Today

- Shapes of solutions for distinct eigenvalues case.
- General solution for complex eigenvalues case.
- Shapes of solutions for complex eigenvalues case.

Shapes of so … e plane

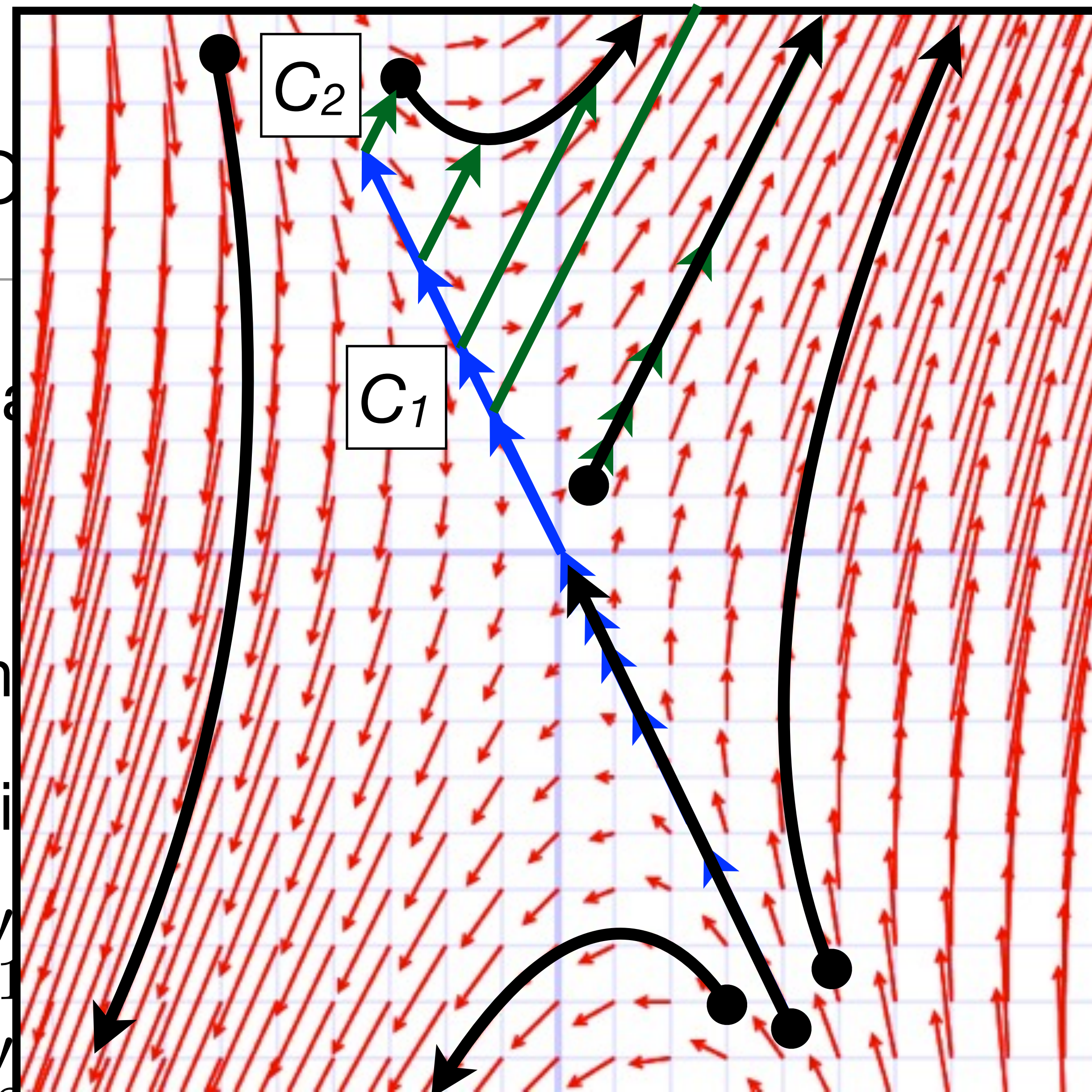
- When matrix A ha … olution to **x**'=A**x** is

- What do solution … phase plane)?

- If the initial condi … on is a straight line. Example:

x_1' … 6

x_2' … $4x_1 + x_2$ … $x_2(0) = -12$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} + \boxed{C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$C_1 = 6, \; C_2 = 0$$

- IVP solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

Shapes of solution curves in the phase plane

- Simple example to show general idea.

$$\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1(t) = C_1 e^{\lambda_1 t} \qquad t = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$x_2(t) = C_2 e^{\lambda_2 t} \qquad t = \frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right)$$

$$\frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right) = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \frac{\lambda_2}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \ln\left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

$$x_2 = C_2 \left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

- Can we plot solutions in x_1-x_2 plane by graphing x_2 versus x_1?

