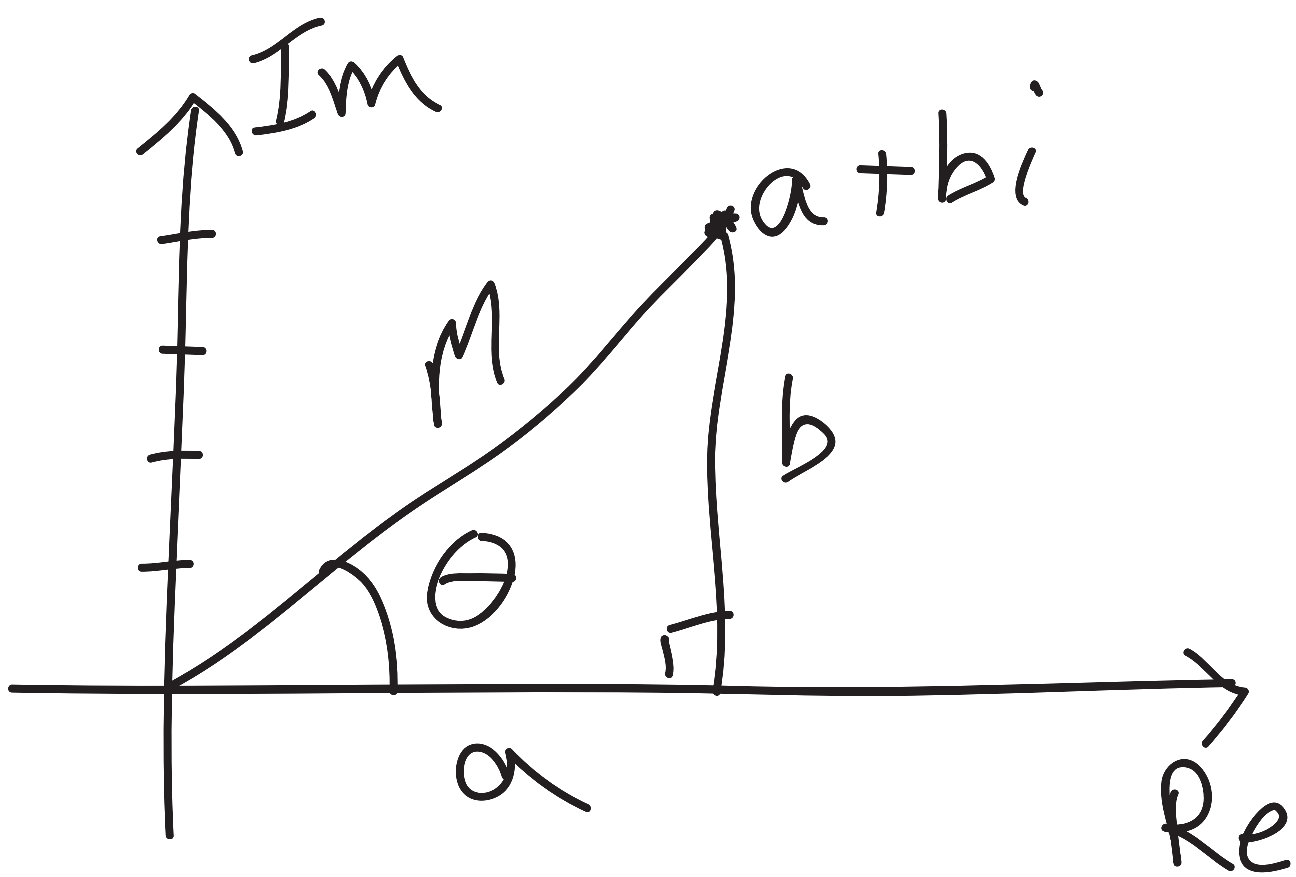
Today

- Euler's formula
- Complex case
- Repeated roots
- The geometry of homogeneous and nonhomogeneous matrix equations
- Solving nonhomogeneous equations
    - Method of undetermined coefficients

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M \cos \theta$$

$$b = M \sin \theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

$$a + bi = M(\cos \theta + i \sin \theta)$$

θ is sometimes called the argument or phase of $a + bi$.

Complex number review

- Toward Euler's formula
  - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

  - What function has Taylor series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ / $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ / $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

★ (A) cos x ★ (C) e^x

★ (B) sin x (D) ln x

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$-1 = i^2$$

$$= 1 + i\theta + (-1)\frac{\theta^2}{2!} + (-1)i\frac{\theta^3}{3!} + (-1)^2\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

