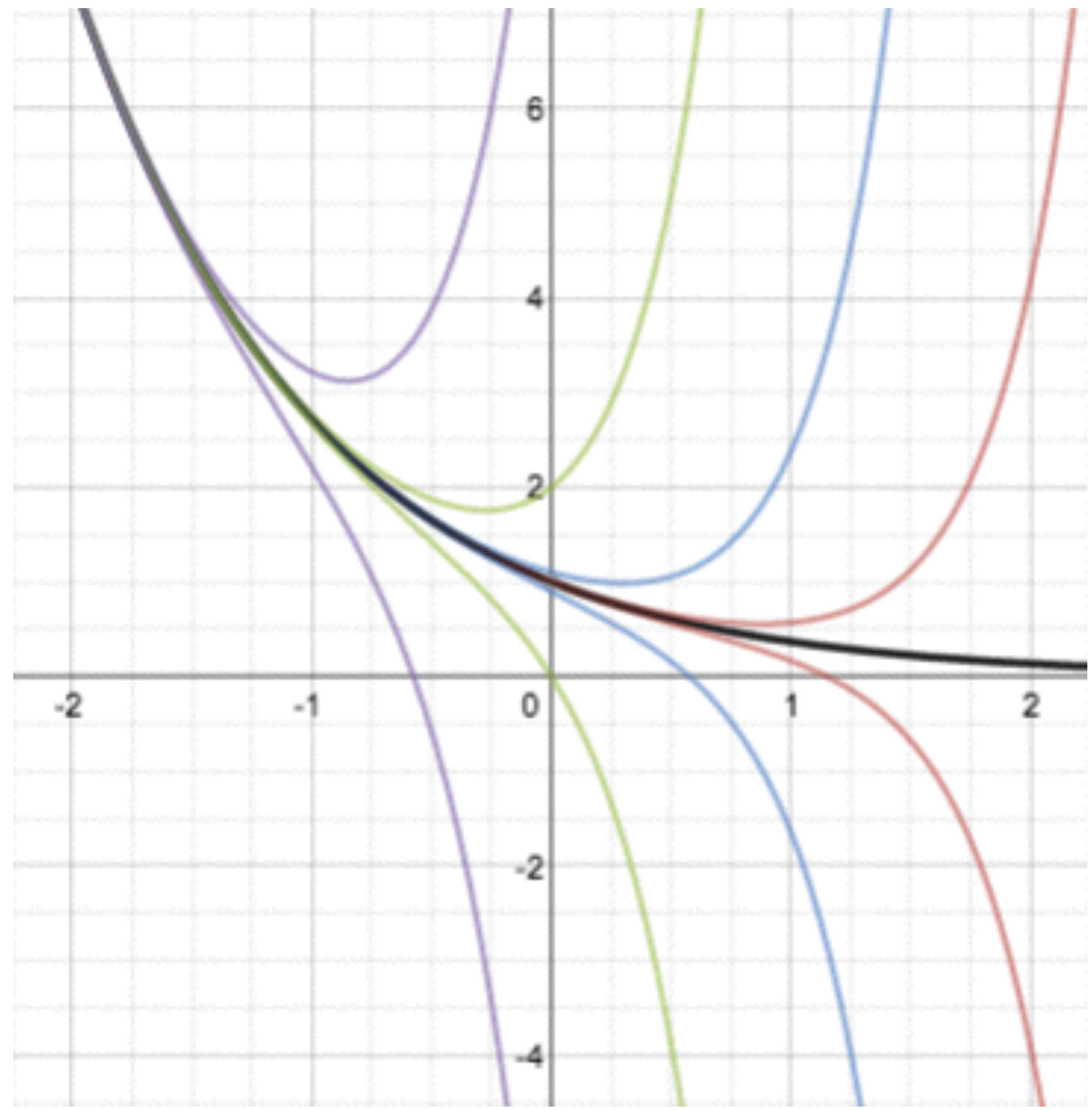

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