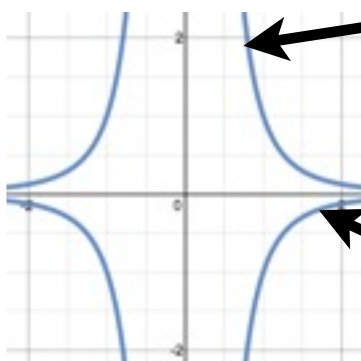
Shapes of solution curves in the phase plane

- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

$$x_2 = C_2 \left(\frac{x_1}{C_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

- For the shape of solutions, we need to know the sign and size of $\dfrac{\lambda_2}{\lambda_1}$.

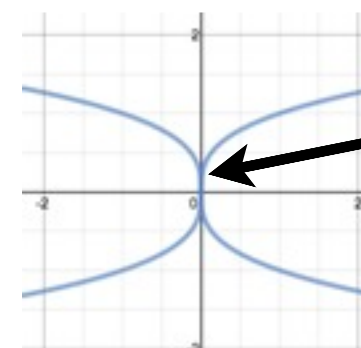
$$\lambda_2 = -3\lambda_1 \qquad x_2 = \frac{C}{x_1^3}$$



far from λ_2 axis

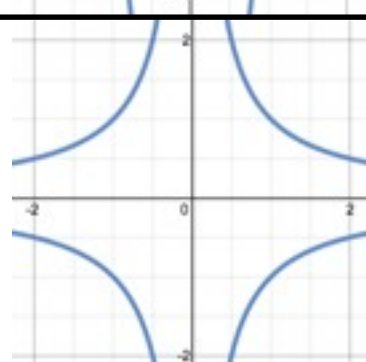
close to λ_1 axis

$$\lambda_2 = \frac{1}{3}\lambda_1 \qquad x_2 = C\sqrt[3]{x_1}$$

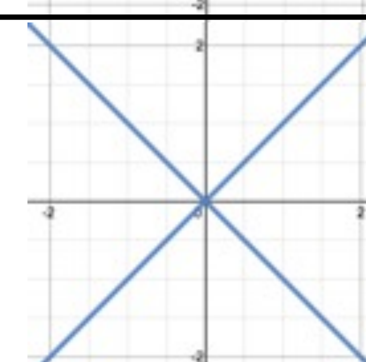


stays near λ_2 axis

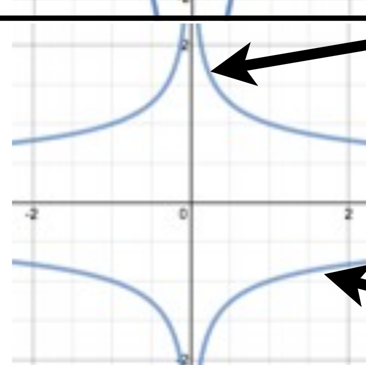
$$\lambda_2 = -\lambda_1 \qquad x_2 = \frac{C}{x_1}$$



$$\lambda_2 = \lambda_1 \qquad x_2 = Cx_1$$



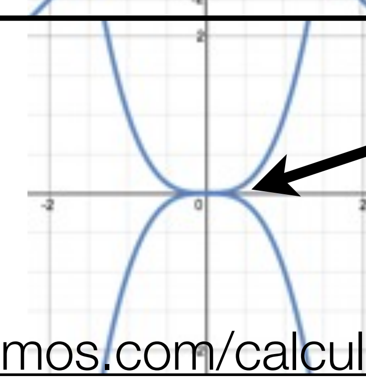
$$\lambda_2 = -\frac{1}{3}\lambda_1 \qquad x_2 = \frac{C}{\sqrt[3]{x_1}}$$



