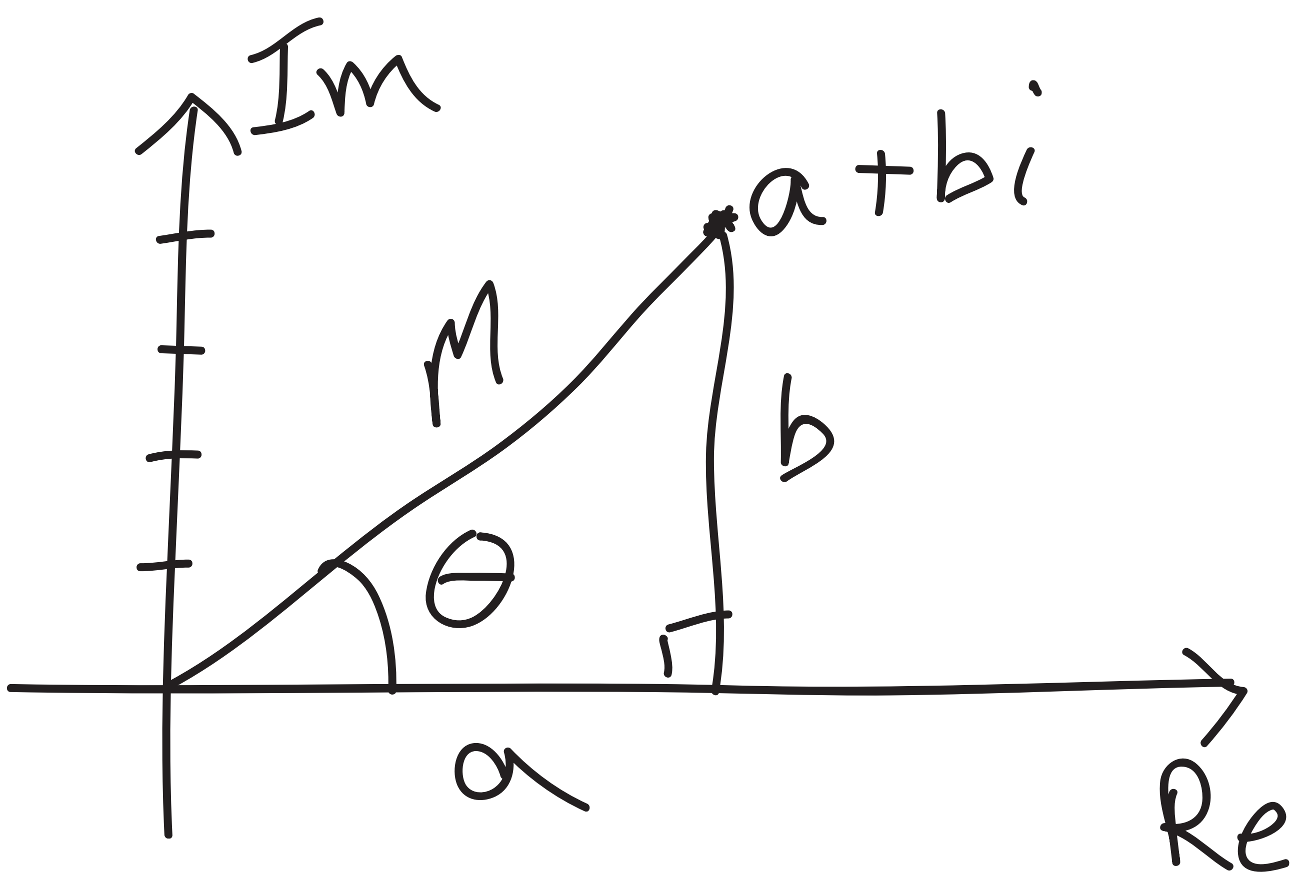
Today

- Midterm 1 postponed to Feb 10 (not updated on calendar yet).
- Solving a second order linear homogeneous equation with constant coefficients
    - complex roots to the characteristic equation,
    - repeated roots to the characteristic equation (Reduction of Order).
- Connections to matrix algebra.
- Solving a second order linear **non**homogeneous equation.

