Today

- Solving ODEs using Laplace transforms
- The Heaviside and associated step and ramp functions
- ODE with a ramped forcing function

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 4y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0$$
$$s^2Y(s) - s - 0 + 4Y(s) = 0$$
$$s^2Y(s) + 4Y(s) = s$$
$$Y(s) = \frac{s}{s^2+4}$$

- To find y(t), we have to invert the transform. What y(t) would have Y(s) as its transform?

- Recall that $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{\omega^2 + s^2}$. So $y(t) = \cos(2t)$.

Solving IVPs using Laplace transforms

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13}$$

$$\lambda = \frac{-6 \pm i\sqrt{52-36}}{2} = -3 \pm 2i$$

- To find y(t), we hav … would have Y(s) as its transform?

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{e^{-3t}\cos t\} = \frac{s+3}{1+(s+3)^2}$$

$$Y(s) = \frac{s+3+3}{s^2+6s+9+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{2}\frac{2}{(s+3)^2+4}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

Solving IVPs using Laplace transforms - complex

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13} = \frac{s+6}{s^2+6s+9+4} = \frac{s+6}{(s+3)^2+4} = \frac{s+3+3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+2^2} + \frac{3}{2}\frac{2}{(s+3)^2+2^2}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

1. Does the denominator have real or complex roots? Complex.
2. Complete the square in the denominator.
3. Put numerator in form (s+α)+β where (s+α) is the completed square.
4. Fix up coefficient of the term with no s in the numerator.
5. Invert.

Solving IVPs using Laplace transforms - real

- Solve the equation $y'' + 6y' + 5y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+5} = \frac{s+6}{s^2+6s+9-4} = \frac{s+6}{(s+3)^2-4} = \frac{s+6}{(s+1)(s+5)}$$

$$= \frac{5}{4} \cdot \frac{1}{s+5} - \frac{1}{4} \cdot \frac{1}{s+1}$$ (partial fraction decomposition)

$$y(t) = \frac{5}{4} \; e^{-5t} \; - \; \frac{1}{4} \; e^{-t}$$

1. Does the denominator have real or complex roots? Real.
2. Factor the denominator (factor directly, complete the square or QF).
3. Partial fraction decomposition.
4. Invert. Recall that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$.

Solving IVPs using Laplace transforms - nonhomog

- What is the transformed equation for the IVP

$$y' + 6y = e^{2t}$$
$$y(0) = 2$$

(A) $Y'(s) + 6Y(s) = \dfrac{1}{s+2}$

(B) $Y'(s) + 6Y(s) = \dfrac{1}{s-2}$

(C) $sY(s) + 2 + 6Y(s) = \dfrac{1}{s+2}$

☆(D) $sY(s) - 2 + 6Y(s) = \dfrac{1}{s-2}$

(E) Explain, please.

$$\mathcal{L}\{y'(t)\} = sY(s) - 2$$
$$\mathcal{L}\{6y(t)\} = 6Y(s)$$
$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$$

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{(2-s)t}\, dt$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Solving IVPs using Laplace transforms

- Find the solution to $y' + 6y = e^{2t}$, subject to IC $y(0) = 2$.

$$sY(s) - 2 + 6Y(s) = \frac{1}{s-2}$$

$$Y(s) = \left(2 + \frac{1}{s-2}\right) / (s+6)$$

$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

$$\downarrow$$

$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

$$y(t) = 2e^{-6t} + \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s+6}\right)$$

$$y(t) = 2e^{-6t} + \frac{1}{8}e^{2t} - \frac{1}{8}e^{-6t}$$

$$\frac{1}{(s-2)(s+6)} = \frac{A}{s-2} + \frac{B}{s+6}$$

$$1 = A(s+6) + B(s-2)$$

$$(s=2) \qquad 1 = 8A$$

$$(s=-6) \qquad 1 = -8B$$

$$y(t) = \frac{15}{8}e^{-6t} + \frac{1}{8}e^{2t}$$

$$y_h(t) = Ce^{-6t}$$

$$C = \frac{15}{8} \qquad y_p(t) = \frac{1}{8}e^{2t}$$

Solving IVPs using Laplace transforms

- With a forcing term, the equation

$$ay'' + by' + cy = g(t)$$

  has Laplace transform

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s)$$

- Solving for Y(s):