close to λ_2 axis

far from λ_1 axis

$$\lambda_2 = 3\lambda_1 \qquad x_2 = Cx_1^3$$



stays near λ_1 axis

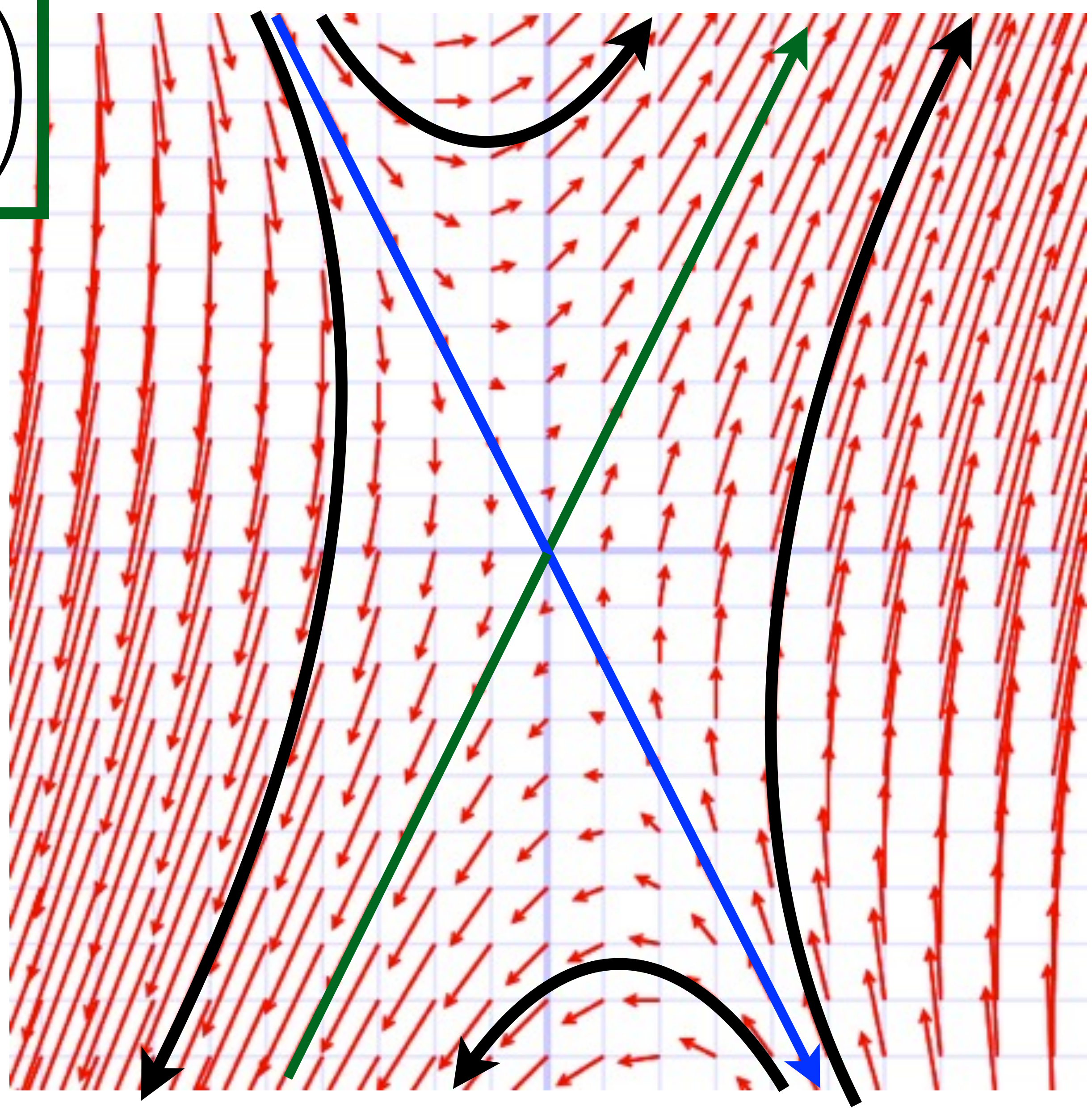
https://www.desmos.com/calculator/c4rhrgotmo

Shapes of solution curves in the phase plane

- With more complicated solutions (evectors off-axis), tilt shapes accordingly.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} + \boxed{C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

- Going forward in time, the blue component shrinks slower than the green component grows so solutions appear closer to blue "axis" than to green "axis"

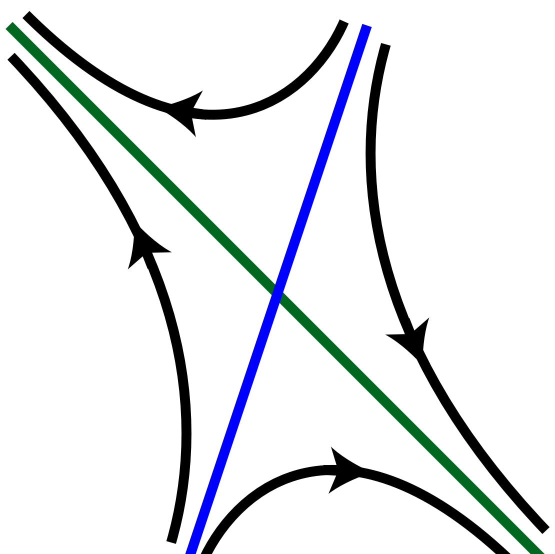


Shapes of solution curves in the phase plane

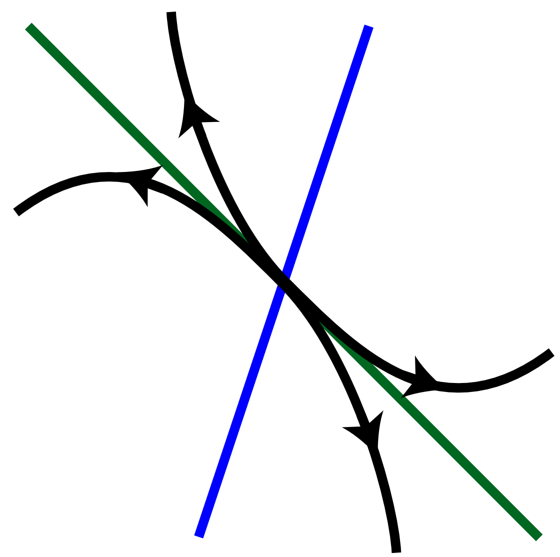
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{3t}\begin{pmatrix}1\\3\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}\ ?$$