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

$$b = M\sin\theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a + bi = M(\cos\theta + i\sin\theta)$$

θ is sometimes called the argument or phase of $a + bi$.

Complex number review

- Toward Euler's formula
  - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

  - What function has Taylor series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ / $\boldsymbol{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$

(A) cos x

(C) e^x

(B) sin x

(D) ln x

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$-1 = i^2$$

$$= 1 + i\theta + (-1)\frac{\theta^2}{2!} + (-1)i\frac{\theta^3}{3!} + (-1)^2\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

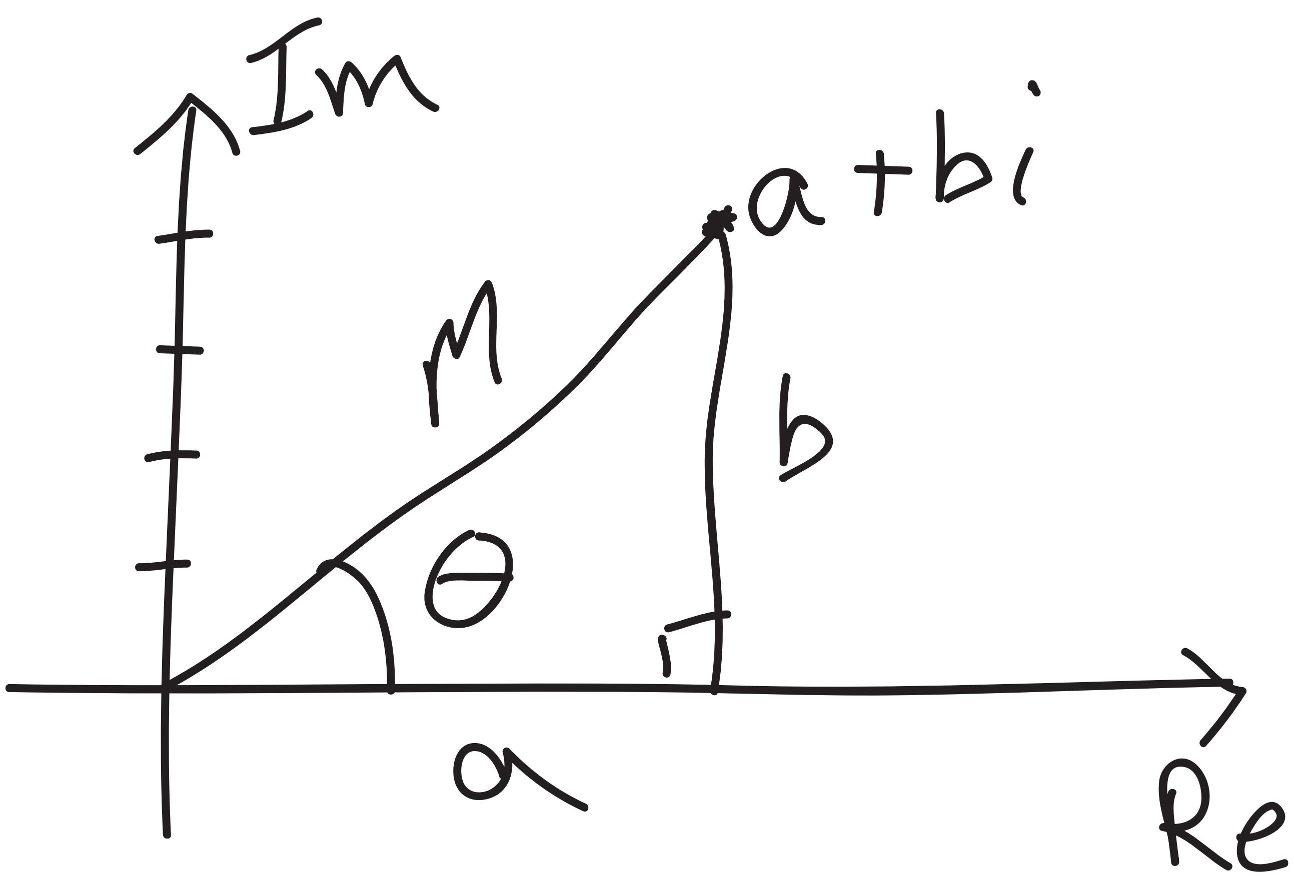
$\cos\theta + i\sin\theta$

Euler's formula:

$= e^{i\theta}$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

$$b = M\sin\theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a + bi = M(\cos\theta + i\sin\theta)$$

$$a + bi = Me^{i\theta}$$

(Polar form makes multiplication much cleaner)

Complex roots (Section 3.3)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- When $b^2 - 4ac < 0$, we get complex roots:

$$\begin{aligned} r_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a} \\ &= \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i \\ &= \alpha \pm \beta i \end{aligned}$$

Complex roots (Section 3.3)

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$
\begin{aligned}
y_1(t) &= e^{(\alpha+\beta i)t} \\
&= e^{\alpha t} e^{i\beta t} \\
&= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) &= e^{(\alpha-\beta i)t} \\
&= e^{\alpha t} e^{-i\beta t} \\
&= e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t)) \\
&= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))
\end{aligned}
$$

Complex roots (Section 3.3)

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$y_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$y_2(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