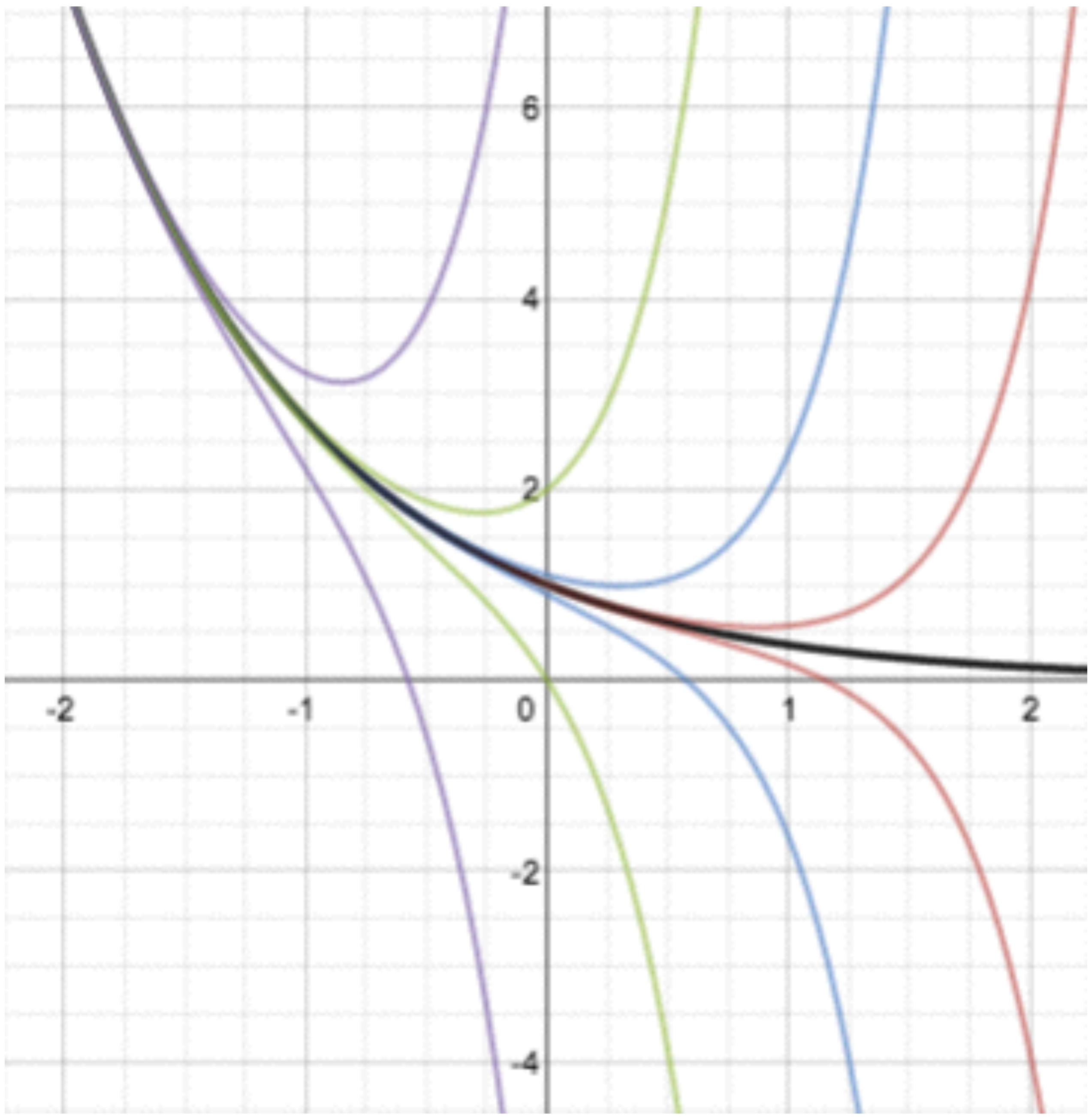

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