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

transform of homogeneous solution with two degrees of freedom (y(0) and y'(0) act like C_1 and C_2.

transform of particular solution

Solving IVPs using Laplace transforms

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

- If denominator has distinct real factors, use PFD and get

$$Y_h(s) = \frac{A}{s-r_1} + \frac{B}{s-r_2} \quad \longrightarrow \quad y_h(t) = Ae^{r_1 t} + Be^{r_2 t}$$

- If denominator has repeated real factors, use PFD and get

$$Y_h(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2} \quad \longrightarrow \quad y_h(t) = Ae^{rt} + Bte^{rt}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Solving IVPs using Laplace transforms

$$Y(s) = \frac{(as+b)y(0)+ay'(0)}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

- Unique real factors, $Y_h(s) = \dfrac{A}{s-r_1} + \dfrac{B}{s-r_2} \quad \rightarrow \quad y_h(t) = Ae^{r_1 t} + Be^{r_2 t}$
- Repeated factor, $Y_h(s) = \dfrac{A}{s-r_1} + \dfrac{B}{(s-r_2)^2} \rightarrow \quad y_h(t) = Ae^{r_1 t} + Bte^{r_1 t}$
- No real factors, complete square, simplify and get

$$Y_h(s) = \frac{As}{(s-\alpha)^2+\beta^2} + \frac{B}{(s-\alpha)^2+\beta^2} \qquad (\ A = ay(0),\ \ B = ay'(0)+by(0)\ )$$

$$Y_h(s) = \frac{A(s-\alpha)+A\alpha}{(s-\alpha)^2+\beta^2} + \frac{B}{(s-\alpha)^2+\beta^2}$$

$$Y_h(s) = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B+A\alpha}{(s-\alpha)^2+\beta^2}$$

$$Y_h(s) = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B+A\alpha}{\beta}\frac{\beta}{(s-\alpha)^2+\beta^2} \qquad \rightarrow y(t) = e^{-\alpha t}\left(A\cos(\beta t) + \frac{B+A\alpha}{\beta}\sin(\beta t)\right)$$

Solving IVPs using Laplace transforms

- Inverting the forcing/particular part $Y_p(s) = \dfrac{G(s)}{as^2 + bs + c}$.
- Usually a combination of similar techniques (PFD, manipulating constants) works.
- Which is the correct PFD form for $Y(s) = \dfrac{s^2 + 2s - 3}{(s-1)^2(s^2+4)}$?

(A) $Y(s) = \dfrac{A}{(s-1)^2} + \dfrac{B}{(s^2+4)}$

(B) $Y(s) = \dfrac{As+B}{(s-1)^2} + \dfrac{Cs+D}{(s^2+4)}$

(C) $Y(s) = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s^2+4)}$

★(D) $Y(s) = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{Cs+D}{(s^2+4)}$

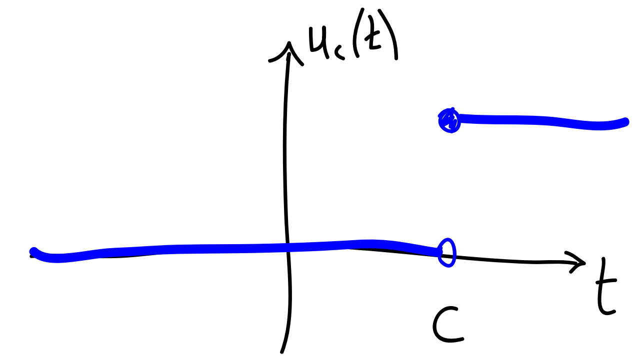
(E) MATH 101 was a long time ago.