- Instead of using these to form the general solution, let's use them to find two real valued solutions:

$$\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{\alpha t}\cos(\beta t)$$

$$\frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = e^{\alpha t}\sin(\beta t)$$

- General solution:

$$y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$

Complex roots (Section 3.3)

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i$, $r_2 = 1 - 2i$

★ (B) $r_1 = -1 + 2i$, $r_2 = -1 - 2i$

(C) $r_1 = 1 - 2i$, $r_2 = -1 + 2i$

(D) $r_1 = 2 + 4i$, $r_2 = 2 - 4i$

(E) $r_1 = -2 + 4i$, $r_2 = -2 - 4i$

Complex roots (Section 3.3)

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

☆ (A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

(D) $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$

Complex roots (Section 3.3)

- Example: Find the solution to the IVP

$$y'' + 2y' + 5y = 0,\ \ y(0) = 1,\ \ y'(0) = 0$$

- General solution: $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

(B) $y(t) = e^{-t}\left(\cos(2t) - \dfrac{1}{2}\sin(2t)\right)$

(C) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

☆ (D) $y(t) = \dfrac{1}{2}e^{-t}(2\cos(2t) + \sin(2t))$

Repeated roots (Section 3.4)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  ii. A repeated real root: $b^2 - 4ac = 0$.

  iii. Two complex roots: $b^2 - 4ac < 0$.

- For case ii ($r_1 = r_2 = r$), we need another independent solution!
- Reduction of order - a method for guessing another solution.

Reduction of order

- You have one solution $y_1(t)$ and you want to find another independent one, $y_2(t)$.
- Guess that $y_2(t) = v(t)y_1(t)$ for some as yet unknown $v(t)$. If you can find $v(t)$ this way, great. If not, gotta try something else.
- Example - $y'' + 4y' + 4y = 0.$ Only one root to the characteristic equation, r=-2, so we only get one solution that way: $y_1(t) = e^{-2t}$.
- Use Reduction of order to find a second solution.

$$y_2(t) = v(t)e^{-2t}$$

- Heuristic explanation for exponential solutions and Reduction of order.