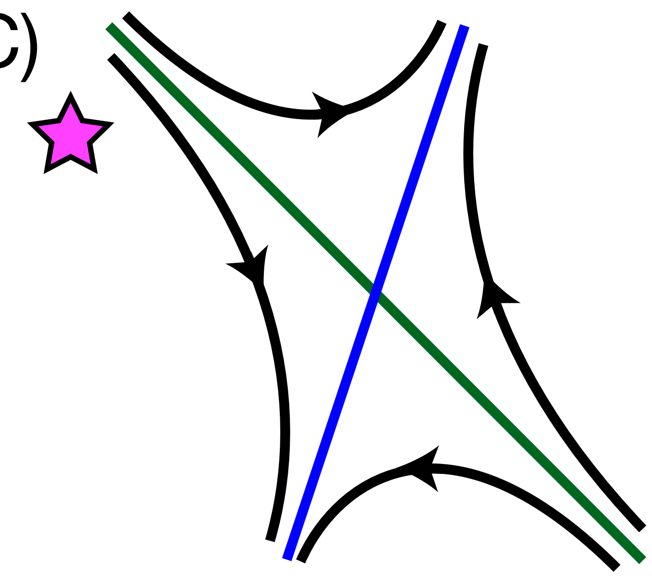
(A)



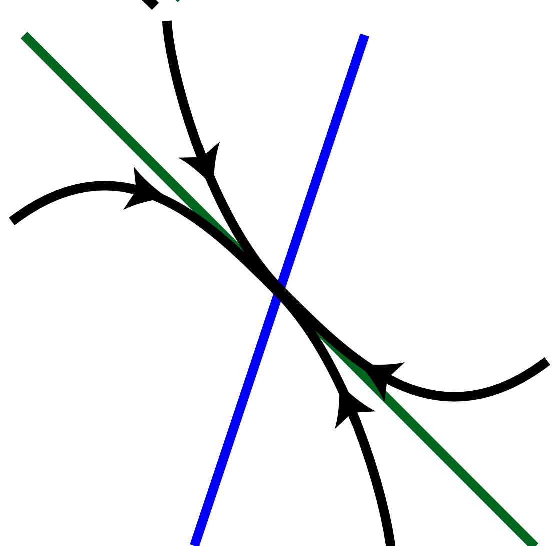
(B)



(C)



(D)



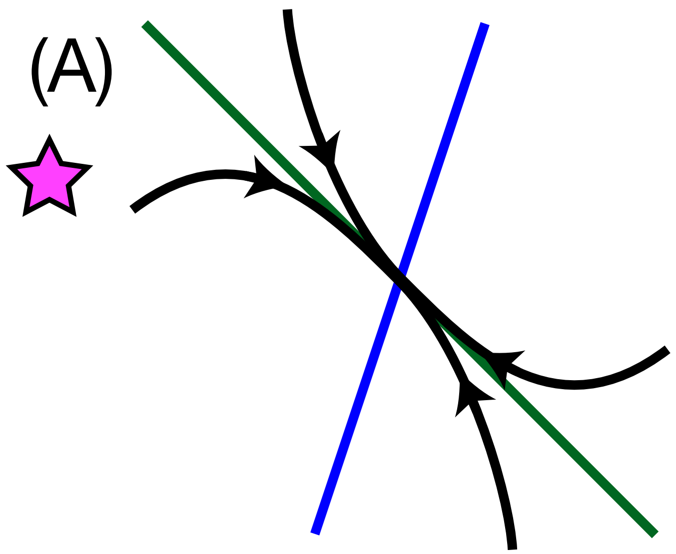
(E) Explain, please.

Shapes of solution curves in the phase plane

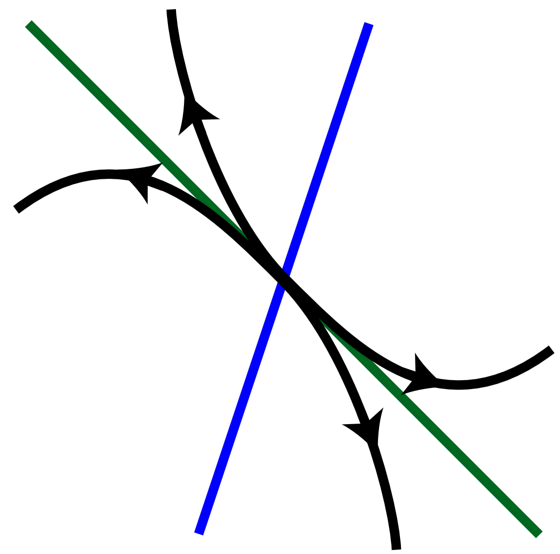
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ ?}$$