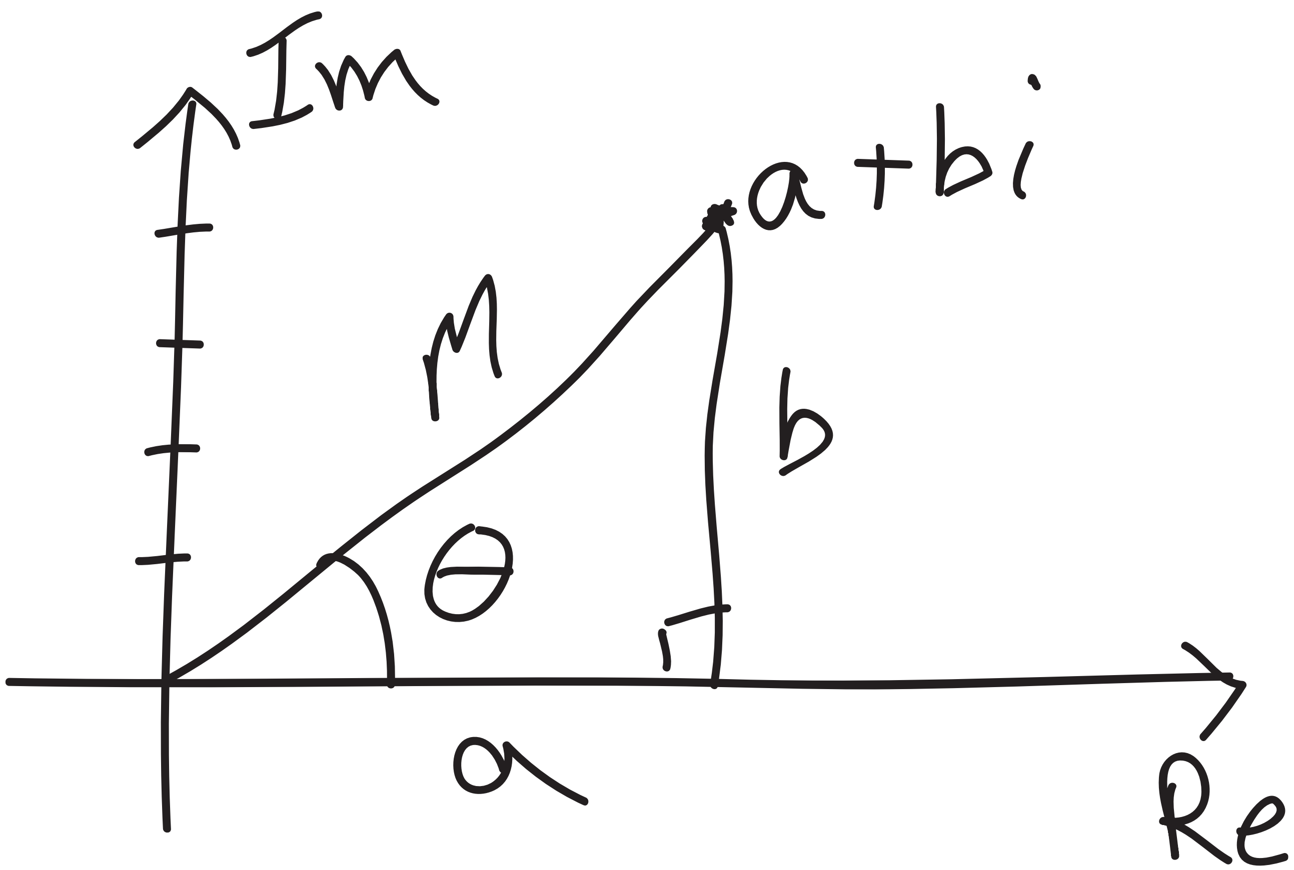
$$\cos\theta + i\sin\theta$$

Euler's formula:

$$= e^{i\theta}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

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$$a + bi = M(\cos\theta + i\sin\theta)$$

$$a + bi = Me^{i\theta}$$

(Polar form makes multiplication much cleaner)

Complex roots

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- When b^2 - 4ac < 0, we get complex roots:

$$\begin{aligned} r_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a} \\ &= \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i \\ &= \alpha \pm \beta i \end{aligned}$$

Complex roots

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$
\begin{aligned}
y_1(t) &= e^{(\alpha+\beta i)t} \\
&= e^{\alpha t} e^{i\beta t} \\
&= e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) &= e^{(\alpha-\beta i)t} \\
&= e^{\alpha t} e^{-i\beta t} \\
&= e^{\alpha t}(\cos(-\beta t) + i \sin(-\beta t)) \\
&= e^{\alpha t}(\cos(\beta t) - i \sin(\beta t))
\end{aligned}
$$

Complex roots

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$y_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$y_2(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

- Instead of using these to form the general solution, let's use them to find two real valued solutions:

$$\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{\alpha t}\cos(\beta t)$$

$$\frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = e^{\alpha t}\sin(\beta t)$$

- General solution:

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)}$$

Complex roots

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i$, $r_2 = 1 - 2i$

(B) $r_1 = -1 + 2i$, $r_2 = -1 - 2i$

(C) $r_1 = 1 - 2i$, $r_2 = -1 + 2i$

(D) $r_1 = 2 + 4i$, $r_2 = 2 - 4i$

(E) $r_1 = -2 + 4i$, $r_2 = -2 - 4i$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

☆ (A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

(D) $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$

Complex roots

- Example: Find the solution to the IVP

$$y'' + 2y' + 5y = 0, \;\; y(0) = 1, \;\; y'(0) = 0$$

- General solution: $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

(B) $y(t) = e^{-t}\left(\cos(2t) - \dfrac{1}{2}\sin(2t)\right)$

(C) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

☆ (D) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) + \sin(2t))$