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$. $\quad y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$

$$\Downarrow \qquad\qquad \Downarrow$$

$$4y_2(t) = 4v(t)e^{-2t} \qquad 4y_2'(t) = 4v'(t)e^{-2t} - 8v(t)e^{-2t}$$

$$y_2''(t) = v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$$

$$\Rightarrow y_2''(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}$$

$$0 = y_2'' + 4y_2' + 4y_2 = v''e^{-2t}$$

$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$ (where $v(t) = C_1 t + C_2$).

$$= (C_1 t + C_2)e^{-2t}$$

$$y(t) = C_1 te^{-2t} + C_2 e^{-2t}$$

$y_2(t)$ $y_1(t)$

Is this the general solution? Calculate the Wronskian:

$$W(e^{-2t}, te^{-2t})(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-4t} \neq 0$$

So yes!

Summary (3.1-3.4)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. (r_1, r_2)

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  ii. A repeated real root: $b^2 - 4ac = 0$. (r)

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

  iii. Two complex roots: $b^2 - 4ac < 0$. ($r_{1,2} = \alpha \pm i\beta$)

$$y = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

(A) $y(t) = C_1 e^{-2t} + C_2 e^{-4t}$

☆ (B) $y(t) = C_1 e^{2t} + C_2 e^{4t}$

(C) $y(t) = e^{2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(D) $y(t) = e^{-2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(E) $y(t) = C_1 e^{2t} + C_2 t e^{4t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 9y = 0$$

(A) $y(t) = C_1 e^{3t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{3t}$

(C) $y(t) = C_1 e^{3t} + C_2 e^{-3t}$

☆ (D) $y(t) = C_1 e^{3t} + C_2 t e^{3t}$

(E) $y(t) = C_1 e^{3t} + C_2 v(t) e^{3t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 10y = 0$$

(A) $y(t) = C_1 e^{3t} + C_2 e^{t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{-t}$

(C) $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$

(D) $y(t) = e^{t}(C_1 \cos(3t) + C_2 \sin(3t))$

★ (E) $y(t) = e^{3t}(C_1 \cos(t) + C_2 \sin(t))$

Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

- An mxn matrix is a gizmo that takes an n-vector and returns an m-vector:

$$\overline{y} = A\overline{x}$$

- It is called a linear operator because it has the following properties:

$$A(c\overline{x}) = cA\overline{x}$$

$$A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$$

- Not all operators work on vectors. Derivative operators take a function and return a new function. For example,

$$z = L[y] = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y$$

- This one is linear because

$$L[cy] = cL[y]$$

$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

Some connections to linear (matrix) algebra

- A homogeneous matrix equation has the form

$$A\overline{x} = \overline{0}$$

- A non-homogeneous matrix equation has the form

$$A\overline{x} = \overline{b}$$

- A homogeneous differential equation has the form

$$L[y] = 0$$

- A non-homogeneous differential equation has the form

$$L[y] = g(t)$$

Some connections to linear (matrix) algebra

Systems of equations written in operator notation.

System of equations	Operator definition	Equation in operator notation
$x_1 + 2x_2 = 4$ $3x_1 + 4x_2 = 7$	$A\overline{x} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	$A\overline{x} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

Some differential equations we've seen, written in operator notation.

