Today

- Solving ODEs using Laplace transforms
- The Heaviside and associated step and ramp functions
- ODE with a ramped forcing function

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 4y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

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$$s^2Y(s) - s - 0 + 4Y(s) = 0$$

$$s^2Y(s) + 4Y(s) = s$$

$$Y(s) = \frac{s}{s^2 + 4}$$

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- Recall that $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{\omega^2 + s^2}$. So $y(t) = \cos(2t)$.

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$$\mathcal{L}\{e^{-3t} \cos t\} = \frac{s+3}{1 + (s+3)^2}$$

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$$\lambda = \frac{-6 \pm i\sqrt{52-36}}{2} = -3 \pm 2i$$

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Solving IVPs using Laplace transforms - real

- Solve the equation $y'' + 6y' + 5y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

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Solving IVPs using Laplace transforms - nonhomog

- What is the transformed equation for the IVP

$$y' + 6y = e^{2t}$$
$$y(0) = 2$$

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{(2-s)t}\, dt$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

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(D) $sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$

(E) Explain, please.

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$$sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$$

$$Y(s) = \left(2 + \frac{1}{s-2}\right) / (s+6)$$

$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

$$\downarrow$$

$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

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$$\frac{1}{(s-2)(s+6)} = \frac{A}{s-2} + \frac{B}{s+6}$$

$$1 = A(s+6) + B(s-2)$$

$$(s=2) \quad 1 = 8A$$

$$(s=-6) \quad 1 = -8B$$

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$$y(t) = 2e^{-6t} + \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s+6}\right)$$

$$\frac{1}{(s-2)(s+6)} = \frac{A}{s-2} + \frac{B}{s+6}$$

$$1 = A(s+6) + B(s-2)$$

$$(s = 2) \qquad 1 = 8A$$

$$(s = -6) \qquad 1 = -8B$$

Solving IVPs using Laplace transforms

- Find the solution to $y' + 6y = e^{2t}$, subject to IC $y(0) = 2$.

$$sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$$

$$Y(s) = \left(2 + \frac{1}{s-2}\right)/(s+6)$$

$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

$$\downarrow$$

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$$y(t) = \frac{15}{8}e^{-6t} + \frac{1}{8}e^{2t}$$

$$y_h(t) = Ce^{-6t}$$

$$C = \frac{15}{8} \qquad y_p(t) = \frac{1}{8}e^{2t}$$

Solving IVPs using Laplace transforms

- With a forcing term, the equation

$$ay'' + by' + cy = g(t)$$

has Laplace transform

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s)$$

- Solving for Y(s):

Solving IVPs using Laplace transforms

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- Solving for Y(s):

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

transform of homogeneous solution with two degrees of freedom (y(0) and y'(0) act like C_1 and C_2.

transform of particular solution

Solving IVPs using Laplace transforms

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

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Solving IVPs using Laplace transforms

- Inverting the forcing/particular part $Y_p(s) = \dfrac{G(s)}{as^2 + bs + c}$.

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(A) $Y(s) = \dfrac{A}{(s-1)^2} + \dfrac{B}{(s^2+4)}$

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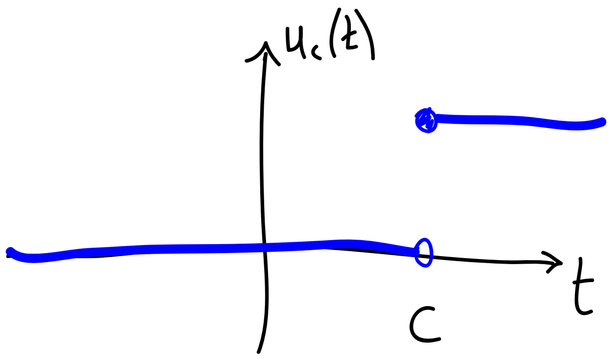
Laplace transforms (so far)

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Step function forcing

- We define the Heaviside function $u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$



- In WW, $u_c(t) = u(t-c) = h(t-a)$

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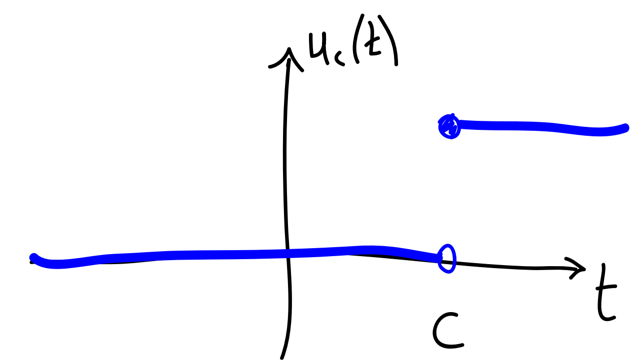
- We use it to model on/off behaviour in ODEs.