Laplace transforms (so far)

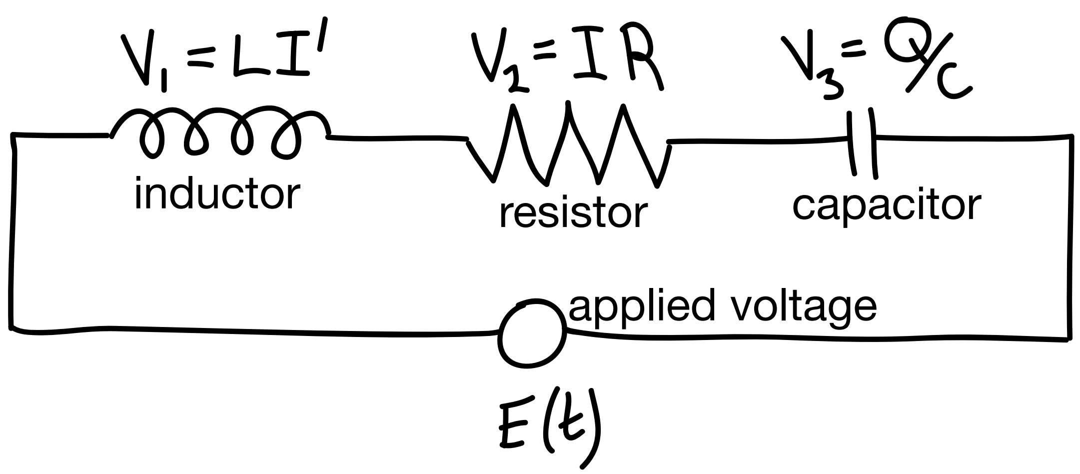
$f(t)$	$F(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Step function forcing

- We define the Heaviside function $u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$

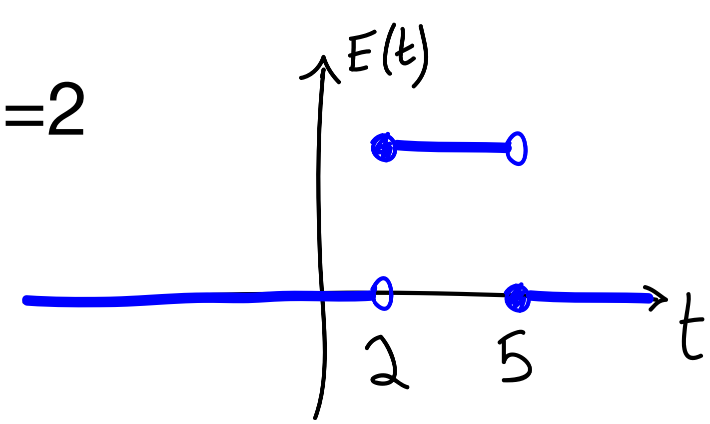


- We use it to model on/off behaviour in ODEs.
- For example, in LRC circuits, Kirchoff's second law tells us that:



$$V_1 + V_2 + V_3 = E(t)$$
$$LI' + IR + \frac{1}{C}Q = E(t)$$
$$I = Q'$$
$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

- If E(t) is a voltage source that can be turned on/off, then E(t) is step-like.
- For example, turn E on at t=2 and off again at t=5:



- In WW, $u_c(t) = u(t-c) = h(t-a)$

Step function forcing

- Use the Heaviside function to rewrite $g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$