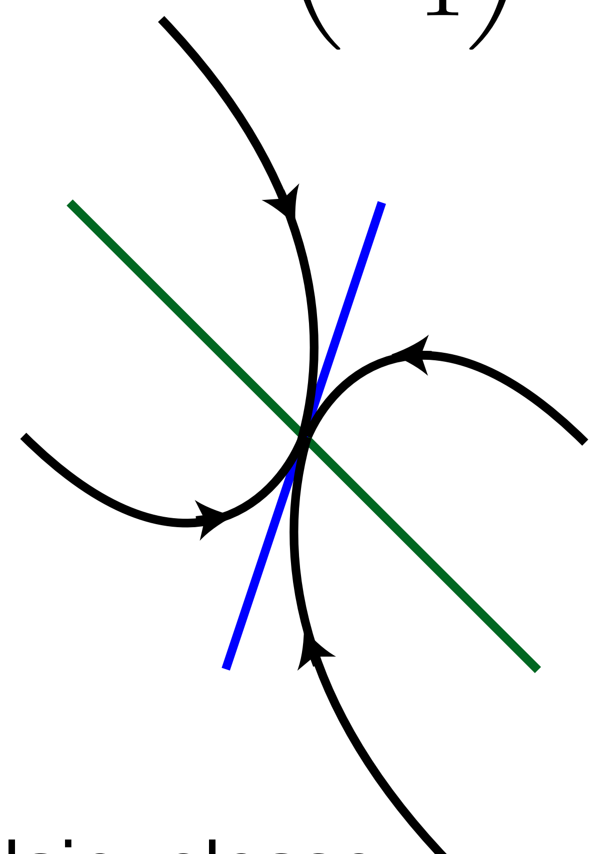
(A)



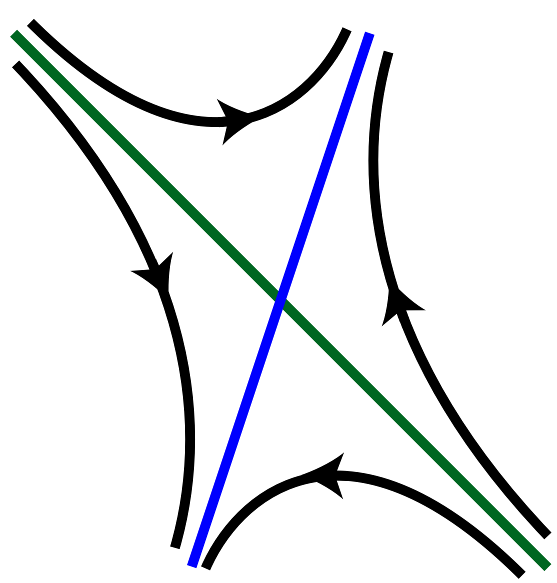
(B)



(C)



(D)



(E) Explain, please.

Complex eigenvalues (7.6) - example

- Find the general solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x}$.
  - The eigenvalues are
  - The eigenvectors are . . .

★ (A) $\lambda = 1 \pm 2i$

(B) $\lambda = -1, 3$

(C) $\lambda = 2 \pm 4i$

(D) $\lambda = -2, 6$

(E) I don't know how to find eigenvalues.

$$A - \lambda_1 I = \begin{pmatrix} 1-(1+2i) & 1 \\ -4 & 1-(1+2i) \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \times \tfrac{1}{2} i$$

$$\sim \begin{pmatrix} -2i & 1 \\ -2i & 1 \end{pmatrix}$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

Complex eigenvalues (7.6) - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$x_1'' = x_1' + x_2'$$

$$x_1'' = x_1' - 4x_1 + x_2$$

$$x_1'' = x_1' - 4x_1 + x_1' - x_1$$

$$x_1'' - 2x_1' + 5x_1 = 0$$

$$r^2 - 2r + 5 = 0$$

$$r = 1 \pm 2i$$

Complex eigenvalues (7.6) - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$r = 1 \pm 2i \qquad x_1(t) = e^t (C_1 \cos(2t) + C_2 \sin(2t))$$

$$\begin{aligned} x_1'(t) = &\, e^t (-2C_1 \sin(2t) + 2C_2 \cos(2t)) \\ &+ e^t (C_1 \cos(2t) + C_2 \sin(2t)) \end{aligned}$$

$$x_2 = x_1' - x_1 = e^t (2C_2 \cos(2t) - 2C_1 \sin(2t))$$

Complex eigenvalues (7.6) - example

- We could just write down a (complex valued) general solution:

$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$x_1(t) = e^t (C_1 \cos(2t) + C_2 \sin(2t))$$

$$x_2(t) = e^t (2C_2 \cos(2t) - 2C_1 \sin(2t))$$

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$

Complex eigenvalues (7.6) - general case

- Find e-values, $\lambda = \alpha \pm \beta i$, and e-vectors, $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
$= \mathbf{a} + i\mathbf{b}$
- Write down solution:

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(\beta t) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin(\beta t) \right) \right.$$
$$\left. + C_2 \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin(\beta t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos(\beta t) \right) \right]$$

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t} \left[C_1 \left(\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t) \right) \right.$$
$$\left. + C_2 \left(\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) \right) \right]$$

Complex eigenvalues (7.6) - example

- Suppose you find eigenvalue $\lambda = 2\pi i$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and, for some initial value problem,

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

- But what about $\lambda_2 = -2\pi i$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$?