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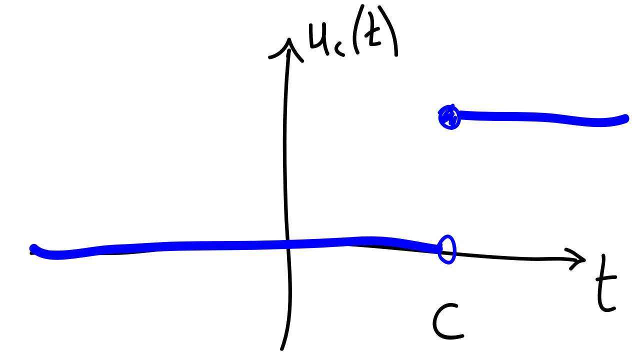

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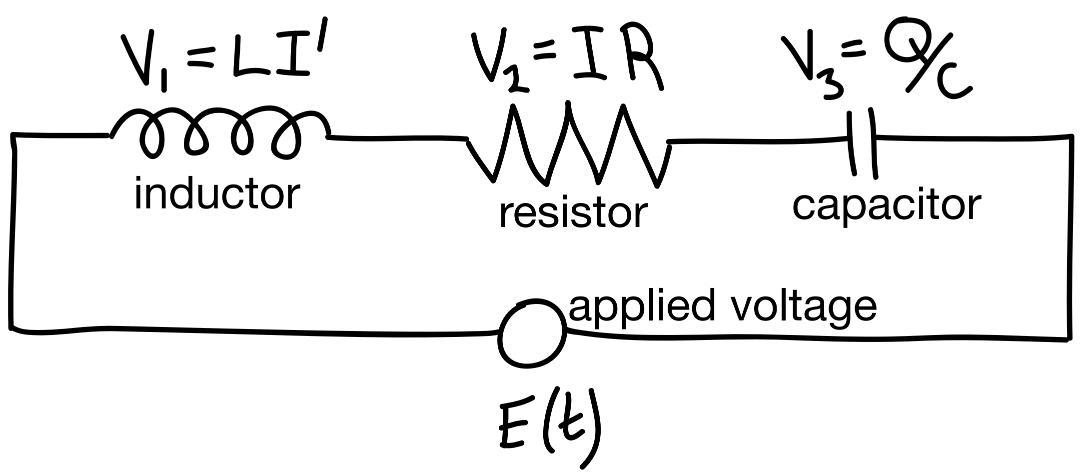
- For example, in LRC circuits, Kirchoff's second law tells us that:



$$V_1 + V_2 + V_3 = E(t)$$

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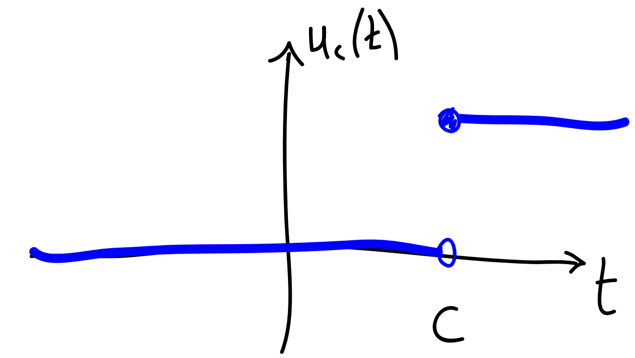

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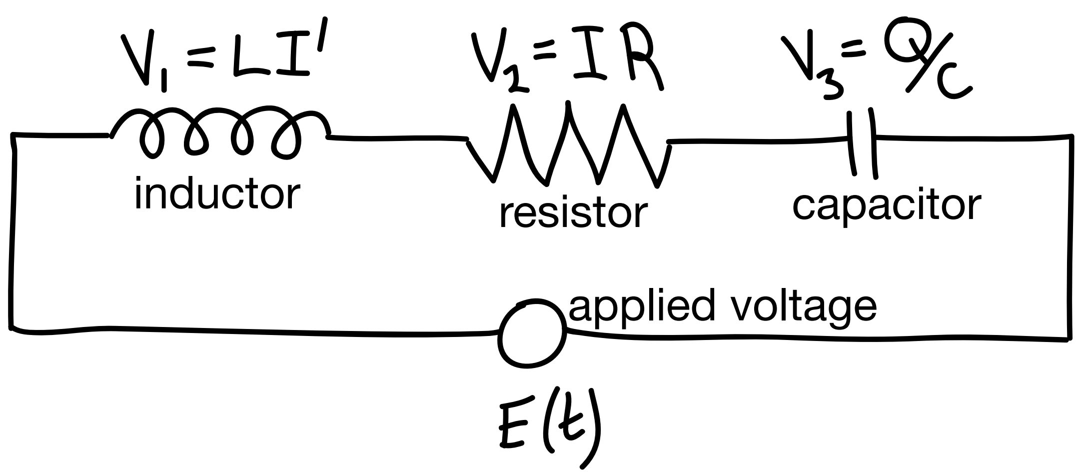

