Deriving the FS coefficient formulae

Define the dot product for periodic functions (with period P)

$$f(x)\circ g(x) = \int_{\substack{\text{one}\\ \text{period}}} f(x)\cdot g(x)\,dx = \int_{-P/2}^{P/2} f(x)g(x)\,dx$$

Let $v_n(x) = \cos\left(\frac{2\pi n x}{P}\right)$, $w_n(x) = \sin\left(\frac{2\pi n x}{P}\right)$, $v_0(x) = 1$. $(n = 1, 2, \cdots)$

Recall (or calculate for yourself) that

$v_0(x)\circ v_0(x) = P$, $v_m(x)\circ v_n(x) = 0$ for $m \neq n$, $v_n(x)\circ v_n(x) = P/2$

$w_m(x)\circ v_n(x) = 0$, $w_m(x)\circ w_n(x) = 0$ for $m \neq n$, $w_n(x)\circ w_n(x) = P/2$

Suppose $f(x)$ can be represented exactly as a FS. Thus

$$f(x) = A_0 v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x) + \sum_{m=1}^{\infty} b_m v_m(x).$$

Find its FS coefficients. As with vectors, use "$\circ$" to find A_0, a_n, b_n.

To find A_0,

$$f(x)\circ v_0(x) = A_0 v_0(x)\circ v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x)\circ v_0(x) + \sum_{m=1}^{\infty} b_m w_m(x)\circ v_0(x) = A_0\cdot P$$

Thus, $A_0 = \frac{1}{P} f(x)\circ v_0(x) = \frac{1}{P}\int_{-P/2}^{P/2} f(x)\,dx.$

To find a_n,

$$f(x)\circ v_n(x) = A_0 v_0(x)\circ v_n(x) + \sum_{m=1}^{\infty} a_m v_m(x)\circ v_n(x) + \sum_{m=1}^{\infty} b_m w_m(x)\circ v_n(x) = a_n \underbrace{v_n(x)\circ v_n(x)}_{P/2}$$

Thus, $a_n = \frac{2}{P} f(x)\circ v_n(x) = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\frac{2n\pi x}{P}\,dx.$

Similarly, $b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\frac{2n\pi x}{P}\,dx$

<u>In many</u> cases, we will have $P = 2L$ (but not always!) so

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$