same thing

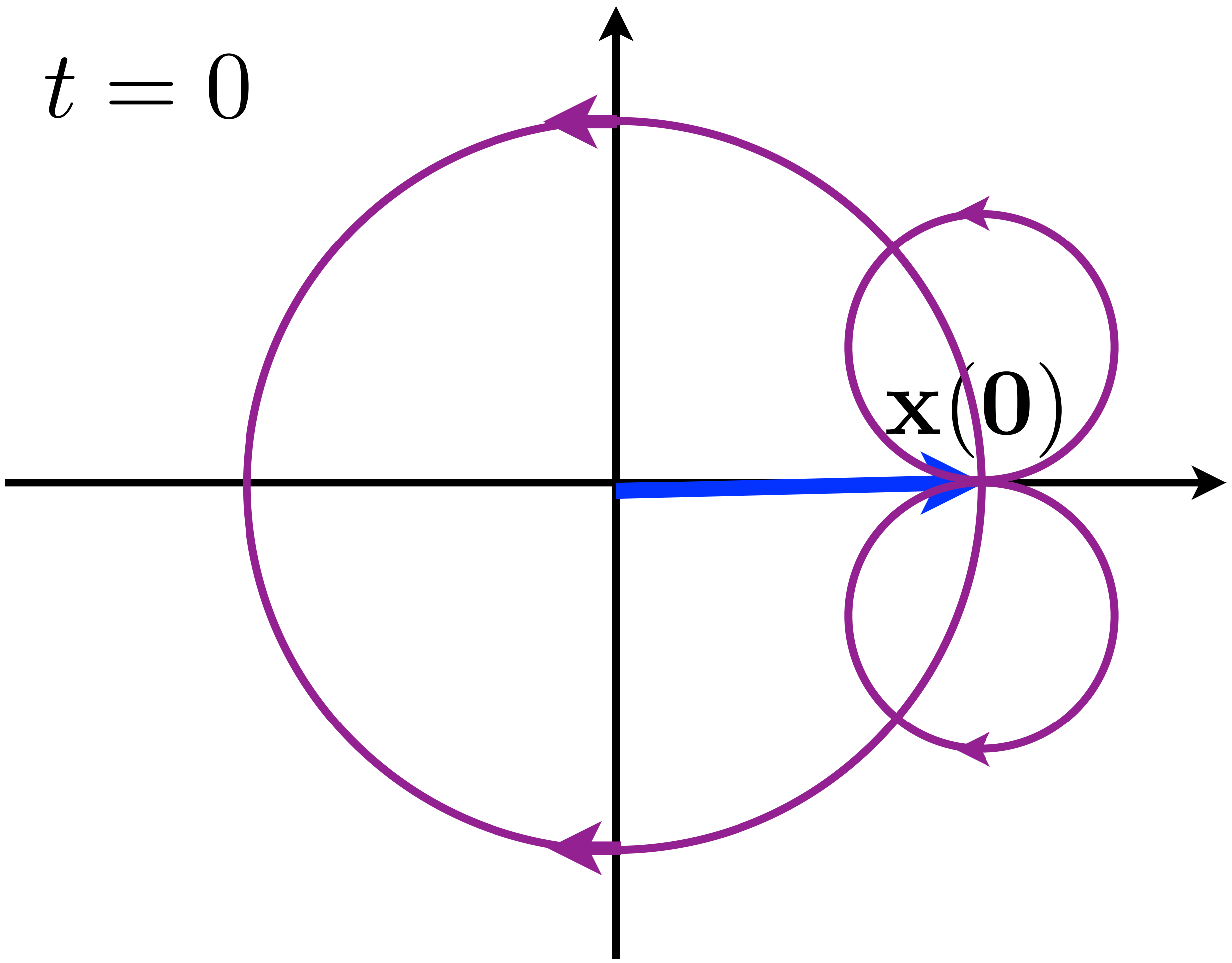
$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(-2\pi t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(-2\pi t)$$

- Note: the initial condition was carefully chosen so that C_1=1 and C_2=0.

$$\mathbf{x(t)} = e^{\alpha t} \left[C_1 \left(\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t) \right) + C_2 \left(\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) \right) \right]$$

Complex eigenvalues (7.6) - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



- What happens as t increases?

★ (A) The vector rotates clockwise.

(B) The vector rotates counter-clockwise.