$$V_1 + V_2 + V_3 = E(t)$$

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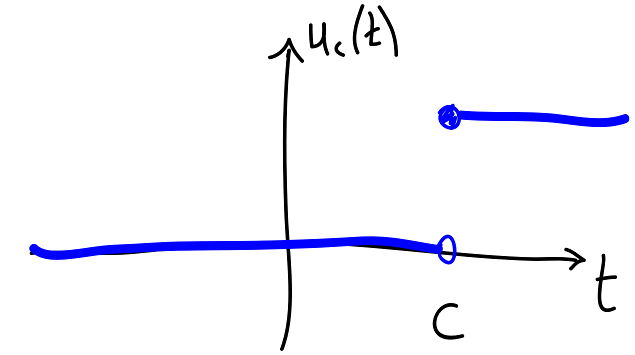

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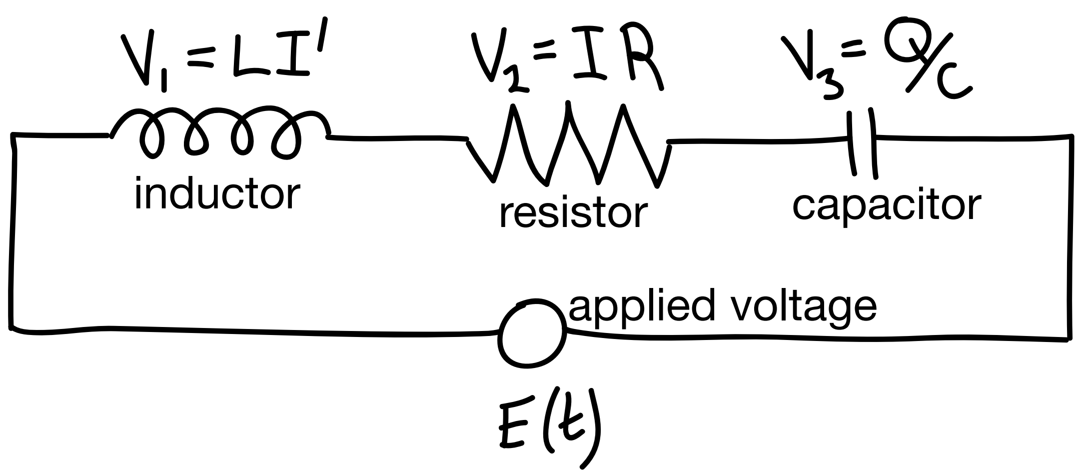

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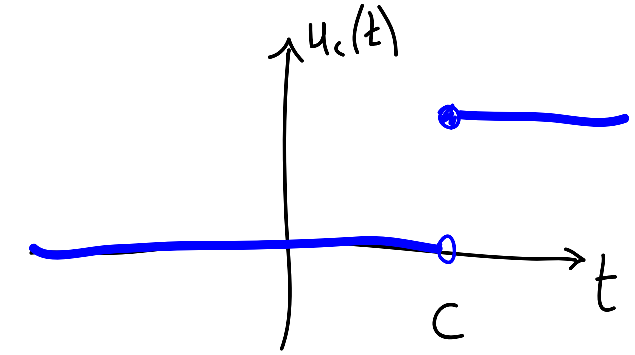

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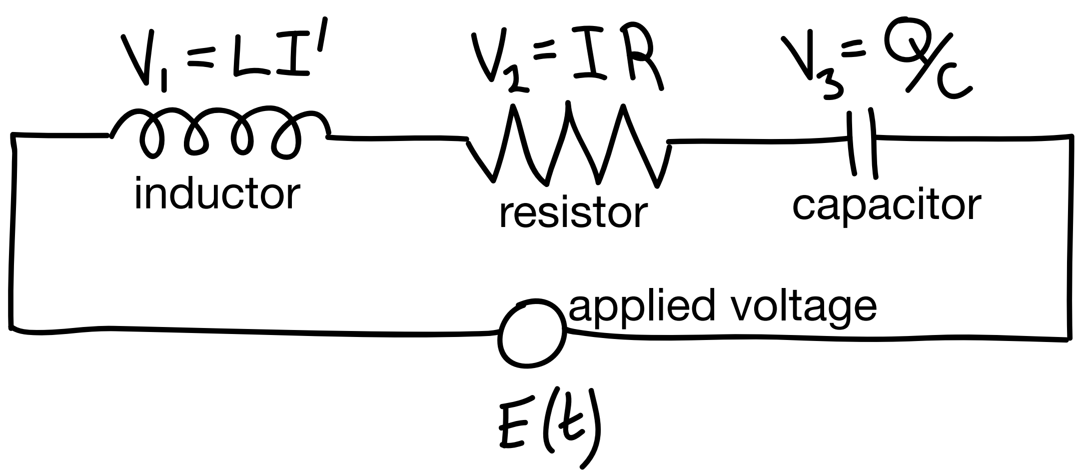

$$V_1 + V_2 + V_3 = E(t)$$

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$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