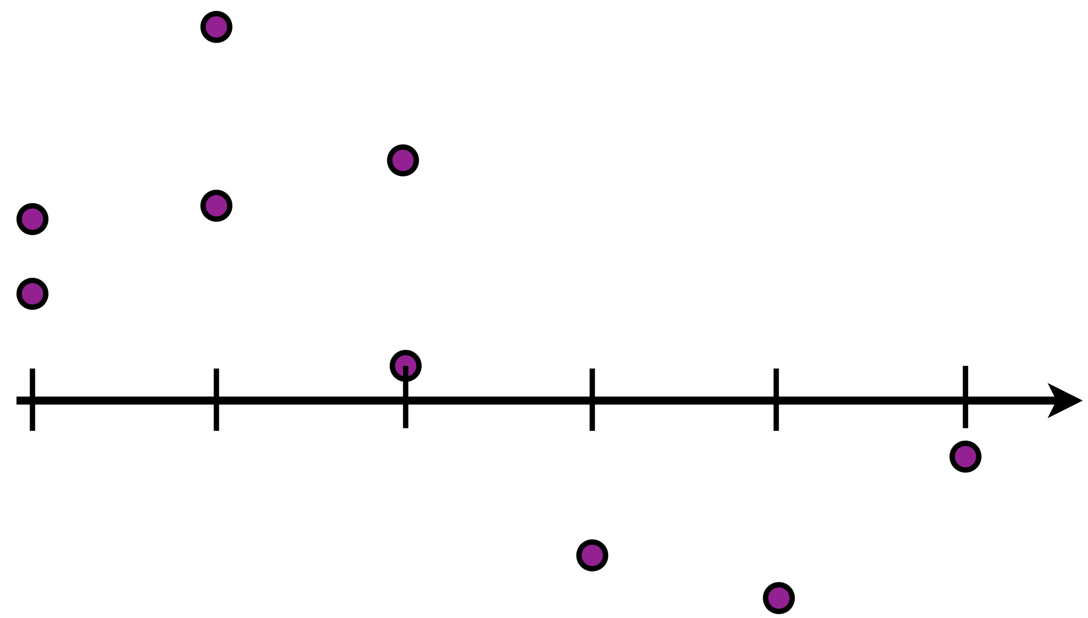
Differential equation	Operator definition	Equation in operator notation
$t\frac{dy}{dt} + 2y = 4t^2$	$L[y] = t\frac{dy}{dt} + 2y$	$L[y] = 4t^2$
$y'' + 4y' + 4y = 0$	$L[y] = y'' + 4y' + 4y$	$L[y] = 0$

Some connections to linear (matrix) algebra

- A more detailed connection between matrix equations and DEs:
  - A vector as a function

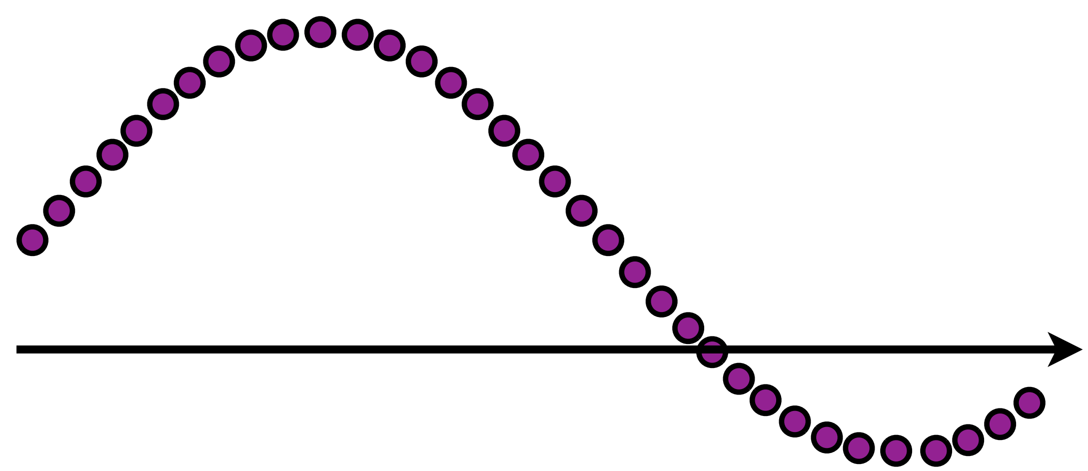
$$\bar{b} = \begin{pmatrix} \sin(1) \\ \sin(2) \\ \sin(3) \\ \sin(4) \\ \sin(5) \\ \sin(6) \end{pmatrix}$$

$$\bar{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$



  - A function is just a vector with an infinite number of entries.

$$y(t) = \sin(t)$$



  - A differential operator is just a really big matrix.