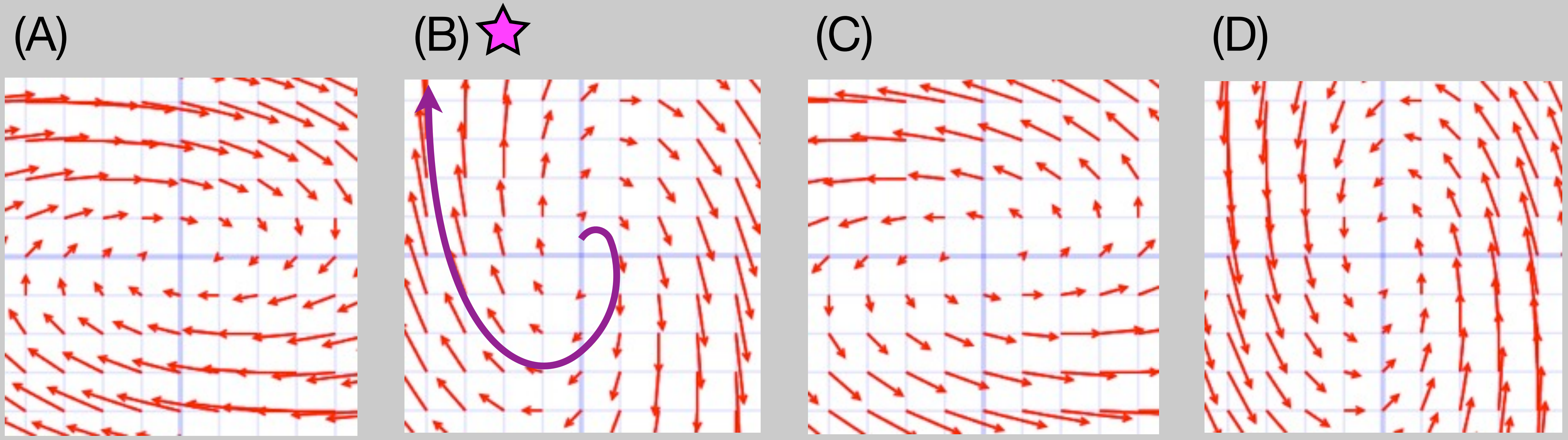
Today

- Systems with complex eigenvalue - example
- Systems with a repeated eigenvalue
- Summary of 2x2 systems with constant coefficients.

Complex eigenvalues (7.6) - example

- Back to our earlier example: $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x}$

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$



(E) Explain, please.

Repeated eigenvalues

- What happens when you get two identical eigenvalues?
- Two cases:

  1. The single eigenvalue has two distinct eigenvectors.
  2. There is only one eigenvector (matrix is defective).

1. $\overline{\mathbf{x}}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \overline{\mathbf{x}}$

2. $\overline{\mathbf{x}}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \overline{\mathbf{x}}$

Repeated eigenvalues

1. $\overline{\mathbf{x}}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \overline{\mathbf{x}}$

$$\det(A - \lambda I) = (\lambda - 3)^2 = 0$$

$$\lambda = 3$$

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v} = 0$$

All vectors solve this so choose any two independent vectors:

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. $\overline{\mathbf{x}}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \overline{\mathbf{x}}$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2$$

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{v} = 0$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<-- only 1 evector!

$$\mathbf{x}(t) = C_1 e^{2t}\mathbf{v} + C_2 e^{3t}(\mathbf{w} + t\mathbf{v})$$

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

$$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Summary - homogeneous 2x2 systems

Steady states - constant solutions (set x'=0 and solve Ax=0).

- For the system of equations $\mathbf{x}' = A\mathbf{x}$, we always have $\mathbf{x}(t) = \mathbf{0}$ as a steady state solution.

- If *A* is singular matrix with $A\mathbf{v} = \mathbf{0}$ then $\mathbf{x}(t) = \mathbf{v}$ is also a steady state solution. In fact, $\mathbf{x}(t) = c\mathbf{v}$ is a steady state for all *c.* It is also an eigenvector associated with eigenvalue $\lambda = 0$.

- If *A* is nonsingular then $\mathbf{x}(t) = \mathbf{0}$ is the only steady state.

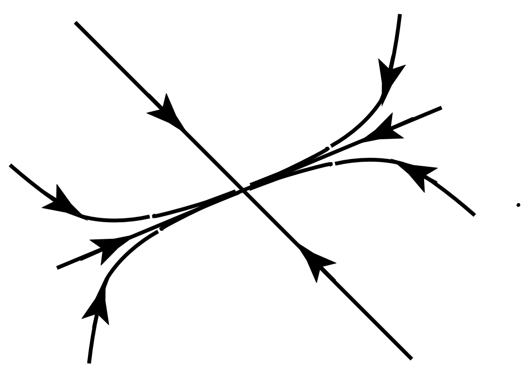
Summary - homogeneous 2x2 systems

Steady states

- Steady states are classified by the nature of the surrounding solutions:

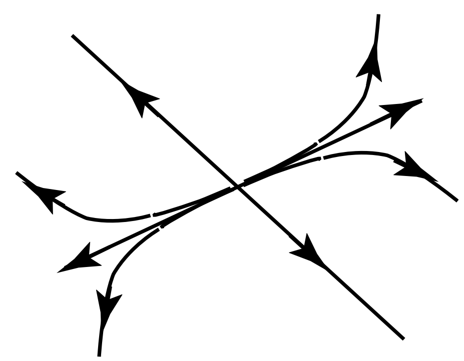
stable node

- real negative evalues



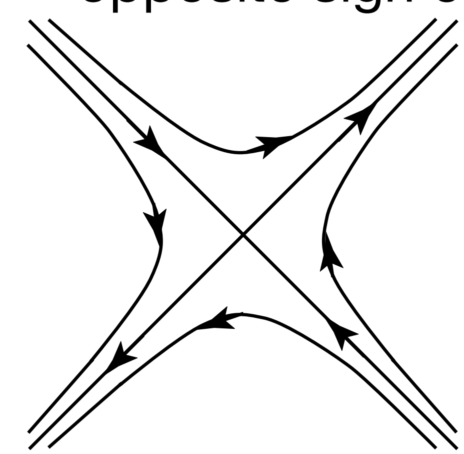
unstable node

- real positive evalues



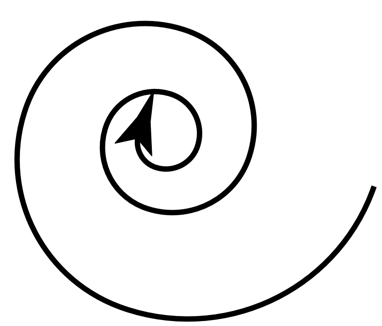
saddle

- opposite sign evalues



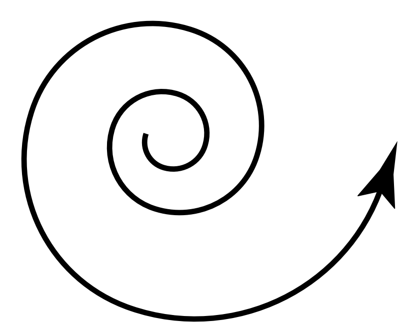
stable spiral

- complex evalues, negative real part



unstable spiral

- complex evalues, positive real part



Summary - homogeneous 2x2 systems

- Quick way to determine how all other solutions behave:

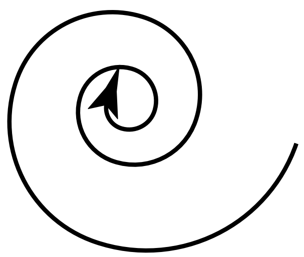
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) \\ &= 0 \end{aligned}$$

Summary - homogeneous 2x2 systems

- When do the solutions spiral IN to the origin?

$$\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$$



ensures negative real part

★(A) $\begin{cases} \mathrm{tr}A < 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$ $\qquad \lambda = \boxed{\dfrac{\mathrm{tr}A}{2}} \pm \boxed{\dfrac{\sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}}$

ensures complex evalue

(B) $\begin{cases} \mathrm{tr}A > 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(C) $\begin{cases} \mathrm{tr}A < 0, \ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

(E) Explain, please.

(D) $\begin{cases} \mathrm{tr}A > 0, \ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

Summary - homogeneous 2x2 systems

- When is the origin a stable node?

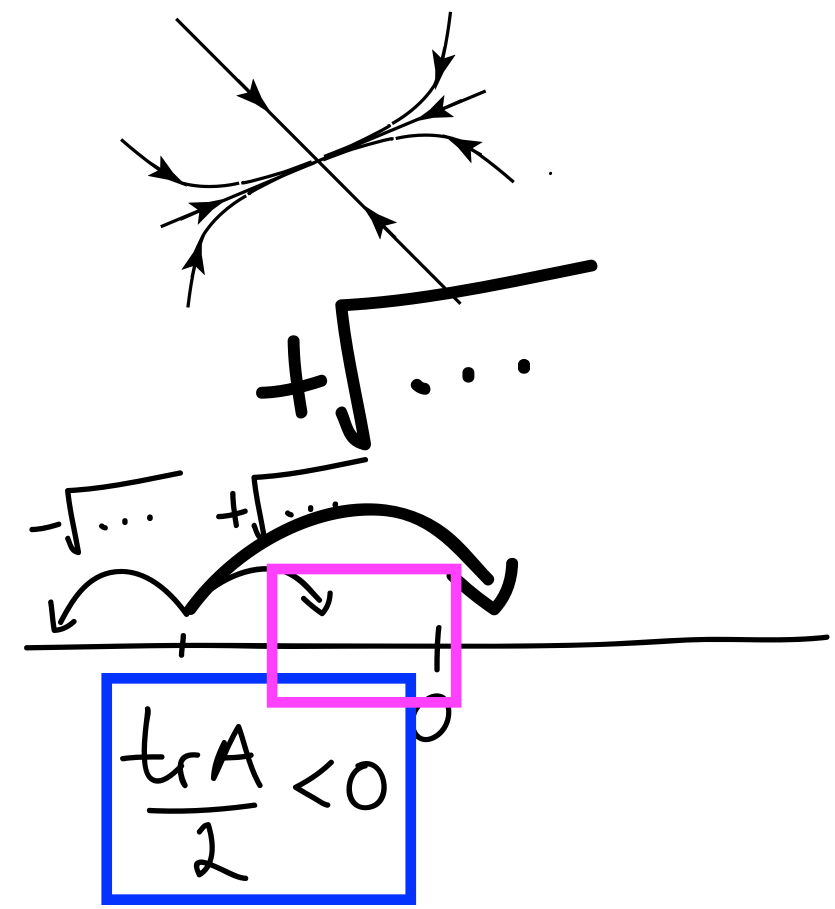
$$\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$$

(A) $\begin{cases} \mathrm{tr}A < 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(B) $\begin{cases} \mathrm{tr}A > 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(C) $\begin{cases} \boxed{\mathrm{tr}A < 0,}\ \boxed{\det(A) > 0} \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

(D) $\begin{cases} \mathrm{tr}A > 0,\ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$



(E) Explain, please.

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$