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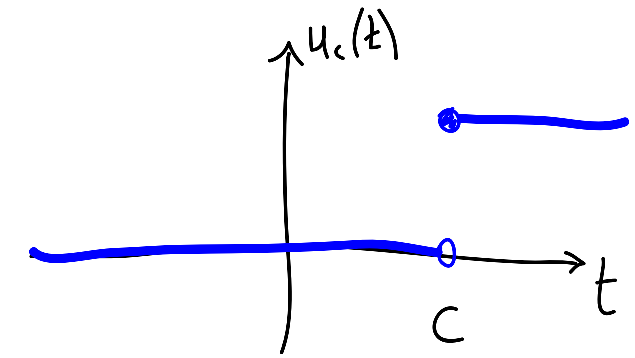

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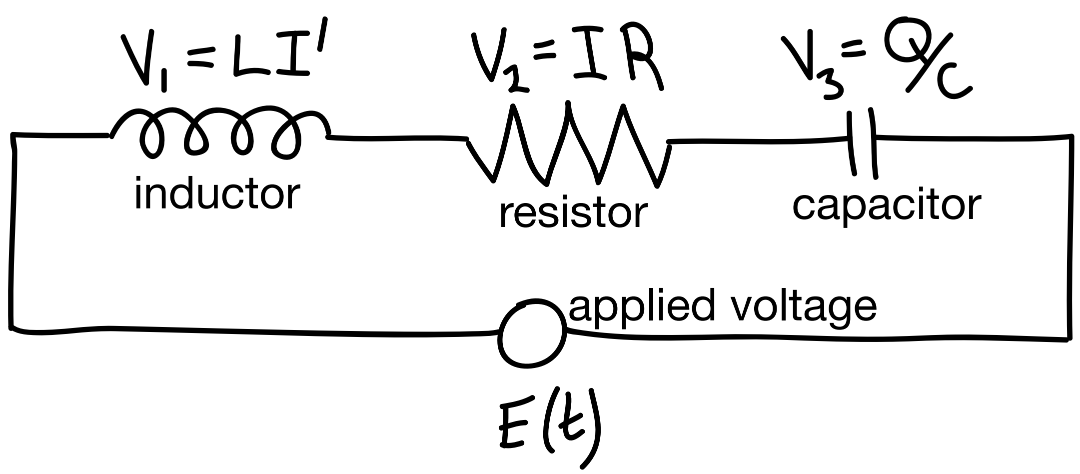

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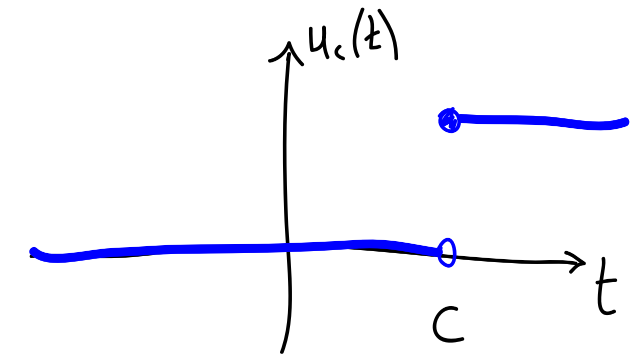
$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

- If E(t) is a voltage source that can be turned on/off, then E(t) is step-like.

Step function forcing

- We define the Heaviside function $u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$



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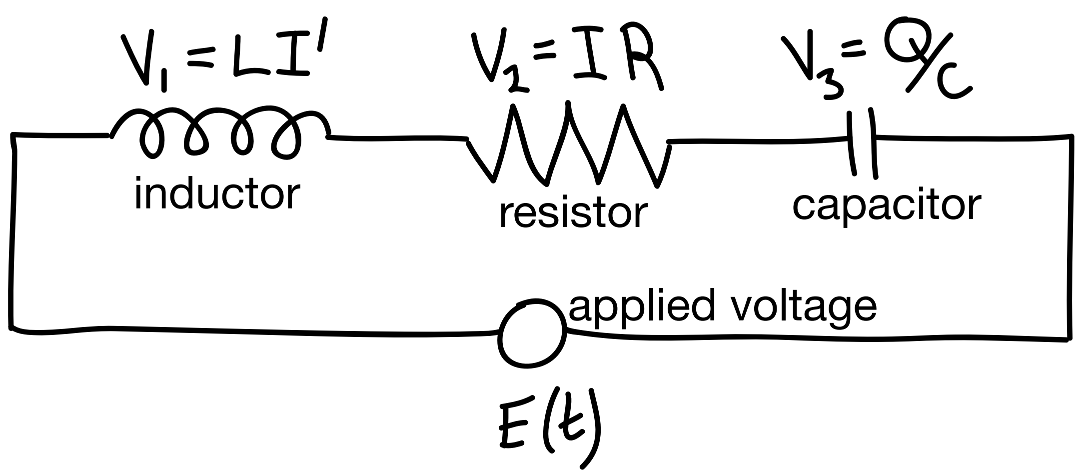