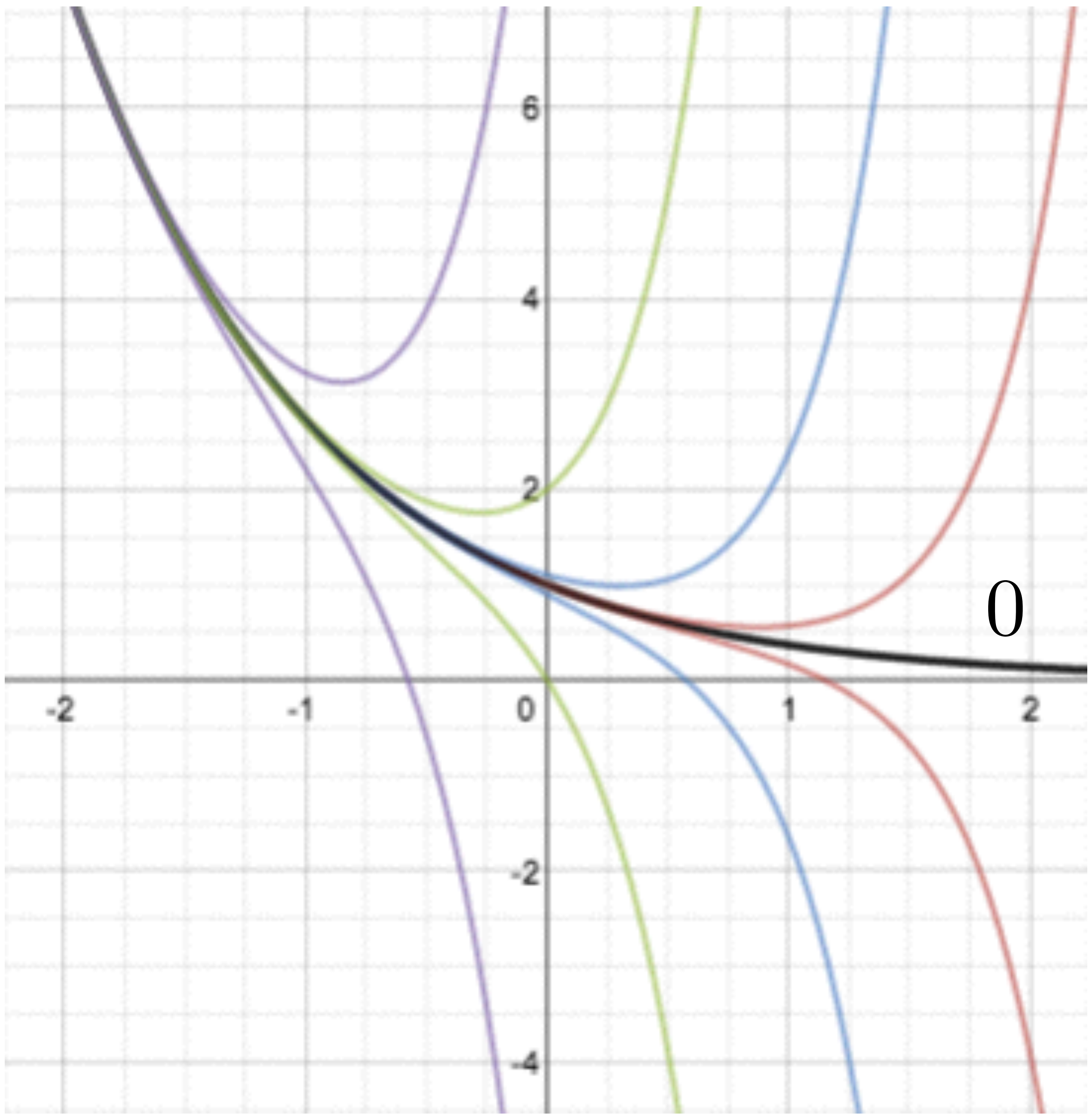

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