MATH 256-201 – Midterm 2 – March 14, 2017.

Surname: ____________ Given name: ____________ Student number: ____________

Tutorial (circle one): T2A-Thomas T2B-Jummy T2C-Sarai T2D-Colin T2E-Xiaowei T2F-Shirin

This midterm has 6 pages including a blank page at the end for rough work. Answers must be justified and work must be shown.

1. (a) **[3 pts]** The regions in the trace-determinant plane shown below are labeled (A) through (E). Match each letter to the classification of the steady state of the equation $\mathbf{x}' = A\mathbf{x}$ (stable node, unstable node, etc.).

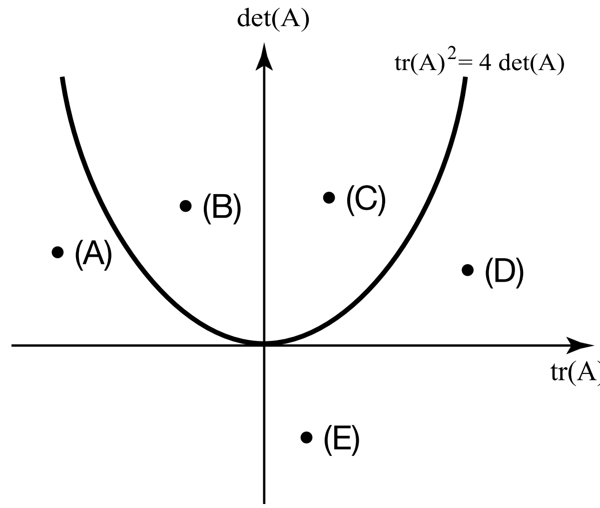
(A)

(B)

(C)

(D)

(E)



(b) **[6 pts]** Consider the system of equations given in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

i. Using the letters from the diagram above, list (in order) the regions of the trace/determinant plane that the system moves through as α goes from $-\infty$ to ∞.

ii. Find the value of α at each transition.

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2. **[5 pts]** Find the general solution to the equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

3. **[3 pts]** Find a particular solution to the equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -6 \end{pmatrix}.$$

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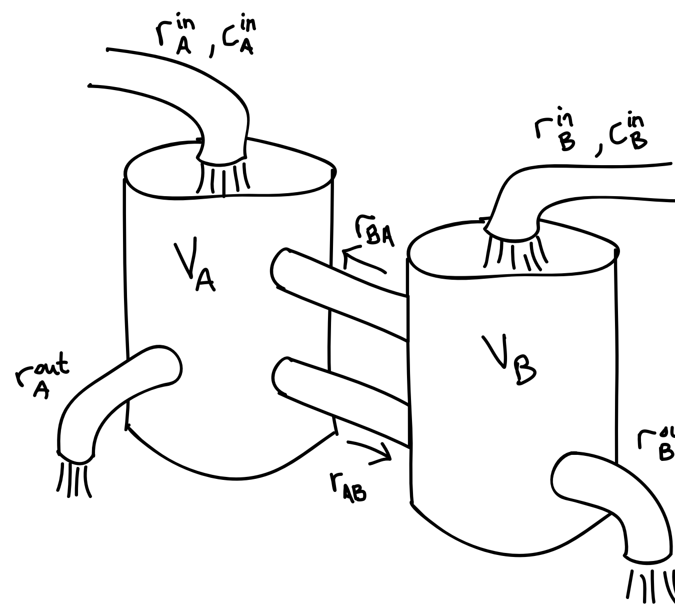
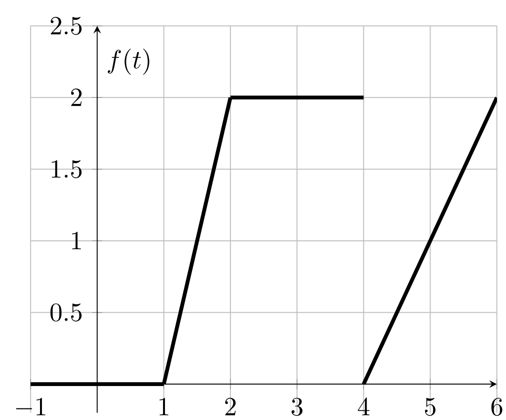
4. **[6 pts]** Write down a system of equations in matrix form for the concentration of salt in the two tanks shown below. The quantities denoted with an r represent the water flow rates in L/min in each pipe, and those denoted with a c represent the concentration flowing through the labeled pipe in g/L, and the V_X represent the volume in Tank X. Assume that the volume in each tank remains constant.



5. **[6 pts]** Write an expression for the function $f(t)$ shown below using Heaviside functions. In your final answer, all terms should be in the form $u_c(t)g(t-c)$ for some g, such that the Laplace transform is easy to compute.



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6. **[4 pts]** What is the Laplace transform of $f(t) = 4u_3(t)(2t-4)$?

7. **[4 pts]** A forced mass-spring system oscillates with position satisfying the equation $mx'' + bx' + kx = F_0\cos(\omega t)$. The forcing frequency ω is chosen to be larger than the natural frequency of the system ($\sqrt{k/m}$). The amplitude of the solution is given by

$$A = \frac{F_0}{\sqrt{b^2\omega^2 + (m\omega^2 - k)^2}}.$$

Suppose that a nearly identical mass-spring system is forced at the same frequency ω with the only difference being that the mass is larger. The amplitude of the solution, A, will differ accordingly. Will this second system oscillate with larger or smaller amplitude? Justify your answer.

Hint: Recall that if the function $p(m)$ is positive and increasing for some value of m, then the function $A(m) = \frac{1}{\sqrt{p(m)}}$ is decreasing for that value of m and vice versa.

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8. **[6 pts]** Calculate the inverse Laplace transform of $F(s) = e^{-7s}\dfrac{s}{s^2+4s+13}$.

9. **[6 pts]** Suppose for the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

where a, b, c, and d are real numbers, one eigenvalue is $-2+3i$ and corresponding eigenvector is $\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$. What is the general solution?

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Work on this page will not be marked unless there is a note on a previous page indicating that this page should be checked.

Laplace transforms

$f(t)$	$F(s)$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$u_c(t)f(t-c)$	$e^{-sc}F(s)$
$\delta(t-c)$	e^{-sc}