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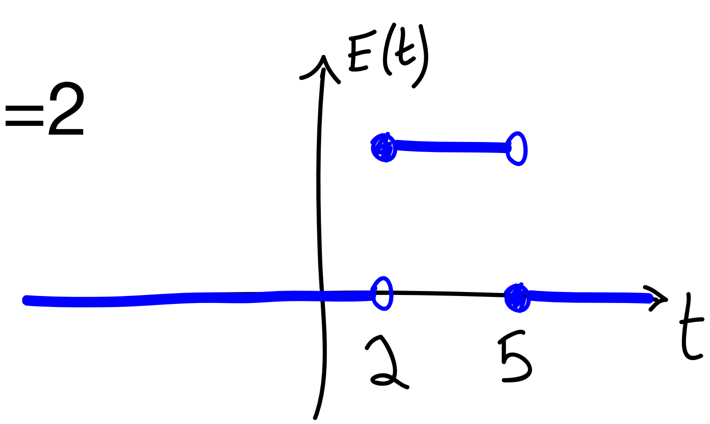
- For example, turn E on at t=2 and off again at t=5:



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- Use the Heaviside function to rewrite $g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$

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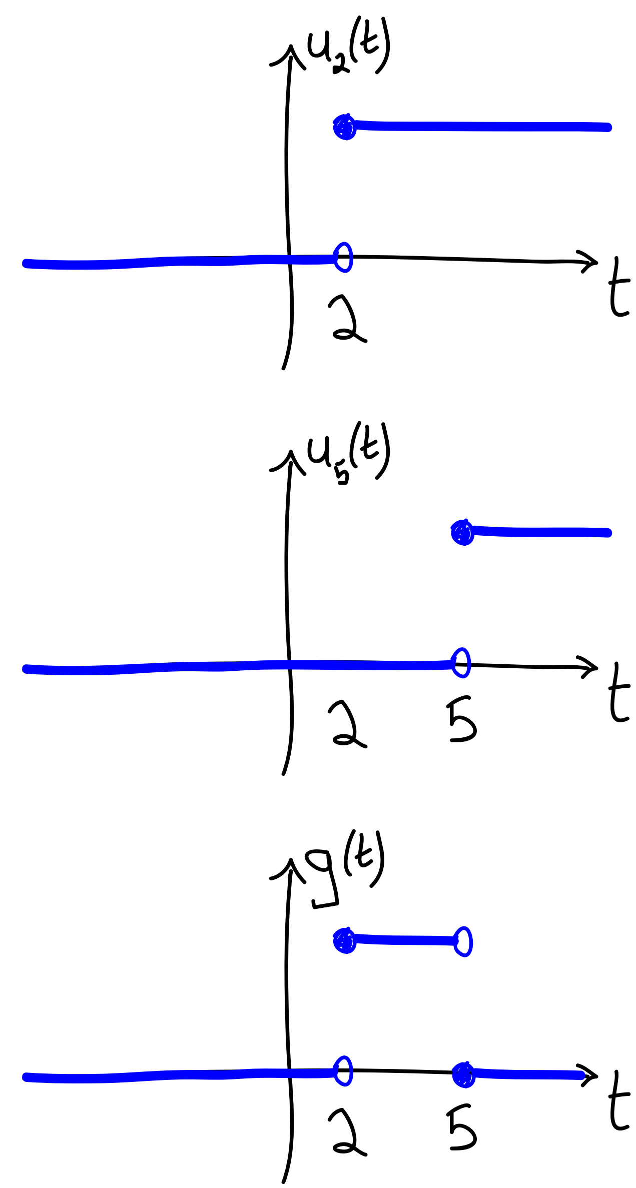