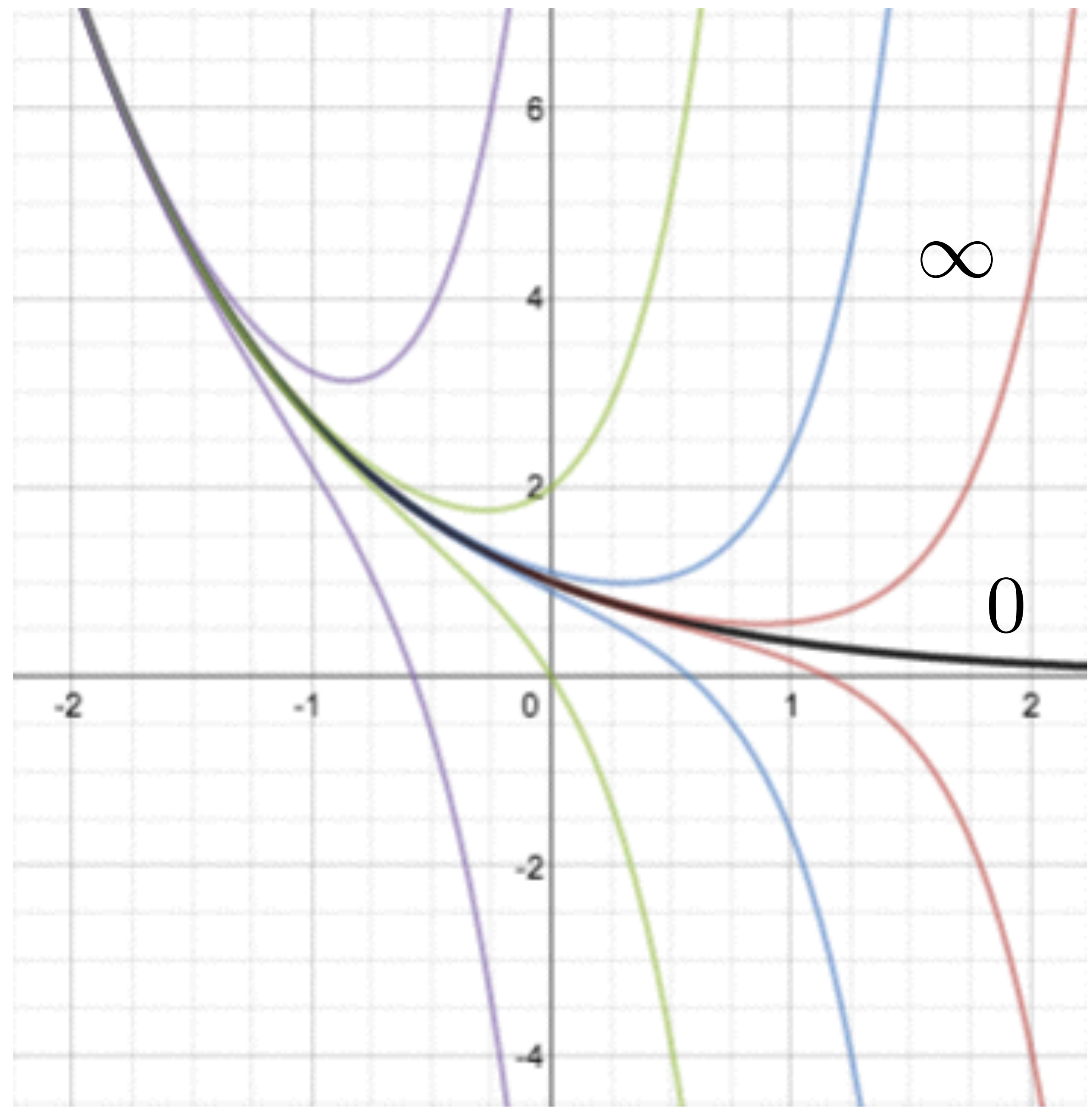

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