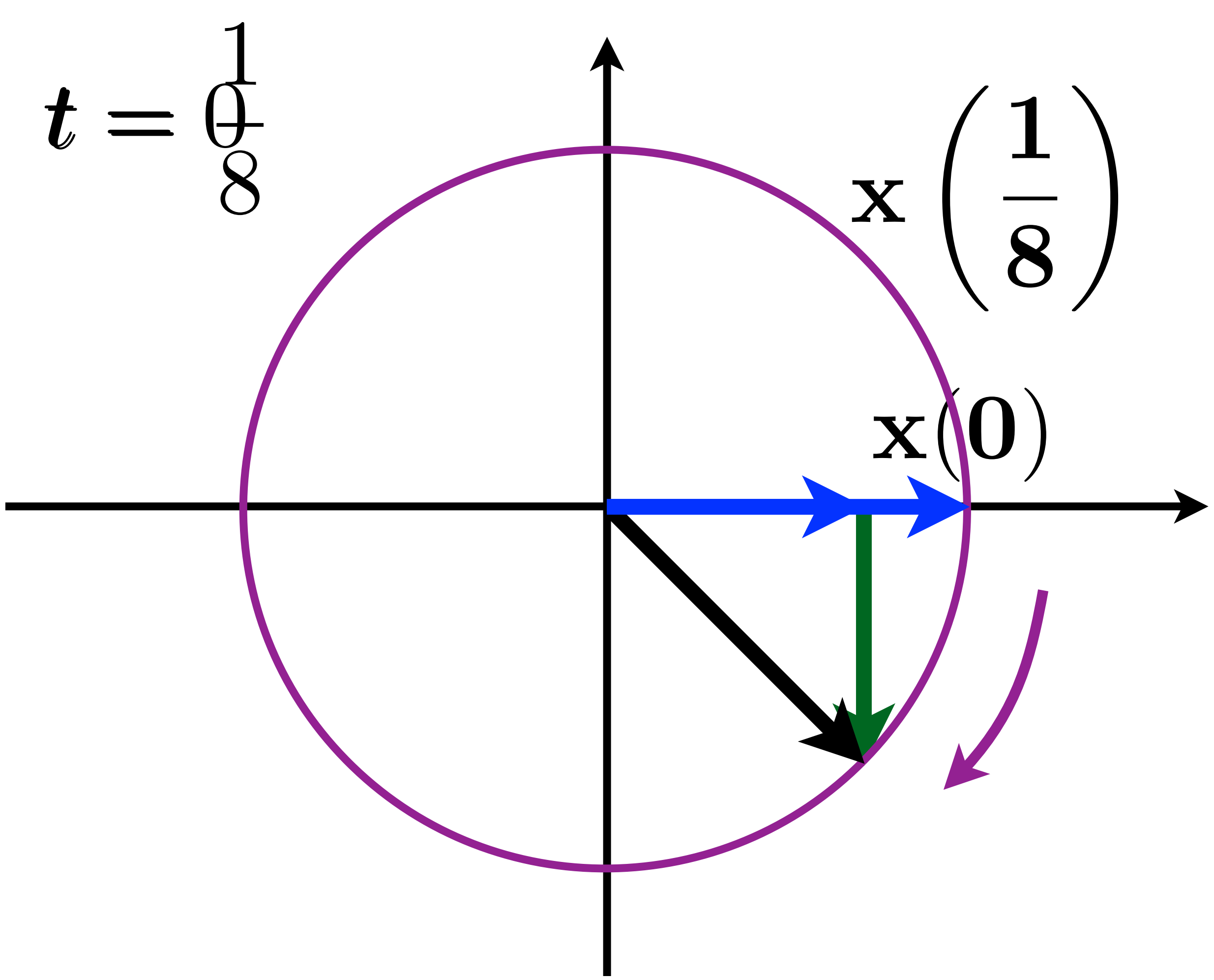
(C) The tip of the vector maps out a circle in the first quadrant.

(D) The tip of the vector maps out a circle in the fourth quadrant.

(E) Explain please.

Complex eigenvalues (7.6) - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos\left(\frac{\pi}{4}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{4}\right)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

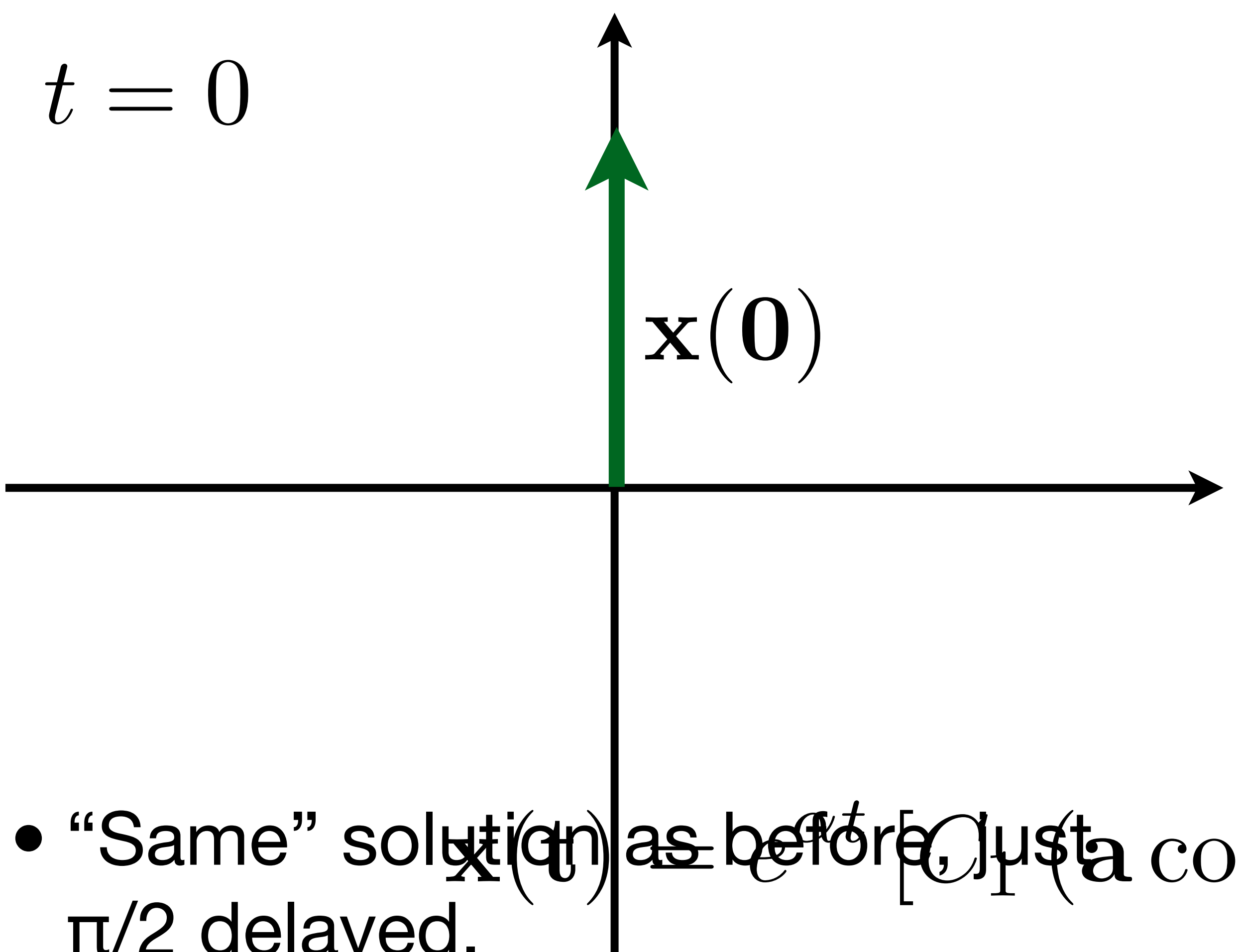
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Complex eigenvalues (7.6) - example