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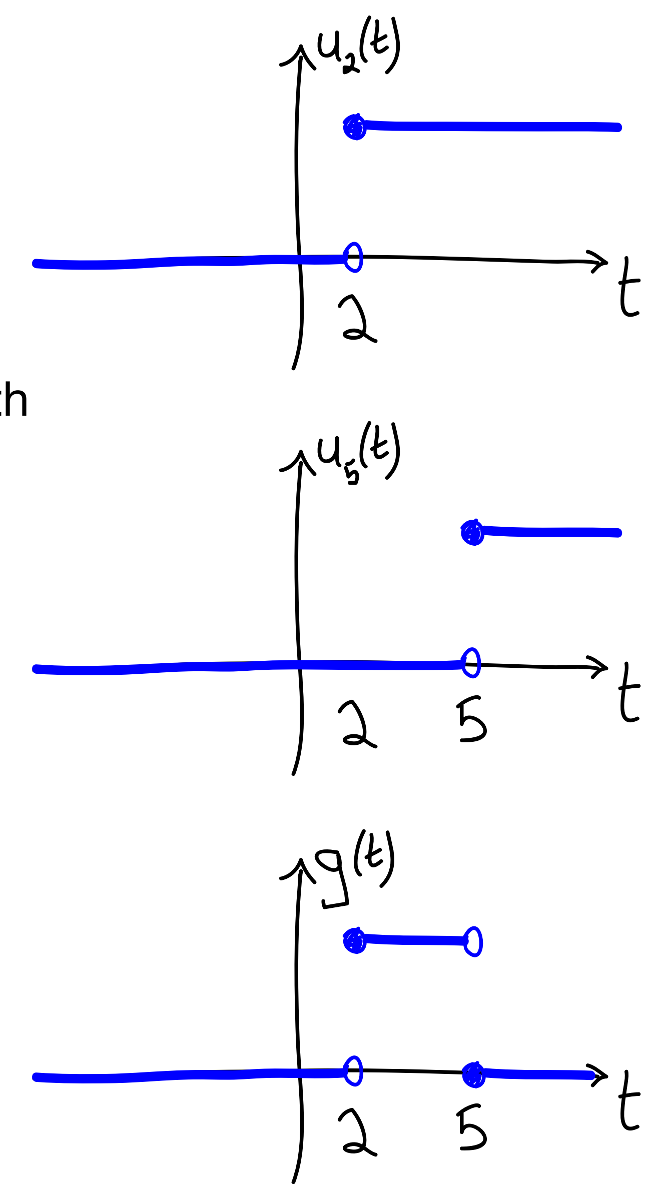

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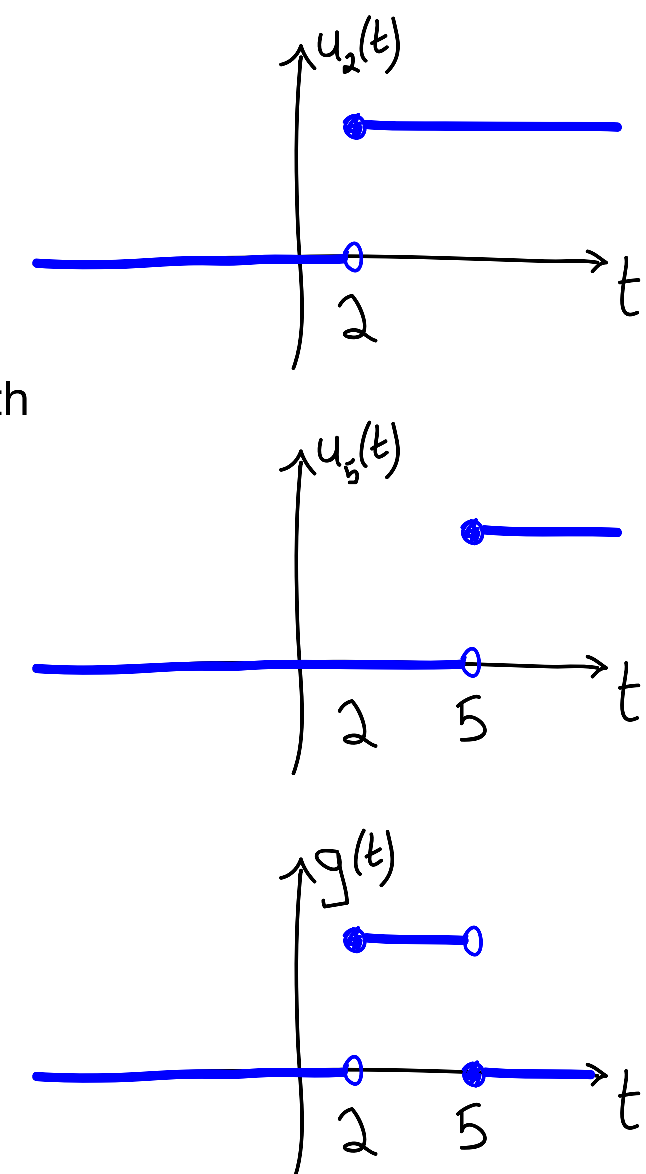

