Last name: $\qquad$ First name: $\qquad$ Student \#: $\qquad$
Tutorial (circle one): T2A - Cole, T2B - Shirin, T2C - Dhananjay, T2D - Xiaoyu, T2E - Catherine, T2F - Will
Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. [ $\mathbf{6} \mathbf{p t s}$ ] Consider the system of equations $\mathbf{x}^{\prime}=A \mathbf{x}$ where

$$
A=\left(\begin{array}{cc}
a & a-3 \\
a-3 & a
\end{array}\right) .
$$

(a) In each row of the table below, give inequalities/equations involving $a$ which ensure that the steady state is of the given type. Write X if there is no value of $a$ that gives the specified classificaiton.

| Type | Condtion(s) on $a$ |
| :--- | :--- |
| unstable node |  |
| unstable spiral |  |
| stable node |  |
| stable spiral |  |
| saddle |  |
| repeated <br> eigenvalue |  |
| zero eigenvalue |  |

2. [ $\mathbf{5} \mathbf{~ p t s ] ~ F i n d ~ t h e ~ g e n e r a l ~ s o l u t i o n ~ t o ~ t h e ~ s y s t e m ~ o f ~ e q u a t i o n s ~} \mathbf{x}^{\prime}=A \mathrm{x}$ where

$$
A=\left(\begin{array}{cc}
2 & -1 \\
1 & 2
\end{array}\right)
$$

3. [7 pts] Suppose tank A and B each hold 5 L . The upper pipe dumping solution into tank A (marked $\mathrm{A}_{\text {in }}$ ) has a salt concentration of $2 \mathrm{~g} / \mathrm{L}$ and the one dumping solution in tank B (marked $\mathrm{B}_{\text {in }}$ ) has a salt concentration of $3 \mathrm{~g} / \mathrm{L}$.
(a) Write down a system of equations for the problem.
(b) Find the steady state of the system.

4. [6 pts] Write an expression for the function $f(t)$ shown below using Heaviside functions. In your final answer, all terms should be in the form $u_{c}(t) g(t-c)$ for some $g$, such that the Laplace transform is easy to compute.

5. [15 pts] Consider the function $m(t)=16+12\left(u_{1}(t)-u_{2}(t)\right)$.
(a) Sketch $m(t)$.
(b) Solve the differential equation $y^{\prime}+4 y=m(t)+3 \delta(t-3)$ with initial condition $y(0)=0$.
(c) Sketch the solution to the equation in (b).
6. [ $\mathbf{5} \mathbf{~ p t s}$ ] The motion of a forced tuning fork satisfies the equation $y^{\prime \prime}+2 y^{\prime}+9 y=5 \cos (\omega t)$. The Method of Undetermined Coefficients gives a particular solution of

$$
y_{p}(t)=A \cos (\omega t)+B \sin (\omega t)
$$

where

$$
A=\frac{5\left(9-\omega^{2}\right)}{4 \omega^{2}+\left(9-\omega^{2}\right)^{2}} \quad \text { and } B=\frac{10 \omega}{4 \omega^{2}+\left(9-\omega^{2}\right)^{2}} .
$$

At what frequency $\omega$ does the tuning fork vibrate with largest amplitude? Using a formula for the amplitude of $y_{p}$ is acceptable. Any claims (e.g. " $\omega=42$ is a max.") must be justified.

Hint: Recall that taking the square root of a function does not change the location of its minima and the reciprocal of a function has maxima wherever the original function has minima.
7. (a) [ $\mathbf{3} \mathbf{p t s}]$ Calculate the inverse transform of $Y(s)=\frac{s}{s^{2}+8 s+20}$.
(b) [ $\mathbf{3} \mathbf{p t s}]$ Give a differential equation and initial values that would have a transformed solution $Y(s)$.
8. [ $\mathbf{5} \mathbf{~ p t s ] ~ C o n s i d e r ~ t h e ~ v e c t o r ~ f i e l d ~ i n ~ t h e ~ f i g u r e ~ b e l o w . ~ T h e ~ a b s o l u t e ~ v a l u e s ~ o f ~ t h e ~ e i g e n v a l u e s ~ a r e ~} 1$ and 3 . The eigenvectors lie along the straight lines. Give an expression for the general solution to the differential equation associated with the vector field. You will have to determine the sign of each eigenvalue and the eigenvector associated with each eigenvalue. The sizes of the vectors are to scale.


This page is for rough work. It will not be marked.

## Laplace transforms

| $f(t)$ | $F(s)$ |
| :--- | :--- |
| 1 | $\frac{1}{s}$ |
| $t^{n}$ | $\frac{n!}{s^{n+1}}$ |
| $\sin (a t)$ | $\frac{a}{s^{2}+a^{2}}$ |
| $\cos (a t)$ | $\frac{s}{s^{2}+a^{2}}$ |
| $e^{a t} f(t)$ | $\frac{F(s-a)}{c} F\left(\frac{s}{c}\right)$ |
| $f(c t)$ | $e^{-s c} F(s)$ |
| $u_{c}(t) f(t-c)$ |  |
| $\delta(t-c)$ | $e^{-s c}$ |

