Today

- Finish up undetermined coefficients (on the midterm, on WW3)
- Physics applications - mass springs (not on midterm, (on WW4)
- Undamped, over/under/critically damped oscillations (maybe Thurs)

Method of undetermined coefficients

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Step 1: Solve the associated homogeneous equation
    $$y'' + 2y' - 3y = 0.$$
    $$y_h(t) = C_1 e^t + C_2 e^{-3t}$$
  - Step 2: What do you have to plug in to $L[\ \cdot\ ]$ to get e^{2t} out?
    - Try $y_p(t) = Ae^{2t}$.
    - $L[y_p(t)] = L[Ae^{2t}] =$
      - (A) $5e^{2t}$
      - (B) $5Ae^{2t}$ ★
      - (C) $4e^{2t}$
      - (D) $4Ae^{2t}$
    - A is an undetermined coefficient (until you determine it).

Method of undetermined coefficients

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Summarizing:
    - We know that, for any C_1 and C_2,

$$L[C_1 e^t + C_2 e^{-3t}] = 0$$

    - We also know that

$$L[Ae^{2t}] = 5Ae^{2t}$$

    - Finally, by linearity, we know that

$$L[C_1 e^t + C_2 e^{-3t} + Ae^{2t}] = 0 + 5Ae^{2t}$$

    - So what's left to do to find our general solution? Pick A =?1/5.

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the solution to the associated homogeneous equation?

☆ (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$

(C) $y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$

(D) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t) + e^t$

(E) Don't know.

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t}$

(B) $y_p(t) = Ae^{-2t}$

☆ (C) $y_p(t) = Ae^t$

(D) $y_p(t) = Ate^t$

(E) Don’t know

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the value of A that gives the particular solution (Ae^t) ?

(A) A = 1

(B) A = 3

(C) A = 1/3

★ (D) A = -1/3

(E) Don't know.

Method of undetermined coefficients

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
    - What is the solution to the associated homogeneous equation?

      (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

      (B) $y_h(t) = C_1 \cos(2t) + C$

      (C) $y_h(t) = C$

      $C_1 e^{2t} + C_2 e^{-2t} + e^t$

      Don't know.

Same as the last example

Method of undetermined coefficients

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t}$

(B) $y_p(t) = Ae^{-2t}$

☆ (C) $y_p(t) = Ate^{2t}$

(D) $y_p(t) = Ae^{t}$

(E) $y_p(t) = Ate^{t}$

$$(Ae^{2t})'' - 4Ae^{2t} = 0 \text{ !}$$

- Simpler example in which the RHS is a solution to the homogeneous problem.

$$y' - y = e^t$$

$$e^{-t}y' - e^{-t}y = 1$$

$$y = te^t + Ce^t$$

- General rule: when your guess at y_p makes LHS=0, try multiplying it by t.

Method of undetermined coefficients

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the value of A that gives the particular solution (Ate^{2t})?

(A) A = 1

(B) A = 4

(C) A = -4

☆ (D) A = 1/4

(E) A = -1/4

$$(Ate^{2t})' = Ae^{2t} + 2Ate^{2t}$$

$$(Ate^{2t})'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$= 4Ae^{2t} + 4Ate^{2t}$$

$$\left(Ate^{2t}\right)'' - 4\left(Ate^{2t}\right) = 4Ae^{2t}$$

Need: $= e^{2t}$

Method of undetermined coefficients

- Example 8. Find the general solution to $y'' - 4y = \cos(2t)$.
  - What is the form of the particular solution?

☆ (A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

☆ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Challenge: What small change to the DE makes (D) correct?

Method of undetermined coefficients

- Example 8. Find the general solution to $y'' + y' - 4y = \cos(2t)$.
  - What is the form of the particular solution?

(A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

★ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Method of undetermined coefficients

- Example 9. Find the general solution to $y'' - 4y = t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = At^3$

(B) $y_p(t) = At^3 + Bt^2 + Ct$

☆ (C) $y_p(t) = At^3 + Bt^2 + Ct + D$

(D) $y_p(t) = At^3 + Be^{2t} + Ce^{-2t}$ ← waste of time including solution to homogeneous eq.

(E) Don't know.