- For more on writing down functions in Heaviside notation see: https://www.youtube.com/watch?v=TGzU5O6csyA

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Recall: $\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty e^{-st}(f(t) + g(t))\ dt$

$$= \int_0^\infty e^{-st} f(t)\ dt + \int_0^\infty e^{-st} g(t)\ dt$$

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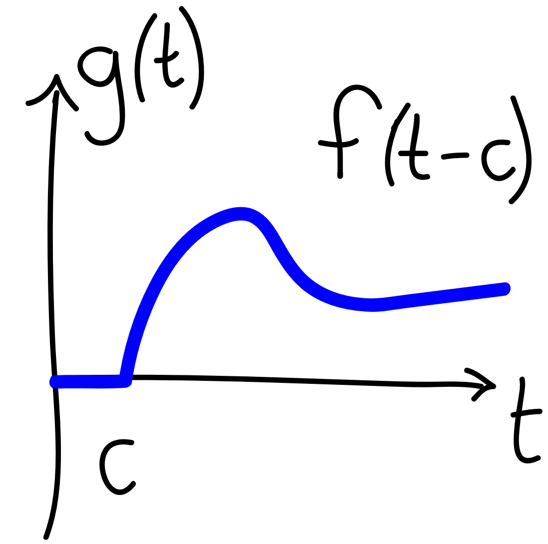

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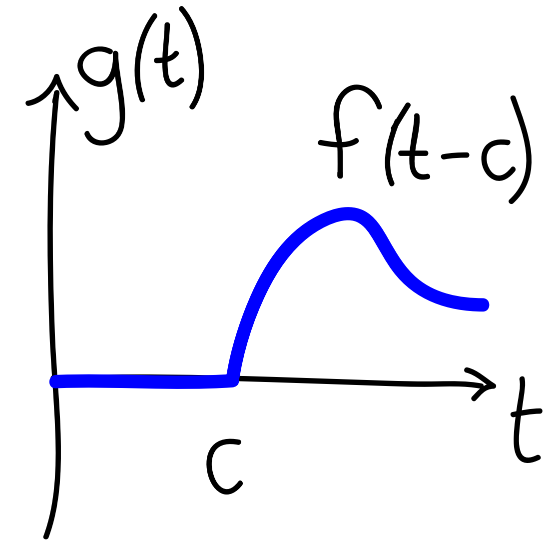

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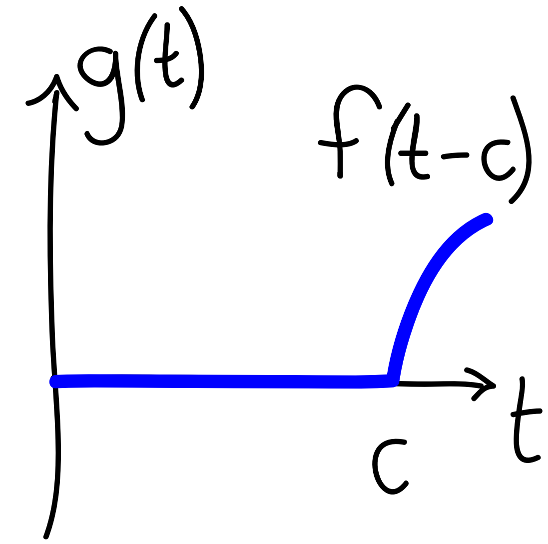

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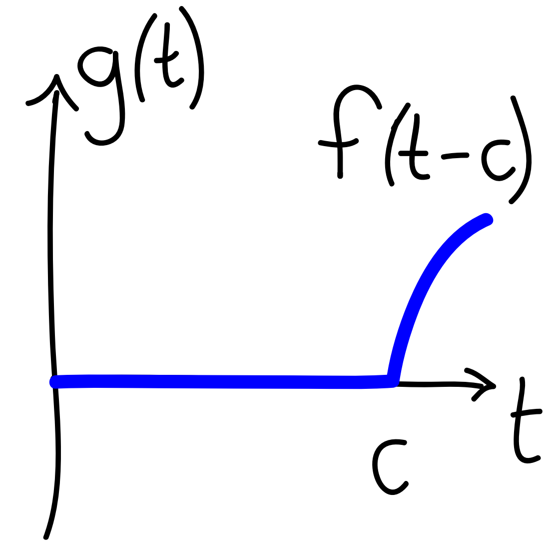