- Same equation, initial condition chosen so that $C_1=0$ and $C_2=1$.

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t)$$

$t = 0$



- What happens as t increases?

(A) The vector rotates clockwise.

(B) The vector rotates counter-clockwise.

(C) The tip of the vector maps out a circle in the first quadrant.

(D) The tip of the vector maps out a circle in the second quadrant.

(E) Explain please.

- “Same” solution as before, just π/2 delayed.

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\right]$$

Complex eigenvalues (7.6) - general case

- Looking at the general solution again...

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right)\right.$$
$$\left. + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\right]$$

- Both parts rotate in the exact same way but the C_2 part is delayed by a quarter phase.

- If an initial condition lies neither parallel to vector **a** nor to vector **b**, C_1 and C2 allow for intermediate phases to be achieved.

- **x(t)** can be rewritten (using trig identities) as

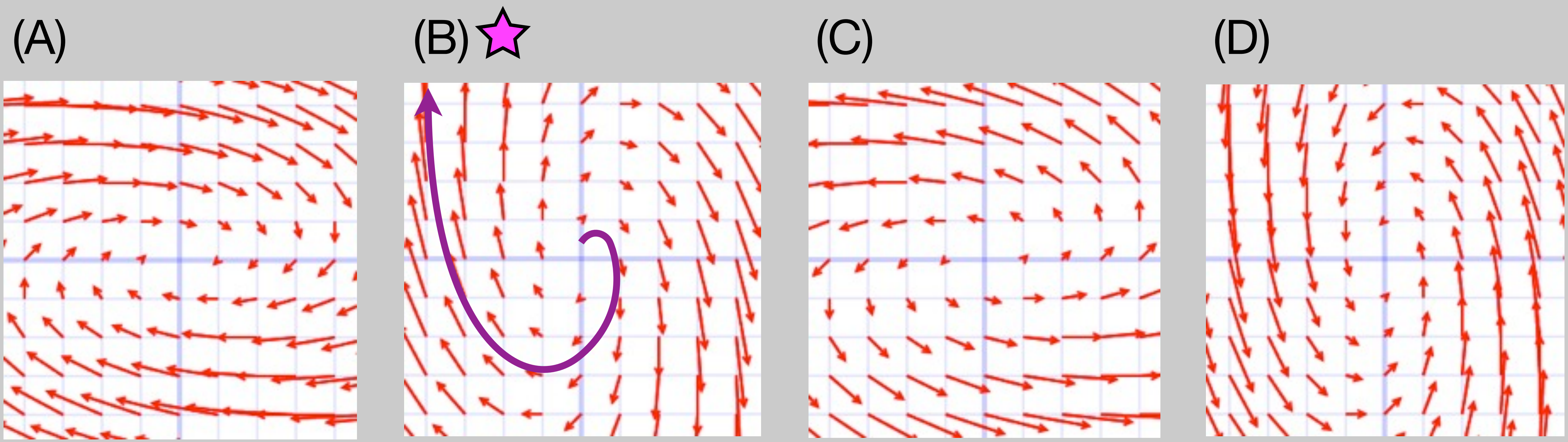
$$\mathbf{x}(\mathbf{t}) = Me^{\alpha t}\left(\mathbf{a}\cos(\beta t - \phi) - \mathbf{b}\sin(\beta t - \phi)\right)$$

where M and ϕ are constants to replace C_1 and C_2.

Complex eigenvalues (7.6) - example

- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$



(E) Explain, please.