(A) $g(t) = u_2(t) + u_5(t)$

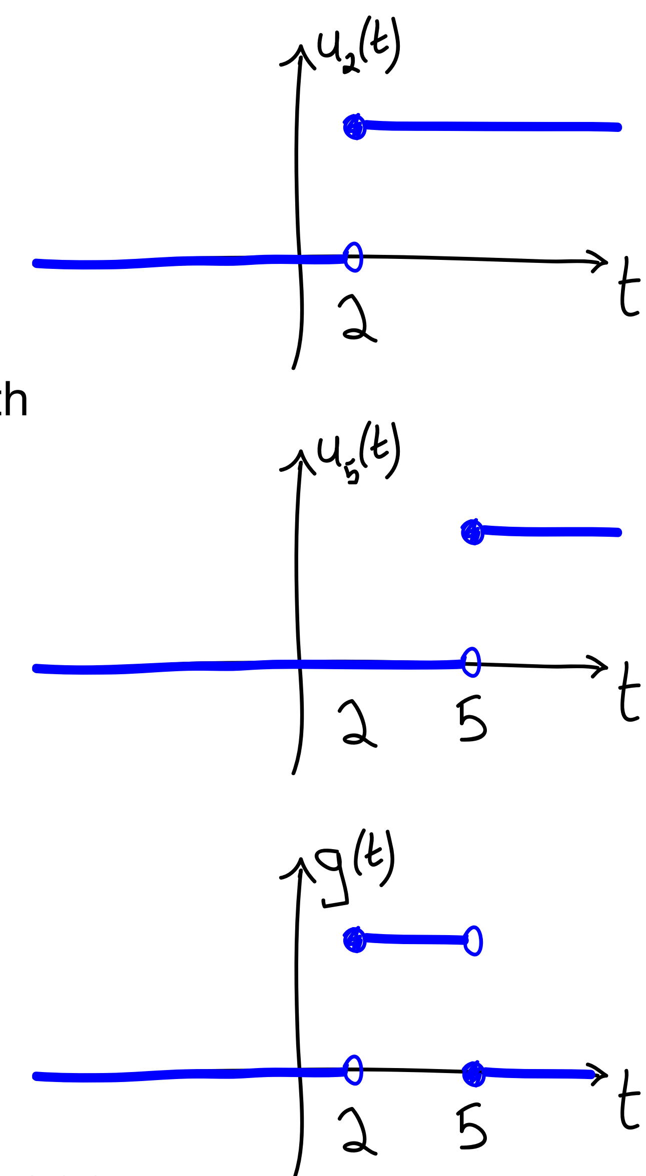
(B) $g(t) = u_2(t) - u_5(t)$

(C) $g(t) = u_2(t)(1 - u_5(t))$

messier with transforms

(D) $g(t) = u_5(t) - u_2(t)$

(E) Explain, please.



- For more on writing down functions in Heaviside notation see: https://www.youtube.com/watch?v=TGzU5O6csyA

Step function forcing

- What is the Laplace transform of

$$g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

$$= u_2(t) - u_5(t) \text{ ?}$$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\ dt$$

$$= \int_c^\infty e^{-st}\ dt \quad = -\frac{1}{s} e^{-st} \Big|_c^\infty = \frac{e^{-sc}}{s} \quad (s > 0)$$

$$\mathcal{L}\{u_2(t) - u_5(t)\} = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} \quad (s > 0)$$

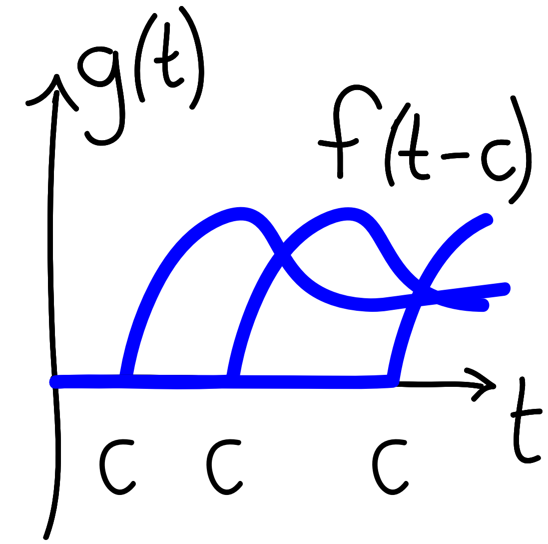
Recall: $\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty e^{-st}(f(t) + g(t))\ dt$

$$= \int_0^\infty e^{-st} f(t)\ dt + \int_0^\infty e^{-st} g(t)\ dt$$

$$= \quad \mathcal{L}\{f(t)\} \quad + \quad \mathcal{L}\{g(t)\}$$

Step function forcing

- Suppose we know the transform of f(t) is F(s).
- It will be useful to know the transform of



$$k(t) = \begin{cases} 0 & \text{for } t < c, \\ f(t-c) & \text{for } t \geq c. \end{cases}$$

$$= u_c(t) f(t-c)$$

$$\mathcal{L}\{k(t)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\ dt$$

$$= \int_c^\infty e^{-st} f(t-c)\ dt \qquad u = t - c,\ du = dt$$

$$= \int_0^\infty e^{-s(u+c)} f(u)\ du$$

$$= e^{-sc} \int_0^\infty e^{-su} f(u)\ du \qquad = e^{-sc} F(s)$$