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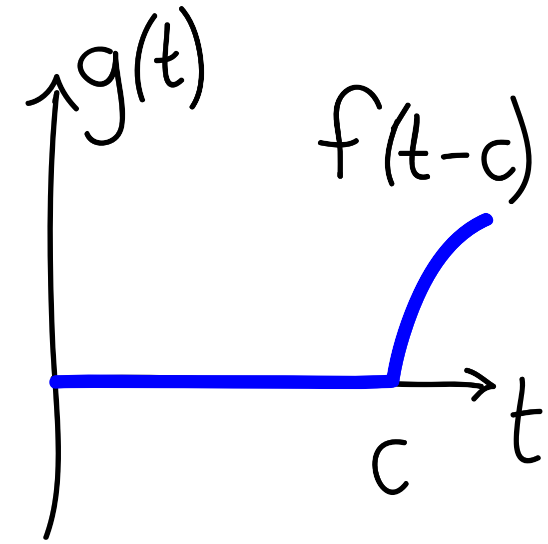

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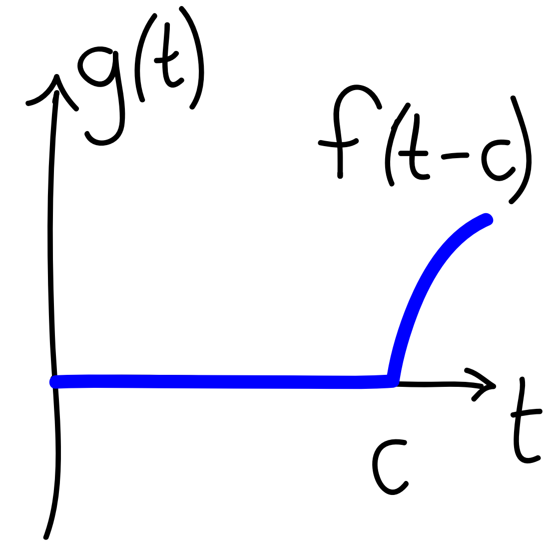

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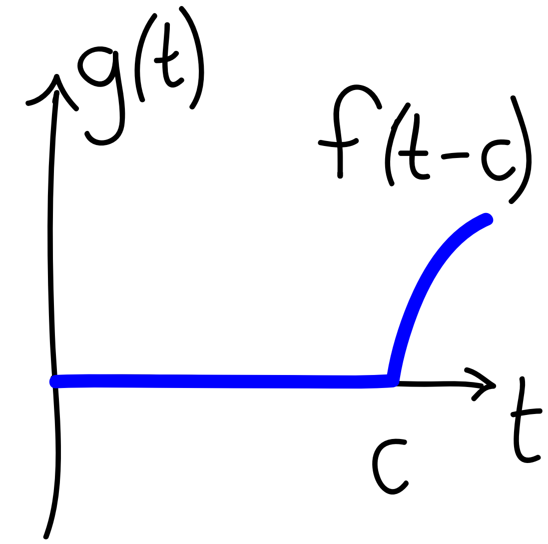

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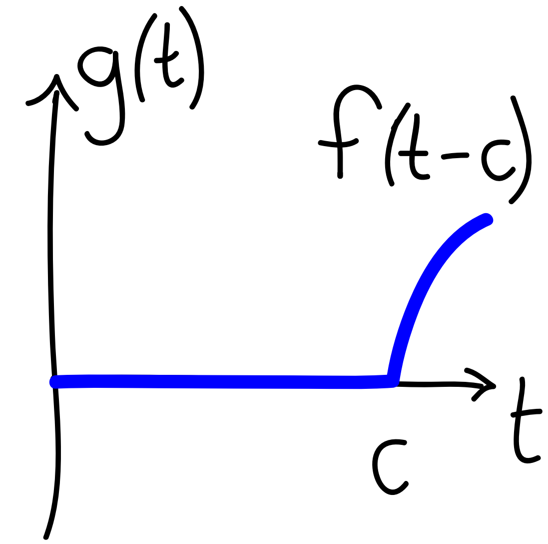

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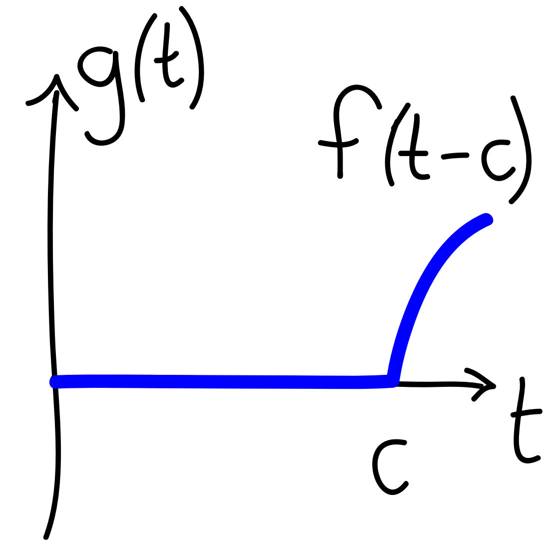

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