Method of undetermined coefficients

- When RHS is sum of terms:

$$y'' - 4y = \cos(2t) + t^3$$

$$y_p(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F$$

or

$$y_{p_1}(t) = A\cos(2t) + B\sin(2t)$$

$$y_{p_2}(t) = Ct^3 + Dt^2 + Et + F$$

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$

Method of undetermined coefficients

- Example. Find the general solution to $y'' + 2y' = e^{2t} + t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt$

(B) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt + E$

☆ (C) $y_p(t) = Ae^{2t} + (Bt^4 + Ct^3 + Dt^2 + Et)$

$y_p(t) = Ae^{2t} + t(Bt^3 + Ct^2 + Dt + E)$

(D) $y_p(t) = Ae^{2t} + Be^{-2t} + Ct^3 + Dt^2 + Et + F$

(E) Don't know / still thinking.

For each wrong answer, for what DE is it the correct form?

Method of undetermined coefficients

- Example. Find the general solution to $y'' - 4y = t^3 e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{2t}$

(B) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t}$

(C) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t} + (Dt^3 + Et^2 + Ft)e^{-2t}$

★ (D) $y_p(t) = (At^4 + Bt^3 + Ct^2 + Dt)e^{2t}$
$y_p(t) = t(At^3 + Bt^2 + Ct + D)e^{2t}$

(E) Don't know / still thinking.

Method of undetermined coefficients

$$y'' + 3y' - 10y = x^2 e^{-5x}$$

$$y = e^{rx}$$
$$r^2 + 3r - 10 = 0$$
$$r = -5, 2$$

$$y_h(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y_p(x) = Ax^2 e^{-5x}$$
$$y_p'(x) = 2Axe^{-5x} - 5Ax^2 e^{-5x}$$
$$y_p''(x) = 2Ae^{-5x} - 10Axe^{-5x} - 10Axe^{-5x} + 25Ax^2 e^{-5x}$$

$$-10y_p(x) = \qquad - 10Ax^2 e^{-5x}$$
$$3y_p'(x) = \qquad 6Axe^{-5x} - 15Ax^2 e^{-5x}$$
$$y_p''(x) = 2Ae^{-5x} - 20Axe^{-5x} + 25Ax^2 e^{-5x}$$

$$2Ae^{-5x} - 14Axe^{-5x} + 0 \quad = x^2 e^{-5x}$$

Can’t find A that works! Need 3 unknowns to match all 3 terms.

Method of undetermined coefficients

$$y'' + 3y' - 10y = x^2 e^{-5x}$$

$$y_h(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y_p(x) = Ax^2 e^{-5x} + Bxe^{-5x} + Ce^{-5x}$$

$$y_p'(x) \text{ involves } x^2, x, 1$$

$$y_p''(x) \text{ involves } x^2, x, 1$$

But e^{-5x} gets killed by the operator so C disappears - only 2 unknowns for matching.

Need 3 unknowns but not including e^{-5x}.

$$\begin{aligned} y_p(x) &= Ax^3 e^{-5x} + Bx^2 e^{-5x} + Cxe^{-5x} \\ &= x(Ax^2 e^{-5x} + Bxe^{-5x} + Ce^{-5x}) \end{aligned}$$

Method of undetermined coefficients

- Summary - finding a particular solution to L[y] = g(t).
  - Include all functions that are part of the g(t) family (e.g. cos **and** sin)
  - If part of the g(t) family is a solution to the homogeneous (h-)problem, use t x (g(t) family).
  - If t x (part of the g(t) family), is a solution to the h-problem, use t^2 x (g (t) family). etc.
  - For sums, group terms into families and include a term for each. You can even find a y_p for each family separately and add them up.
  - Works for products of functions - be sure to include the whole family!
  - Never include a solution to the h-problem as it won't survive L[ ]. Just make sure you aren't missing another term somewhere.

Method of undetermined coefficients

- Do lots of these problems and the trends will become clear.

- Try different y_ps and see what goes wrong - this will help you see what must happen when things go right.

- Two crucial facts to remember

    - If you try a form and you can make LHS=RHS with some choice for the coefficients then you're done.

    - If you can't, your guess is most likely missing a term(s).